Data Structures

(Feodor F. Dragan)
Department of Mathematics and Computer Science
Kent State University

Fall, 2000

CHAPTER 8

Recursion

Contents

- What is Recursion?
- Proofs by Mathematical Induction
- Numerical Applications
- Divide-and-Conquer Algorithms

What is Recursion?

- A recursive function directly or indirectly calls itself
- Very powerful problem-solving tool
- Many applications in CS

The Factorial Function

$$n! = \begin{cases} 1 & if \;\; n = 0 \\ n \cdot (n-1)! & if \;\; n > 0 \end{cases}$$

Iterative Definition

```
prod=1;
for (x=n; x>0; x--)
    prod *=x;
return(prod);
```

Recursive Definition

```
long fact( long n )
{   if (n<=1)
            return 1;
    else
            return( n* fact( n-1 ));
}
```

5!=5*4!
4!=4*3!
3!=3*2!
2!=2*1!
1!=1*0!
0!=1

Proofs by Mathematical Induction

• MI is an important proof technique used to establish theorems that hold for positive integers

A proof by induction shows that the theorem is true for some simple cases and then shows how to extend the range of true cases indefinitely.

Theorem 1. *For any integer* $N \geq 1$,

$$\sum_{i=1}^{N} i = 1 + 2 + \cdots + N = \frac{N(N+1)}{2} \qquad (1)$$

Proof:

- Equation (1) is true for $N = 1$
- Suppose that it is true for all N, $1 \leq N \leq k$
- We will prove that it is true for $N = k+1$

In a proof by induction, the basis is the easy case that can be shown by hand.

The inductive hypothesis assumes that the theorem is true for some arbitrary case.

We need to prove that, under this assumption, theorem is true for the next case.

$$\sum_{i=1}^{k+1} i = (k+1) + \sum_{i=1}^{k} i = (k+1) + \frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Theorem 2. *For any integer* $N \geq 1$,

$$\sum_{i=N}^{1}(-1)^{N-i}i^2 = N^2 - (N-1)^2 + (N-2)^2 - \cdots (-1)^{N-1} = \frac{N(N+1)}{2} \qquad (2)$$

Proof:

- **Bases:** Equation (2) is true for $N = 1$
- **Progress:** Suppose that it is true for $N = k$, i.e.,

$$\sum_{i=k}^{1}(-1)^{k-i}i^2 = \frac{k(k+1)}{2}$$

We must show that $(N = k+1)$

$$\sum_{i=k+1}^{1}(-1)^{k+1-i}i^2 = \frac{(k+1)(k+2)}{2}$$

Indeed, $\sum_{i=k+1}^{1}(-1)^{k+1-i}i^2 = (k+1)^2 - \{k^2 - (k-1)^2 + \cdots\}$

$$= (k+1)^2 - \left\{\sum_{i=k}^{1}(-1)^{k-i}i^2\right\}$$

$$= (k+1)^2 - \frac{k(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Recursion

```
// Recursive routine to compute sum of first n integers  (bad use of recursion)
long s( int n)
{
        if ( n==1 )
           return 1;
        else
           return s( n-1 ) + n;
}
```

```
s(5)=5+s(4)
      s(4)=4+s(3)
            s(3)=3+s(2)
                  s(2)=2+s(1)
                        s(1)=1
```

- How it works
  - Function calling and return sequence are stack operations
  - Always have at least one base case that can be solved without recursion
  - Any recursive call must progress toward a base case
- Recursion can always be removed by using explicit stack
  - Recursive algorithm can be rewritten by iteration (to save space); see Fibonacci series (next slide)
- Too much recursive calls can be dangerous
  - Do not use recursion as a substitute for a simple loop

Fibonacci Sequences: Another bad example

$$f(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2 \end{cases}$$

```
if (n<=1)
        return ( n ) ;
f1=0;
f2=1;
for ( i=2; i<=n; i++ )
{
        x=f1;
        f1=f2;
        f2=x+f1;
}
 return ( f2 );
```

```
// compute the nth Fibonacci number
// bad example of recursion use

long fib ( int n )
{
if  (n<=1)
        return  n;
else
        return fib( n-1 ) + fib (n-2 );
}
```

1. f(7)=f(5)+f(6)
2. f(6)=f(4)+f(5)
3. f(5)=f(3)+f(4)
4. f(4)=f(2)+f(3)
5. f(3)=f(1)+f(2)
6. f(2)=f(1)+f(0)
7. 1 0

- The major drawback of this routine is redundant calls
  - each call to *f(n-1)* or *f(n-2)* makes a call to *f(n-3)* ➔3 calls to *f(n-3)*
  - each call to *f(n-2)* or *f(n-3)* makes a call to *f(n-4)* ➔ 5 calls to *f(n-4)*
  - ... .
- In general, let *C(N)* be the number of calls to *fib* during the evaluation of *fib(n)*
- Then,

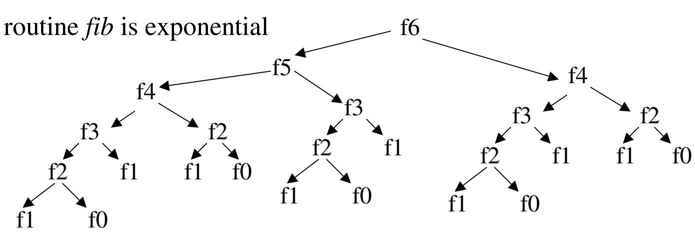
$$C(1) = C(0) = 1$$

$$C(N) = C(N-1) + C(N-2) + 1$$

By induction, we can easily verify that for *N>2,* the solution to this recurrence relation is

$$C(N) = f(N+2) + f(N-1) - 1$$

- The recursive routine *fib* is exponential



Four Basic Rules of Recursion

- *m* **Base case(s):** You must always have some base cases, which can be solved without recursion.
- **Making Progress:** For the cases that are to be solved recursively, the recursive call must always be to a case that makes progress toward a base case.
- **Design rule:** Assume that all recursive calls work and correctly perform current needed calculations (... to prove correctness of the algorithm by induction).
- **Compound Interest Rule:** Never duplicate work by solving the same instance of a problem in separate recursive calls.

Binary Search

- Search is a common activity in computing
- Linear or sequential search is not efficient for large files
- For large data set, binary search is more suitable

```
binsearch( a, x, low, high)
int a[];
int x;
int high, low;
{
        int mid;
        if ( low > high )
           return( -1);
        mid=(low+high)/2;
        return( x==a[mid] ? mid : x< a[mid] ? binsearch (a, x, low, mid-1) :
              binsearch (a, x, mid+1, high) );
}
```

- To make it more efficient, variables *a* and *x* can be eliminated and externally referenced by the main program. Therefore the function would be *binsearch (low, high)*
- An important requirement for a recursive algorithm is to avoid infinite sequence of calls. To do this one or more nonrecursive exit arguments should be presented in the program.

Indirect recursion

- A recursive function may call itself indirectly via another function
- More than two routines may participate in a *recursive chain*

```
a ( formal parameters )
{
        .
        .
        .
    b ( arguments );
}
```

```
b ( formal parameters )
{
        .
        .
        .
    a ( arguments );
}
```


Numerical Applications

Modular Exponentiation

- How to compute $X^N \pmod{P}$ efficiently?
- One way is

```
result =1;
for (i=0; i<N; i++) { result = (result * X) mod P }
```

- If N is huge, doing N multiplications is impractical
- A faster algorithm is based on the following recurrence relation

$$X^N = \begin{cases} (X \cdot X)^{\lfloor N/2 \rfloor} & \text{if } N \text{ is even} \\ X \cdot (X \cdot X)^{\lfloor N/2 \rfloor} & \text{if } N \text{ is odd} \end{cases}$$

Modular Exponentiation Routine

```
// Compute x^n ( mod p ).
// HugeInt: must have copy constructor, operator=,
//              conversion from int, *, /, %, ==, and !=
//              Assumes that p is not zero and power( 0, 0, p ) is 1
template <class HugeInt>
HugeInt power( const HugeInt & x, const HugeInt & n , const HugeInt & p )
{
            if ( n == 0 )
                        return 1;
            HugeInt tmp = power( ( x * x ) % p, n / 2, p );
            if( n % 2 != 0 )
                        tmp = ( tmp * x ) % p;
            return tmp;
}
```

- Time complexity
  - Let *M(N)* be the # of multiplications used by *power*
  - $M(N) \le M\lfloor N/2 \rfloor + 2$ Why?
    - If N is even, *1*+ those done recursively
    - If N is odd, *2*+ those done recursively
  - *M(0)=0*
  - Using halving principle, on average *M(N)=(3/2)log N*
- Exponentiation can be done in logarithmic number of multiplications

Greatest Common Divisor (gcd)

- The largest integer that divides two integers?
- $\gcd(A,B) \equiv \gcd(A-B,B)$ Way?
  - if *D* divides *A* and *B,* it must divide *A-B*
  - if *D* divides *A-B* and *B,* it must divide *A*
- **Euclid's Algorithm:**
  - repeatedly subtract *B* from *A* until *A<B (A=A-B)*
  - repeatedly subtract *A* from *B* until *B<A (B=B-A)*
  - repeat again until *B=0*
  - then *gcd(A,0)=A*
- Example
  - *A=18, B=4*
  - *A=18* ➔ *A=14* ➔ *A=10* ➔ *A=6* ➔ *A=2*
  - *B=4* ➔ *B=2* ➔ *B=0*
  - *gcd(2,0)=2* ➔ *gcd(A,B)=2*

• In a more efficient way

$$gcd\ (A, B) = gcd\ (B, (A \bmod B)) \Rightarrow gcd\ (18,4) = gcd\ (4,2) = 2$$

• This recursive equation gives an algorithm

```
// Returns the greatest common divisor of a and b

template <class HugeInt>
HugeInt gcd( const HugeInt & a, const HugeInt & b )
{
        if ( b==0 )
           return a;
        else
           return gcd( b, a%b);
}
```

• The number of recursive calls is proportional to the logarithm of A

since $A \bmod B$ is $< A/2$ (consider two cases: $B \le A/2$ and $B>A/2$)

Divide-and-Conquer Algorithms

• Divide: Smaller problems are solved recursively (except, of course, base cases)

• Conquer: Solution to the original problem is formed from the solutions of the subproblems

• Inherently recursive

Maximum Contiguous Subsequence Sum

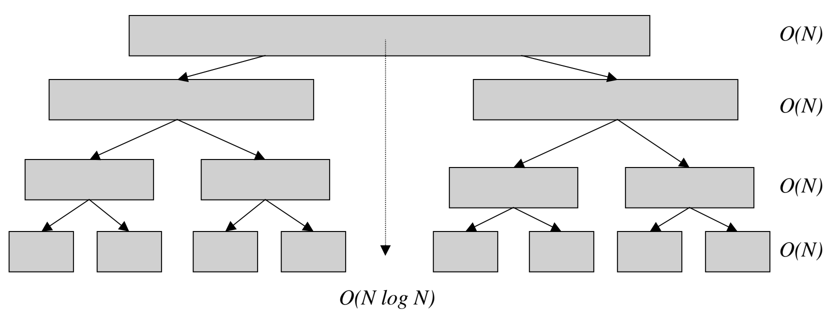
- Divide the sequence into two halves
- The MCSS can reside
  1. in the 1st half 2. in the 2nd half 3. in both
- In case 3, the last element of the 1st half and the first element of the 2nd half are parts of MCSS

  using two N/2 loops starting from these points, we can find two MCSS from each half, and then add them ➔ $O(N)$
- Cases 1 and 2 can be done recursively
- Choose the max among the three cases

```
// Recursive O( N log N ) maximum contiguous subsequence sum algorithm.
template <class Comparable>
Comparable maxSubSum( const vector<Comparable> & a, int left, int right )
{
        Comparable maxLeftBorderSum = 0, maxRightBorderSum = 0;
        Comparable leftBorderSum = 0, rightBorderSum = 0;
        int center = ( left + right ) / 2;
        if( left == right ) // Base Case.
            return a[ left ] > 0 ? a[ left ] : 0;
        Comparable maxLeftSum = maxSubSum( a, left, center );
        Comparable maxRightSum = maxSubSum( a, center + 1, right );
        for( int i = center; i >= left; i-- )
        {
                leftBorderSum += a[ i ];
                if( leftBorderSum > maxLeftBorderSum )
                    maxLeftBorderSum = leftBorderSum;
        }
        for( int j = center + 1; j <= right; j++ )
        {
                rightBorderSum += a[ j ];
                if( rightBorderSum > maxRightBorderSum )
                    maxRightBorderSum = rightBorderSum;
        }
        return max3( maxLeftSum, maxRightSum, maxLeftBorderSum + maxRightBorderSum );
}
// Public driver.
template <class Comparable>
Comparable maxSubsequenceSum( const vector<Comparable> & a )
{ return a.size( ) > 0 ? maxSubSum( a, 0, a.size( ) - 1 ) : 0; }
```


- Time analysis ➔ *O(N log N)*

Assuming N is a power of 2, the solution to the equation $T(N) = 2\,T(N/2) + N$, with initial condition $T(1) = 1$ is $T(N) = N \log N + N$



Trace of recursive calls for recursive maximum contiguous subsequence sum algorithm

Homework

Problem 8.27 (a), page 317