Data Structures

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CHAPTER 10
Randomization

Contents

- Introduction
- Pseudorandom Numbers
- Generating Pseudorandom Numbers Algorithms
- Nonuniform Random Numbers
- How to Generate a Random Permutation
- Randomized Algorithms

Introduction

- Many applications require random numbers
  - Cryptography, simulation, even searching and sorting
  - Programming testing

Definition 1. A permutation of 1, 2, ... , N is a sequence on N integers that includes each of 1, 2, ... , N exactly once.

- How do we test a sorting algorithm
- generate test data; random permutations, sort them
- Simulation is another important application
- Gaming, Randomized Algorithms

Generating Pseudorandom Numbers Algorithms

- The following algorithms differ in
  - Technique
  - Speed of Generation
  - Length of the repeatability cycle
  - Ease of programming

Random Number Generators

- No true randomness
- *Pseudorandom numbers* appear to be random
  - Have many properties of random numbers
- Hard to find good number generator
- How to flip a coin? ➔ look for the last digit of the clock
  - How fast is the clock?
- *Uniform random numbers:* each number occurs with the same probability
- Typically a random sequence, rather than one random number, is required

The Mid-Square Method

- Developed in mid 1940's by John von Neumann
- It starts with some initial number (seed)
- The number is then squared
- The middle digits of this square are used as the next number in the sequence

- Example:

$$\begin{aligned}
x_0 &= 3187 \\
(3187)^2 = 10156969 &\Rightarrow x_1 = 1569 \\
(1569)^2 = 02461761 &\Rightarrow x_2 = 4617 \\
(4617)^2 = 21316689 &\Rightarrow x_3 = 3166 \\
(3166)^2 = 10023566 &\Rightarrow x_4 = 0235 \\
(0235)^2 = 00055225 &\Rightarrow x_5 = 0552 \\
(0552)^2 = 00304704 &\Rightarrow x_6 = 3047 \\
(3047)^2 = 09284209 &\Rightarrow x_7 = 2842
\end{aligned}$$

- This process could continue to generate;
  0769, 5913, 9635, 8332, 4222, 8542

- This technique has a number of shortcomings:
  1. Short repeatability period
  2. Whenever zero is generated, all succeeding numbers will be zero
- The Problem with Mid-Square method led to development of alternate algorithms

$$\begin{aligned}
x_0 &= 44 \\
(44)^2 = 1936 &\Rightarrow x_1 = 93 \\
(93)^2 = 8649 &\Rightarrow x_2 = 64 \\
(64)^2 = 4096 &\Rightarrow x_3 = 09 \\
(09)^2 = 0081 &\Rightarrow x_4 = 08 \\
(08)^2 = 0064 &\Rightarrow x_5 = 06 \\
(06)^2 = 0036 &\Rightarrow x_6 = 03 \\
(03)^2 = 0009 &\Rightarrow x_7 = 00 \\
(00)^2 = 0000 &\Rightarrow x_7 = 00
\end{aligned}$$

Linear Congruential Generator

- Random sequence $$X_{i+1} = A\,X_i (\bmod M) \qquad (1)$$
- X_0 is the *seed* ➔ carefully initialized
- If A and M are carefully chosen, any other seed satisfying $A \le X_0 \le M$ is equally valid
- If M is prime, X_i is never 0
- $M = 11,\ A = 7,\ X_0 = 1$ ➔ $\underbrace{7,5,2,3,10,4,6,9,8,1}, \underbrace{7,5,2,3,10,\ldots}$

- The sequence repeats after *M-1=10* numbers
- The length of a sequence until a number is repeated is called a *period*
- A *full period* can be obtained if *A* and *M* are carefully chosen
  ➔ *full-period linear congruential generator*
- If *M* is prime, several choices of *A* give a full period, *M-1*
- Suppose *M=11, A=5* ➔ $\underbrace{5,3,4,9,1}, \underbrace{5,3,4,9,1}, \underbrace{5,3,4,\ldots}$
- If $M = 2^{31} - 1 = 2{,}147{,}483{,}647$, which is a prime number, *(1)* gives a large cycle
  - *A=48,271* is one of the many good choices

- In 32-bit machine multiplication of M will result overflow
- How to fix this?

Let M=AQ+R (2)

Then, $X_{i+1} = AX_i (\mathrm{mod}\, M) = AX_i - M\left\lfloor \frac{AX_i}{M} \right\rfloor = AX_i - M\left\lfloor \frac{X_i}{Q} \right\rfloor + M\left\lfloor \frac{X_i}{Q} \right\rfloor - M\left\lfloor \frac{AX_i}{M} \right\rfloor$

$$= AX_i - M\left\lfloor \frac{X_i}{Q} \right\rfloor + M\left(\left\lfloor \frac{X_i}{Q} \right\rfloor - \left\lfloor \frac{AX_i}{M} \right\rfloor\right)$$

Since $X_i = Q\underbrace{\left\lfloor \frac{X_i}{Q} \right\rfloor}_{quotient} + \underbrace{X_i \,\mathrm{mod}\, Q}_{remainder}$

$$X_{i+1} = A\left(Q\left\lfloor \frac{X_i}{Q} \right\rfloor + X_i \,\mathrm{mod}\, Q\right) - M\left\lfloor \frac{X_i}{Q} \right\rfloor + M\left(\left\lfloor \frac{X_i}{Q} \right\rfloor - \left\lfloor \frac{AX_i}{M} \right\rfloor\right)$$

$$= (AQ - M)\left\lfloor \frac{X_i}{Q} \right\rfloor + A(X_i \,\mathrm{mod}\, Q) + M\left(\left\lfloor \frac{X_i}{Q} \right\rfloor - \left\lfloor \frac{AX_i}{M} \right\rfloor\right)$$

Since by eqn (2) AQ - M= - R,

$$X_{i+1} = A(X_i \,\mathrm{mod}\, Q) - R\left\lfloor \frac{X_i}{Q} \right\rfloor + M\left(\left\lfloor \frac{X_i}{Q} \right\rfloor - \left\lfloor \frac{AX_i}{M} \right\rfloor\right)$$

The term $\delta(X_i) = \left\lfloor \frac{X_i}{Q} \right\rfloor - \left\lfloor \frac{AX_i}{M} \right\rfloor$ is either 0 or 1, because both terms are integers and their difference lies between 0 and 1, thus

$$\underline{\underline{X_{i+1} = A(X_i \,\mathrm{mod}\, Q) - R\left\lfloor \frac{X_i}{Q} \right\rfloor + M\delta(X_i)}}$$

```
// Random class interface
// CONSTRUCTION: with (a) no initializer or (b) an integer
//     that specifies the initial state of the generator
// ******************PUBLIC OPERATIONS*********************
//     Return a random number according to some distribution
// int randomInt( )                    --> Uniform, 1 to 2^31-1
// double randomReal( )                --> Uniform, 0..1
// int randomInt( int low, int high )  --> Uniform Low..High
// int poisson( double expectedVal )   --> Poisson
// double negExp( double expectedVal )  --> Negative exponential
class Random
{  public:
   Random( int initVal = 1 ) : state ( initVal )  { }
// Uniform distributions
   int randomInt( );            // 1 to 2^31 - 1
   double randomReal( );        // >0.0 to <1.0
   int randomInt( int low, int high );
// Non-uniform distribution
   int poisson( double expectedVal );
   double negExp( double expectedVal );

   private:
   int state;     static const int A;     static const int M;     static const int Q;     static const int R;
};
```

Interface for the ***Random*** class

Implementation of some of the ***Random*** class

```
const int Random::A = 48271;
const int Random::M = 2147483647;
const int Random::Q = M / A;
const int Random::R = M % A;

// randInt returns the next random number and updates state.
int Random::randomInt( )
{
   int tmpState = A * ( state % Q ) - R * ( state / Q );
   if( tmpState >= 0 )
      state = tmpState;
   else
      state = tmpState + M;
   return state;
}

// Return a random double in 0..1.
double Random::randomReal( )
{  return randomInt( ) / 2147483647.0;  }

// Return random int in low..high
int Random::randomInt( int low, int high )
{   return randomInt( ) % ( high - low + 1 ) + low;  }
```

Linear Congruential Generator Improved

- Many Libraries have generators based on the functions

$$X_{i+1} = (A X_i + C)(\mathrm{mod}\, 2^B)$$

where B is the machine word size and C is odd.

- Consider a sequence generated by equation

$$X_{i+1} = (A X_i + C)(\mathrm{mod}\, M)$$

  - if $C=0$, then the method is called a ***multiplicative congruential technique***
  - if $C \neq 0$, then the method is called a ***mixed congruential technique***

- The selection of values for A, C, M, X_0 has a dramatic effect on the length of the period of the generated sequence

- Example: Let $A=2$, $C=3$, $M=10$, $X_0=0$, then

$$\begin{array}{ll}
x_0 = 0 & \\
x_1 = (2\times 0+3) \bmod 10 = 3 & x_5 = (2\times 5+3) \bmod 10 = 3 \\
x_2 = (2\times 3+3) \bmod 10 = 9 & x_6 = (2\times 3+3) \bmod 10 = 9 \\
x_3 = (2\times 9+3) \bmod 10 = 1 & x_7 = (2\times 9+3) \bmod 10 = 1 \\
x_4 = (2\times 1+3) \bmod 10 = 5 & x_8 = (2\times 1+3) \bmod 10 = 5
\end{array}$$

Additive Congruential Generator

- Its seed is a sequence of n numbers $X_1, X_2, \ldots, X_n$
  - This sequence can be generated using some other technique

- Then, $X_j = (X_{j-1} + X_{j-n}) \bmod M$

- The main advantage is its speed
- It can yield periods greater than M
- The theoretical behavior of this technique is not well understood ➔ careful validation is necessary
- Example: Let M=10, and extend the sequence 1, 2, 4, 8, 6

$$
\begin{array}{ll}
x_1 = 1 & x_6 = (x_5 + x_1) \bmod 10 = (6+1) \bmod 10 = 7 \\
x_2 = 2 & x_7 = (x_6 + x_2) \bmod 10 = (7+2) \bmod 10 = 9 \\
x_3 = 4 & x_8 = (x_7 + x_3) \bmod 10 = (9+4) \bmod 10 = 3 \\
x_4 = 8 & x_9 = (x_8 + x_4) \bmod 10 = (3+8) \bmod 10 = 1 \\
x_5 = 6 & x_{10} = (x_9 + x_5) \bmod 10 = (1+6) \bmod 10 = 7 \\
 & x_{11} = (x_{10} + x_6) \bmod 10 = (7+7) \bmod 10 = 4 \\
 & x_{12} = (x_{11} + x_7) \bmod 10 = (4+9) \bmod 10 = 3 \\
 & x_{13} = (x_{12} + x_8) \bmod 10 = (3+3) \bmod 10 = 6 \\
 & x_{14} = (x_{13} + x_9) \bmod 10 = (6+1) \bmod 10 = 7 \\
 & x_{15} = (x_{14} + x_{10}) \bmod 10 = (6+7) \bmod 10 = 4
\end{array}
$$

Quadratic Congruential Generator

- Is used when M is a power of 2
- It has a longer period than Mid-Square
- The equation is

$$X_{i+1} = (X_i(X_i + 1)) \bmod M$$

- The seed X_0 must satisfy the relation X_0 *mod 4 = 2*
- Example: Let *M=16* and X_0=2,

$$\begin{aligned}
x_0 & & = 2 \\
x_1 &= (2(3)) \bmod 16 &= 6 \\
x_2 &= (6(7)) \bmod 16 &= 10 \\
x_3 &= (10(11)) \bmod 16 &= 14 \\
x_4 &= (14(15)) \bmod 16 &= 2 \\
x_5 &= (2(3)) \bmod 16 &= 6 \\
x_6 &= (6(7)) \bmod 16 &= 10 \\
x_7 & & = 14 \\
x_8 & & = 2
\end{aligned}$$

Nonuniform Random Numbers

- Many events do not happen in uniform fashion
  - recall, in *uniform distribution* each number occurs with the same probability
- A uniform random number generator can be used to generate random numbers that satisfy other distributions.

Poisson Distribution

- Occurs in simulations: used as the arrival process in most models
- k= # of packets (customers) arrived, λ = average arrival rate

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 0,1,\ldots$$

- The Poisson random number can be generated by
  - Starting with *n=0*, generate random number $u_n,\ 0 \le u_n \le 1,$ and compute $\prod_{i=0}^{n} u_i$,

    such that $\prod_{i=0}^{n} u_i \le e^{-\lambda} < \prod_{i=0}^{n-1} u_i$
  - Return *n* as the *Poisson(λ).*

Negative Exponential Distribution

- Interarrival times

$$P[t = x] = \lambda e^{-\lambda x},$$

- For Poisson process with mean $1/\lambda$, the interarrival time can be computed as

$$\delta t = -\frac{\ln(u)}{\lambda}.$$

```
// Return random number according to Poisson distribution.
int Random::poisson( double expectedValue )
{  double limit = -expectedValue;
   double sum = log( randomReal( ) );
   int count;
   for( count = 0; sum > limit; count++ )
      sum += log( randomReal( ) );
   return count;
}
```

Poisson distribution

```
// Return random number according to negative exponential distribution.
double Random::negExp( double expectedValue )
{       return - expectedValue * log( randomReal( ) ); }
```

Negative exponential distribution

using the ***Random*** class

```
int main( )
{
   Random r;
   const int SIZE = 15;
   vector<int> a( SIZE );
   int i;

   cout << "Random ints between " << SIZE << " and " << (SIZE + 5) << endl;
   for( i = 0; i < SIZE; i++ )
      cout << r.randomInt( SIZE, SIZE + 5 ) << endl;

   for( i = 0; i < SIZE; i++ )
      a[ i ] = 0;

   for( i = 0; i < 100000; i++ )
      a[ r.poisson( 1.6 ) ]++;

   cout << endl << "Poisson distribution with expected balue of 1.6." << endl;
   for( i = 0; i < a.size( ); i++ )
      cout << a[ i ] << endl;

   return 0;
}
```

Generating a Random Permutation

- How to generate a random permutation?

  The problem is to generate a random permutation of 1, 2, ..., N with all permutations being equally likely.

- Example: (1,8,4,6,2,3,7,5) is a random permutation of (1,2,3,4,5,6,7,8)

- One way: for *a[i]* take randomly a number from 1, 2, ..., N which is not equal to *a[j]* for all $j<i$. ➔ is not a good idea (?) (what is the time bound ?, improvements ?)

- A better way: Randomly swap elements in (1,2,...,N-1,N)

  - swap *a[i]* with *a[j]* where *j* is a random integer from *(0,i)*
  - A random permutation can be generated in linear time, using only one random number per item
  - Are all permutations equally likely? (Yes and No)

Random Permutation

```
// Generate a random permutation of 1..n.
void permute( vector<int> & a )
{
    Random r( (int) time( 0 ) ); // Random object; seed set by time
    const int n = a.size( );

    for( int i = 0; i < n; i++ )
        a[ i ] = i + 1;

    for( int j = 1; j < n; j++ )
    {
        // Swap with random object
        int pos = r.randomInt( 0, j );

        int tmp = a[ j ];
        a[ j ] = a[ pos ];
        a[ pos ] = tmp;
    }
}
```

Randomized Algorithms

- A randomized algorithm uses random numbers rather than deterministic decisions to control branching.

- The running time depends on the random numbers that occur, as well as the particular input.

- Randomized Quick_Select (Median) is statistically guaranteed to work in linear time.

  - Quick_Select we considered before had time bounds quadratic in worst case and linear in average case.
  - Try to make expected running time as an average case running time:
    1. make from your input a random input ➔ permute
    2. choose the pivot randomly instead of deterministically
    3. take a random item in the array and swap with the item in position low; take a random item in the array and swap with the item in position high; take a random item in the array and swap with the item in the middle position; then continue as usual.