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Homogeneously orderable graphs

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Abstract

In this paper we introduce homogeneously orderable graphs which are a common generalization of distance-hereditary graphs, dually chordal graphs and homogeneous graphs. We present a characterization of the new class in terms of a tree structure of the closed neighborhoods of homogeneous sets in 2-graphs which is closely related to the defining elimination ordering.

Moreover, we characterize the hereditary homogeneously orderable graphs by forbidden induced subgraphs as the house-hole-domino-sun-free graphs.

The local structure of homogeneously orderable graphs implies a simple polynomial-time recognition algorithm for these graphs.

Finally, we give a polynomial-time solution for the Steiner tree problem on homogeneously orderable graphs which extends the efficient solutions of that problem on distance-hereditary graphs, dually chordal graphs and homogeneous graphs.

1. Introduction

Several important graph classes have a certain kind of tree structure which can be formulated in terms of hypergraph (namely hypertree) properties. Among them are the well-known chordal graphs (dual hypertrees of maximal cliques), the dually chordal graphs (hypertrees of maximal cliques, dual hypertrees of closed neighbourhoods [6, 14]) and the distance-hereditary graphs (dual hypertrees of maximal cographs [23, 24]).

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The tree structure of the last two classes turned out to be useful especially for some distance and domination-like problems (cf. e.g. [11, 7, 8, 5, 12, 13]). The characterization of distance-hereditary graphs as dual hypertrees of cographs in [24] is used in [25] for designing efficient algorithms solving various Hamiltonian problems.

In [10] homogeneous graphs as a generalization of distance-hereditary graphs are introduced which lead to a polynomial-time algorithm for the Steiner tree problem on these graphs.

In this paper we define a new class of graphs which is a common generalization of distance-hereditary graphs, dually chordal graphs and homogeneous graphs. (For a recent survey on special graph classes cf. [4]). We present a characterization of the new class in terms of a tree structure of the closed neighbourhoods of homogeneous sets in 2-graphs which is closely related to the defining elimination ordering.

Moreover, we characterize the hereditary homogeneously orderable graphs by forbidden induced subgraphs as the house-hole-domino-sun-free graphs.

Finally, we give a polynomial-time solution for the Steiner tree problem on homogeneously orderable graphs which extends the efficient solutions of that problem on distance-hereditary graphs, dually chordal graphs and homogeneous graphs.

2. Preliminaries

Throughout this paper all graphs $G = (V, E)$ are finite, undirected, simple (i.e. loop-free and without multiple edges) and connected.

The *(open) neighbourhood* of a vertex v of G is $N(v) := \{u \in V\colon uv \in E\}$. The *closed neighbourhood* of v is $N[v] := N(v) \cup \{v\}$. For a vertex set $U \subseteq V$ let

$$N(U) := \bigcup_{u \in U} N(u) \setminus U \quad \text{and} \quad N[U] := \bigcup_{u \in U} N[u].$$

A nonempty set $U \subseteq V$ is *homogeneous* in $G = (V, E)$ iff all vertices of U have the same neighbourhood in $V \setminus U$:

$$N(u) \cap (V \setminus U) = N(v) \cap (V \setminus U) \text{ for all } u, v \in U,$$

i.e. any vertex $w \in V \setminus U$ is adjacent to either all or none of the vertices from U.

A homogeneous set H is *proper* iff $|H| < |V|$. Trivially, for each $v \in V$ the singleton $\{v\}$ is a proper homogeneous set. Note also that for a subset $V' \subset V$ if a set $H \subseteq V'$ is homogeneous in G then it is homogeneous also in the induced subgraph $G(V')$ but not vice versa.

A *path* is a sequence of vertices $v_1, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $i = 1, \ldots, k-1$; its *length* is k. A graph G is *connected* iff for any pair of vertices of G there is a path in G joining both vertices. The maximal-induced connected subgraphs of G are called *connected components*.

The *distance* $d_G(u, v)$ of vertices u, v is the minimal length of any path connecting these vertices. Obviously, d_G is a metric on G. If no confusion can arise we will omit the index G.

The *kth neighbourhood* $N^k(v)$ of a vertex v of G is the set of all vertices of distance k to v:

$$N^k(v) := \{u \in V\colon d_G(u, v) = k\}.$$

For convenience we denote by $N_F^k(v)$ the intersection $N^k(v) \cap F$, where $F \subseteq V$.

The *disk* of radius k centred at v is the set of all vertices of distance at most k to v:

$$D(v, k) := \{u \in V\colon d_G(u, v) \leqslant k\} = \bigcup_{i=0}^{k} N^i(v).$$

Note that $N[v] = D(v, 1)$ – a simple identity frequently used in this paper. Analogously to neighbourhoods of sets we define for $U \subseteq V$

$$D(U, k) := \bigcup_{u \in U} D(u, k).$$

For convenience we denote by $D_F(U, k)$ the intersection $D(U, k) \cap F$, where $F \subseteq V$.

The *kth power* G^k of a graph $G = (V, E)$ is the graph with vertex set V and edges between vertices u, v with distance $d_G(u, v) \leqslant k$.

Let $e(v)$ denote the *eccentricity* of vertex $v \in V$:

$$e(v) := \max \{d(v, u)\colon u \in V\}.$$

Then, the *radius* $rad(G)$ of G is the minimum over all eccentricities $e(v)$, $v \in V$, whereas the *diameter* $diam(G)$ of G is the maximum over all eccentricities $e(v)$ for v in V.

In the sequel a subset U of V is a *k-set* iff U induces a clique in the power G^k, i.e. for any pair x, y of vertices of U we have $d_G(x, y) \leqslant k$. A graph G is a *k-graph* iff $diam(G) \leqslant k$. If U is a subset of $V(G)$ then U is called k-graph of G iff the induced subgraph G_U is a k-graph (i.e. $diam(G_U) \leqslant k$) and for any set $U' \supset U$ holds $diam(G_{U'}) \geqslant k + 1$, i.e. G_U is a maximal induced subgraph of diameter $\leqslant k$ in G. Thus each k-graph is a k-set but the converse is in general not true as pointed out in Fig. 1 for $k = 2$.

Let U_1, U_2 be disjoint subsets of V. If every vertex of U_1 is adjacent to every vertex of U_2 then U_1 and U_2 form a *join*, denoted by $U_1 \bowtie U_2$.

Let $H = (V, \mathscr{E})$ be a hypergraph, i.e. $\mathscr{E}$ is a set of subsets of V. Throughout this paper all hypergraphs are assumed to be reduced, i.e. no hyperedge is properly contained in another one.

For every vertex $v \in V$ let $\mathscr{E}(v) := \{e \in \mathscr{E}\colon v \in e\}$ be the set of hyperedges incident to vertex v. Then the *dual hypergraph* H^* of H is the hypergraph with vertex set $\mathscr{E}$ and hyperedges $\mathscr{E}(v)$, $v \in V$.

The *line graph* $L(H)$ is the intersection graph of the hyperedges, the *2-section graph* $2SEC(H)$ is the graph with vertex set V, where two vertices are adjacent iff there is a hyperedge in H containing both.

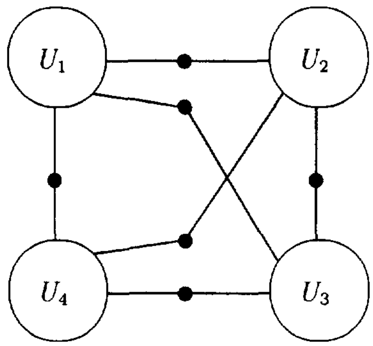

Fig. 1. $U = U_1 \cup U_2 \cup U_3 \cup U_4$ is a 2-set but not a 2-graph.

The following properties are well-known.

Proposition 2.1. *Let $H = (V, \mathscr{E})$ be a hypergraph. Then*
(i) *$(H^*)^*$ is isomorphic to H and*
(ii) *$L(H)$ is isomorphic to $2SEC(H^*)$.*

Let $\mathscr{N}(G) = \{N[v]: v \in V\}$ be the *neighbourhood hypergraph* of G and let $\mathscr{C}(G) = \{C$: C is a maximal clique in $G\}$ be the *clique hypergraph* of G.

Tree structure of a hypergraph can be defined as follows: A hypergraph $H = (V, \mathscr{E})$ is a *hypertree* iff there is a tree T with vertex set V such that any hyperedge e of H induces a subtree in T. A hypergraph $H = (V, \mathscr{E})$ is a *dual hypertree* iff H^* is a hypertree.

Hypertrees and dual hypertrees are closely related to chordal graphs – the graphs which do not contain any chordless cycle of length $\geqslant 4$. Walter et al. (cf. [17]) have shown that a graph is chordal iff it is the intersection graph of subtrees of a tree. The constructive proof shows that we can use the maximal cliques as the vertex set of a representing tree model. Hence we can conclude

Theorem 2.2 *A graph G is chordal iff its clique hypergraph is a dual hypertree.*

Let $\mathscr{M}$ be a set system over a set E. The system $\mathscr{M}$ has the *Helly property* (or for short: $\mathscr{M}$ is Helly) iff each subsystem of pairwise intersecting sets of $\mathscr{M}$ has a nonempty common intersection. A hypergraph $H = (V, \mathscr{E})$ is *Helly* iff $\mathscr{E}$ has the Helly property.

A hypergraph H is *conformal* iff any clique of $2SEC(H)$ is contained in some hyperedge. It is well-known that a hypergraph H is conformal iff its dual H^* has the Helly property.

Theorem 2.3 (Duchet [15]; Flament [16]). (i) *A hypergraph H is a hypertree iff H is Helly and L(H) is chordal.*
(ii) *A hypergraph H is a dual hypertree iff H is conformal and 2SEC(H) is chordal.*

Next we recall the definition and some characterizations of dually chordal graphs. A vertex u is a *maximum neighbour* of a vertex v iff $D(u,1) = D(v,2)$. A *maximum neighbourhood ordering* of a graph G is a sequence $(v_1, \dots, v_n)$ such that for all $i = 1, \dots, n$ the vertex v_i has a maximum neighbour in $G_i := G_{\{v_i, \dots, v_n\}}$.

Theorem 2.4 (Brandstädt et al. [6]). *The following conditions are equivalent:*
(i) *G has maximum neighbourhood ordering.*
(ii) *The clique hypergraph $\mathscr{C}(G)$ is a hypertree.*
(iii) *The neighbourhood hypergraph $\mathscr{N}(G)$ is a hypertree.*

Due to condition (ii) of Theorem 2.4 graphs with maximum neighbourhood ordering are called *dually chordal*. In [11, 7, 8] efficient algorithms for various distance and domination-like problems are given using this hypertree structure.

Finally, we recall the definition and some characterizations of distance-hereditary graphs. An induced subgraph H of G is *isometric* iff the distances $d_H(u,v)$ of any vertices u, v in H are the same as in G. A graph G is *distance-hereditary* iff each connected induced subgraph H is isometric. This graph class was introduced in [21]. Some characterizations and a linear-time recognition algorithm are given in [1, 9, 18].

The following characterizations are due to [24]: A vertex v is called *2-simplicial* iff the disk $D(v,2)$ induces a cograph in G. Hereby a *cograph* is a P_4-free graph, i.e. a connected cograph is a hereditary 2-graph. An ordering $\tau = (v_1, \dots, v_n)$ of the vertices of G is a *2-simplicial ordering* iff for every index $i = 1, \dots, n$ the vertex v_i is 2-simplicial in $G_i := G_{\{v_i, \dots, v_n\}}$. A 2-simplicial vertex v is *d-extremal* iff $e(v) = diam(G)$. Analogously, we can define a *d-extremal ordering*. Let $\mathscr{C}\mathscr{C}(G)$ denote the set of maximal connected cographs of G.

Theorem 2.5 (Nicolai [24]). *Let $G = (V, E)$ be a graph. Then the following conditions are equivalent:*
(i) *G is distance-hereditary.*
(ii) *The cograph-hypergraph $\mathscr{C}\mathscr{C}(G)$ of G is a dual hypertree.*
(iii) *G has a 2-simplicial ordering.*
(iv) *G has a d-extremal ordering.*

Moreover, d-extremal vertices have nice local properties:

Proposition 2.6 (Nicolai [24]). *Let G be a distance-hereditary graph and v be a d-extremal vertex. Then there is a set $S \subseteq N(v)$ which is homogeneous (in G) and dominates $D(v,2)$.*

Note, that in dually chordal graphs a maximum neighbour v of a vertex u dominates $D(u,2)$ too. Thus, we can generalize these properties in the following way:

A vertex v of $G = (V, E)$ with $|V| > 1$ is *h-extremal* iff (the subgraph induced by) $D(v,2)$ contains a proper homogeneous dominating set. More exactly: There is a proper subset $H \subset D(v, 2)$ which is homogeneous in G and for which $D(v, 2) \subseteq D(H, 1)$ holds. A sequence $\sigma = (v_1, \dots, v_n)$ is a *h-extremal ordering* iff for any $i = 1, \dots, n-1$ the vertex v_i is h-extremal in $G_i := G_{\{v_i, \dots, v_n\}}$. A graph G is *homogeneously orderable* iff G has a h-extremal ordering. Thus, we immediately obtain

Corollary 2.7. *Dually chordal and distance-hereditary graphs are homogeneously orderable graphs.*

Sometimes we will write $\sigma = ((v_1, H_1), \dots, (v_n, H_n))$ to emphasize the homogeneous dominating sets H_j for v_j in G_j.

Now we present two lemmata which will be used frequently in the sequel.

Lemma 2.8. *If H is a proper homogeneous set in G and $x, y \in D(H, 1)$ then $d(x, y) \leqslant 2$.*

Proof. Since H is proper and G is connected there must be a vertex v_H of $V \backslash H$ adjacent to any vertex of H.

If both x and y are in $N(H)$ then by the definition of a homogeneous set both vertices are adjacent to each vertex of H. If both vertices are within H they are adjacent to v_H. Finally, if one vertex is in H and the other one in $N(H)$ they are adjacent. □

We immediately conclude

Corollary 2.9. *If v is h-extremal in G then $D(v, 2)$ is a 2-graph.*

Lemma 2.10. *Let v be a vertex in a graph $G = (V, E)$.*

(i) *If $e(v) \geqslant 2$ and v is h-extremal then there is a proper homogeneous set $H \subseteq N(v)$ which dominates $D(v, 2)$.*

(ii) *If $e(v) = 1$ then G is homogeneously orderable with h-extremal ordering $\sigma = ((v_1, \{v\}), \dots, (v_{n-1}, \{v\}))$, where $V = \{v_1, \dots, v_{n-1}, v\}$.*

Proof. For point (ii) there is nothing to show. So let $e(v) \geqslant 2$ and let H be a proper homogeneous dominating set in $D(v, 2)$. If $v \notin H$ then v must be dominated by some $h \in H$. Since H is homogeneous we immediately conclude $H \subseteq N(v)$.

Now consider the case $v \in H$. Assume first that $N^3(v) \neq \emptyset$ and let $u \in N^3(v)$. Since H is homogeneous, $v \in H$ and $vx \notin E$ for any neighbour x of u in $N^2(v)$ no one of these neighbours x is in $N(H)$. But H dominates $D(v, 2)$, hence $N(u) \cap N^2(v) \subset H$. But now H is not homogeneous.

Finally, assume that $N^3(v) = \emptyset$, i.e. G is a 2-graph. Since v is in the homogeneous set H and v is not adjacent to any vertex of $N^2(v)$, but H dominates $D(v, 2)$, the second neighbourhood $N^2(v)$ must be completely contained in H. But now, $N(v) \backslash H$ is homogeneous in G and dominates $D(v, 2)$, so we have the desired set. □

Therefore, for a homogeneously orderable graph G with h-extremal ordering $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$ we will assume $H_j \subseteq N_{G_j}(v_j)$ for $j = 1, \ldots, n-1$ in the sequel.

3. Homogeneously orderable graphs and corresponding hypergraphs

Recall that $F \subseteq V$ is a k-graph iff $diam(G_F) \leqslant k$ and for any set $U \supset F$ $diam(G_U) > k$ holds. Denote by $2\mathcal{G}(G)$ the set of all 2-graphs of G and by $\mathcal{H}(G)$ the set of all maximal proper homogeneous sets of G.

Let $\mathcal{DH}(G) := \{D_F(H, 1)\colon F \in 2\mathcal{G}(G) \text{ and } H \in \mathcal{H}(F)\}$. We will show that a graph G is homogeneously orderable iff the hypergraph $\mathcal{DH}(G)$ is a dual hypertree. Note that this equivalence does not hold for the 2-graph hypergraph $2\mathcal{G}(G)$ instead of $\mathcal{DH}(G)$. Indeed, consider the chordless 5-cycle C_5 which is a 2-graph. Thus, the (reduced) 2-graph hypergraph contains only one hyperedge and hence is a dual hypertree. On the other hand, any proper homogeneous set of a C_5 is a singleton, and any pair of vertices is contained in the neighbourhood of a homogeneous set. Thus, the 2-section graph $2SEC$ $(\mathcal{DH}(C_5))$ is complete, but there is no proper homogeneous set dominating the whole cycle. Consequently, $\mathcal{DH}(C_5)$ is not conformal and thus it is not a dual hypertree. Moreover, no vertex of a C_5 is h-extremal.

To prove that $\mathcal{DH}(G)$ of a homogeneously orderable graph G is a dual hypertree we use Theorem 2.3 (ii), i.e. we show that $2SEC(\mathcal{DH}(G))$ is chordal and $\mathcal{DH}(G)$ is conformal.

First we prove the chordality of the 2-section graph.

Lemma 3.1. *For any graph G we have $2SEC(\mathcal{DH}(G)) = G^2$.*

Proof. (1) Let xy be an edge in $2SEC(\mathcal{DH}(G))$. Then by definition there must be a hyperedge $D_F(H, 1)$ containing both vertices. From Lemma 2.8 we obtain $d(x, y) \leqslant 2$, hence these vertices are adjacent in G^2.

(2) Let xy be an edge in G^2, that is $d_G(x, y) \leqslant 2$. Consider a 2-graph F of G containing both vertices, and if $d_G(x, y) = 2$ a vertex w which is adjacent to both. Obviously, there is a proper homogeneous set H containing x for $d_G(x, y) = 1$ and containing w for $d_G(x, y) = 2$, respectively. In both cases $\{x, y\}$ is a subset of $D_F(H, 1)$. □

The following straightforward lemma will be used frequently in the sequel.

Lemma 3.2. *Let G be graph and let v be a h-extremal vertex of G with $e(v) \geqslant 2$. Then $G \backslash \{v\}$ is an isometric subgraph of G. In particular, we have $G^2 \backslash \{v\} = (G \backslash \{v\})^2$.*

Proof. Since $e(v) \geqslant 2$ we can choose a homogeneous set $H \subseteq N(v)$ dominating $D(v, 2)$ due to Lemma 2.10. Thus, the distances in $G \backslash \{v\}$ are the same as in G. □

Lemma 3.3. *For any homogeneously orderable graph G with h-extremal ordering σ the square G^2 is chordal and σ is a perfect elimination ordering of G^2.*

Proof. Let G be a homogeneously orderable graph with h-extremal ordering $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$. We prove that v_1 is simplicial in G^2. Let x, y be neighbours of v_1 in G^2. Hence, $d_G(x, v_1) \leqslant 2$ and $d_G(y, v_1) \leqslant 2$, i.e. both x and y are contained in $D_G(v_1, 2)$ which is dominated by H_1. Thus, Lemma 2.8 implies $d_G(x, y) \leqslant 2$. Therefore, x and y are adjacent in G^2 and $D_{G^2}(v_1, 1)$ is complete, that is v_1 is simplicial in G^2.

If $e(v_1) \geqslant 2$ we can proceed by induction on the position in σ due to the preceding lemma. Otherwise, $G = D(v_1, 1)$ and G^2 is complete. □

The following immediate consequence of Lemma 3.3 and Corollary 2.7 was known already from papers about distance-hereditary and dually chordal graphs.

Corollary 3.4. *If G is a distance-hereditary or dually chordal graph then G^2 is chordal.*

Now we prove the conformality of $\mathscr{DH}(G)$.

Lemma 3.5. *If G is homogeneously orderable then $\mathscr{DH}(G)$ is conformal.*

Proof. Let $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$ be a h-extremal ordering of G. Furthermore, let $C = \{c_1, \ldots, c_k\}$ be a maximal clique in $2SEC(\mathscr{DH}(G)) = G^2$ such that $c_1 = v_l$ is the leftmost vertex of C with respect to σ. Since C is maximal in G^2 it cannot be completely contained in $D_{G_i}(v_i, 2)$ for $i = 1, \ldots, l-1$ implying $e_{G_i}(v_i) \geqslant 2$ for $i = 1, \ldots, l-1$. Thus, by Lemma 3.2 C is a clique in $(G_l)^2$, i.e. for all $i, j = 1, \ldots, k$ we have $d_{G_l}(c_i, c_j) \leqslant 2$. Since $v_l = c_1$ is h-extremal in G_l we immediately conclude $C \subseteq D(H_l, 1) = D_{G_l}(v_l, 2)$. Suppose $D_{G_l}(v_l, 2)$ is not a 2-graph of G. Then it must be properly contained in a 2-graph F of G. But F induces a clique in G^2 contradicting the maximality of C. Thus, $D_{G_l}(v_l, 2)$ is a 2-graph of G and C is contained in a hyperedge. □

Summarizing the above results we obtain

Corollary 3.6. *If G is homogeneously orderable then $\mathscr{DH}(G)$ is a dual hypertree.*

Lemma 3.7. *Let $\mathscr{DH}(G)$ be a dual hypertree with a hyperedge $D(H, 1) = G$. Then G is homogeneously orderable and, with $H := \{u_1, \ldots, u_k\}$, $V \backslash H := \{v_1, \ldots, v_l\}$, both*

$$\sigma = ((u_1, V\backslash H), \ldots, (u_{k-1}, V\backslash H), (v_1, \{u_k\}), \ldots, (v_l, \{u_k\}))$$

and

$$\sigma' = ((v_1, H), \ldots, (u_{l-1}, H), (u_1, \{v_l\}), \ldots, (u_k, \{v_l\}))$$

are h-extremal orderings of G.

Proof. Follows immediately from $D(H, 1) = G$. □

In order to prove the converse, i.e. if $\mathscr{DH}(G)$ is a dual hypertree then G is homogeneously orderable, we introduce another hypergraph. Consider a 2-graph F which is dominated by some homogeneous (in F) set H, i.e. $F = D_F(H, 1)$. Then F is splitted into two joined sets, namely H and $N_F(H)$: $F = H \bowtie N_F(H)$. In general, a set $U \subseteq V$ is *join-splitted* iff U is the join of two nonempty sets, i.e. $U = U_1 \bowtie U_2$. Since any edge of a graph is a join-splitted set each connected graph can be covered by join-splitted sets. Thus, we can define the hypergraph $\mathscr{SH}(G)$ of the maximal join-splitted sets of G, and immediately obtain the following:

Lemma 3.8. *For any graph G we have $2SEC(\mathscr{SH}(G)) = G^2$.*

So we get

Theorem 3.9. *The following conditions are equivalent for a graph G:*
(i) *G is homogeneously orderable.*
(ii) *$\mathscr{DH}(G)$ is a dual hypertree.*
(iii) *G^2 is chordal and every maximal 2-set of G is join-splitted (and hence a 2-graph).*
(iv) *$\mathscr{SH}(G)$ is a dual hypertree.*

Proof. (i) ⇒ (ii) Follows from the preceding results.

(ii) ⇒ (iii) By Lemma 3.1 the square G^2 is chordal. Let S be a maximal 2-set in G. Thus, S is a maximal clique in G^2, and the conformality of $\mathscr{DH}(G)$ implies that $S \subseteq D_F(H, 1)$ for some 2-graph F of G and some homogeneous set H of F. From the maximality of S we conclude that $S = D_F(H, 1)$ and hence $S = F$. But now $S = H \bowtie N_S(H)$, so we are done.

(iii) ⇒ (i) Let v be a simplicial vertex of the chordal graph G^2. If $e(v) = 1$ we are done by Lemma 2.10. So let $e(v) \geqslant 2$. We show that v is h-extremal in G. Since v is simplicial in G^2 the disk $D(v, 2)$ is complete in G^2. Thus that disk is a maximal 2-set in G and hence join-splitted, say $D(v, 2) = X \bowtie Y$. W.l.o.g. assume $v \in X$ implying $Y \subseteq N(v)$. Therefore, Y is the desired homogeneous set dominating $D(v, 2)$, and v is h-extremal. By Lemma 3.2 $(G\backslash\{v\})^2$ is chordal, and obviously each maximal 2-set of $G\backslash\{v\}$ is join-splitted.

(iv) ⇔ (iii) By Lemma 3.8 and Theorem 2.3 statement (iv) is a reformulation of (iii). □

Corollary 3.10. *If G is homogeneously orderable then for each perfect elimination ordering $\sigma = (v_1, \ldots, v_n)$ of G^2 and $k(\sigma) := \min\{i\colon G^2_{\{v_i, \ldots, v_n\}}$ is complete$\}$ there exists a h-extremal ordering τ of G such that $\tau(i) = \sigma(i)$ for $i = 1, \ldots, k(\sigma) - 1$.*

Corollary 3.11. *If G is a homogeneously orderable graph and v is an arbitrary vertex of G then there is a h-extremal ordering σ of G with v at the end.*

Proof. Recall that for chordal graphs each vertex can be placed at the end of a perfect elimination ordering. Let τ be such a perfect elimination ordering of G^2. Thus, the index of v in τ is at least $k(\tau)$. By the above corollary and by Lemma 3.7 we are done. □

Recall that in distance-hereditary graphs each maximal 2-set is a 2-graph and each 2-graph is a cograph, i.e. a hereditary 2-graph. Here, in homogeneously orderable graphs each maximal 2-set is a join-splitted 2-graph.

4. Homogeneous reductions and extensions

In [10] the authors generalize distance-hereditary graphs. Recall that any distance-hereditary graph can be generated from a single vertex by a sequence of the following three one-vertex extensions. Let $G' = (V', E')$ be a graph, $x' \in V'$ and $x \notin V'$. Define $G := (V' \cup \{x\}, E' \cup E_x)$ where E_x is defined as follows:
PV $E_x := \{xx'\}$ – x is a pendant vertex (leaf) to x,
FT $E_x := \{xy\colon y \in N_{G'}(x')\}$ – x and x' are false twins,
TT $E_x := \{xy\colon y \in D_{G'}(x', 1)\}$ – x and x' are true twins.
It is obvious that in the case of twin operations $\{x, x'\}$ forms a homogeneous set in G. Now, in [10] instead of twins (as a special kind of homogeneous sets) arbitrary homogeneous sets are used.

Let H be a proper homogeneous set of G containing at least two vertices and let $v_H \in H$. Then the graph $H\ Red(G, H, v_H)$ obtained from G by deleting $H \backslash \{v_H\}$, i.e. contracting H to a representing vertex v_H, will be called the *homogeneous reduction* of G (via H).

Conversely, the *homogeneous extension* $H\ Ext(G, v, H)$ of G via a graph H in v with $V(H) \cap V(G) = \emptyset$ is the graph obtained by substituting v by H such that the vertices of H have the same neighbours outside of H as v had in G.

Thus, in distance-hereditary graphs the FT operation is the homogeneous extension of G' in x' via the non-edge $\{x, x'\}$. For a TT operation G' is homogeneously extended in x' via the edge $\{xx'\}$.

In what follows, we want to clarify the relations between some graph classes.

In the sense of [10] a graph G is a *homogeneous graph* iff the iterated reduction via proper homogeneous sets of 2-connected components leads to a tree.

A more natural generalization of distance-hereditary graphs is the following: G is in $\Gamma_{\{PV, HExt\}}(K_1)$ iff G can be generated from a single vertex by a sequence of PV operations and homogeneous extensions. Obviously, these graphs are homogeneous. Note that this inclusion is proper: Consider the graph in Fig. 2 which does neither contain a pendant vertex nor a nontrivial proper homogeneous set. Thus, that graph is not in $\Gamma_{\{PV, HExt\}}(K_1)$. On the other hand, there are two 2-connected components with cutvertex x. The vertex sets of the components minus $\{y_i\}$ are homogeneous and hence this graph is reducible to a P_3.

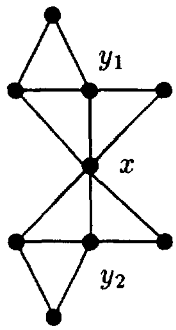

Fig. 2. A homogeneous graph which is not in $\Gamma_{\{PV,HExt\}}(K_1)$.

Lemma 4.1. *If $H \in \mathscr{H}(G)$ and S is a maximal 2-set of G such that $H \cap S \neq \emptyset$ then $H \subset S$.*

Proof. First suppose $S \subseteq H$. Since G is connected and H is proper there must be a vertex w in $V \backslash H$ such that $H \subseteq N(w)$ implying $S \subseteq N(w)$, a contradiction. Thus, $S \backslash H \neq \emptyset$. Next suppose $H \backslash S \neq \emptyset$. Define $S' := H \cup S$. We prove that S' is a 2-set which contradicts the maximality of S. Let $w \in H \cap S$ and consider two vertices x, y of S'. If both vertices are in S then $d(x, y) \leqslant 2$. Now let $x \in S \backslash H$ and $y \in H \backslash S$. If x is adjacent to w then x must be adjacent to y, since H is homogeneous in G. Otherwise there must be a vertex v adjacent to both x and w. Note that v cannot be in H. Thus, $yv \in E$ and $d(x, y) = 2$. Finally, let both x and y be in H and choose a vertex $v \in S \backslash H$. If $wv \in E$ then both x and y must be adjacent to v. Otherwise there is a vertex u in $V \backslash H$ adjacent to w and v. Hence, x and y are adjacent to u too. So we are done. □

Lemma 4.2. *Let G be a homogeneously orderable graph, H be a nontrivial proper homogeneous set of G and v any vertex of H. Then the graph $G \backslash \{v\}$ is homogeneously orderable too.*

Proof. It is easy to see that $G \backslash \{v\}$ is an isometric subgraph of G. Thus $(G \backslash \{v\})^2 = G^2 \backslash \{v\}$ which is chordal. Consider an arbitrary maximal 2-set S in $G \backslash \{v\}$. If S is not maximal in G then $S' := S \cup \{v\}$ is a maximal 2-set in G. Therefore, S' is join-splitted in G, say $S' = X \bowtie Y$ with $v \in Y$. If Y contains at least two vertices we are done. So let $Y = \{v\}$. By the maximality of S' the preceding lemma implies $H \subset S'$. Since H is homogeneous and $X \subseteq N(v)$ we can split S' by $X \backslash \{H\} \bowtie H$. Consequently, S is join-splitted in $G \backslash \{v\}$ because H is nontrivial. The assumption follows from Theorem 3.9. □

Lemma 4.3. *If G is chordal then so is the graph obtained from G by adding a true twin v to some vertex x of G.*

Lemma 4.4. *Let G be a homogeneously orderable graph and H be a proper homogeneous set of G. Then any graph* $G + v := (V \cup \{v\}, E \cup \{vx\colon x \in N_G(H)\} \cup E')$, *where* $E' \subseteq \{vx\colon x \in H\}$, *is homogeneously orderable too.*

Proof. Since $N_{V \setminus H}(v) = N(H)$ it is easy to see that G is an isometric subgraph of $G + v$, and $G^2 = (G + v)^2 \setminus \{v\}$. So $(G + v)^2$ can be obtained from the chordal graph G^2 by adding the true twin v to some vertex x of H since $D_G(v, 2) = D_G(x, 2)$. Hence $(G + v)^2$ is chordal by Lemma 4.3.

Consider an arbitrary maximal 2-set S in $G + v$. If S does not contain v then S is a maximal 2-set in G and hence is join-splitted. Otherwise $S' := S \setminus \{v\}$ is a maximal 2-set in G, and Lemma 4.1 implies $H \subset S$. Therefore, S' is join-splitted, i.e. $S' = X \bowtie Y$. If H is completely contained in one of the splitting sets we can add v to this one to obtain a splitting for S in $G + v$. So assume $H \cap X \neq \emptyset$ and $H \cap Y \neq \emptyset$. But in this case we can split S into $X \setminus H$ and $Y \cup H$, thus we are again in the preceding case. By Theorem 3.9 we are done. □

Corollary 4.5. (1) *If* $H \in \mathscr{H}(G)$ *and* $G' := H\ Red(G, H, v_H)$ *then* G' *is homogeneously orderable too.*

(2) *If* $v \in V(G)$, *H an arbitrary graph and* $G' := H\ Ext(G, v, H)$ *then* G' *is homogeneously orderable too.*

Proof. Follows immediately from the preceding two lemmata. □

Thus, we can summarize our results to

Corollary 4.6. *Homogeneously orderable graphs are closed under homogeneous extensions, homogeneous reductions and under deleting and adding of a vertex with maximum neighbour.*

Proof. The first two points follow directly from the above corollary. To show the third let G be a homogeneously orderable graph and $\sigma = (v_1, \ldots, v_n)$ be a h-extremal ordering of G. Furthermore, let $y \notin G$ be a vertex with maximum neighbour $x \in G$. Obviously, $\{x\}$ is a homogeneous set dominating $D(y, 2)$. Thus, $\tau = (y, v_1, \ldots, v_n)$ is a h-extremal ordering of the new graph. □

Theorem 4.7. *The homogeneously orderable graphs are exactly those graphs which can be generated from a single vertex by adding a vertex with maximum neighbour and by homogeneous extensions, i.e.* $\Gamma_{\{MN, HExt\}}(K_1)$ *is the class of homogeneously orderable graphs.*

Proof. By Corollary 4.6 any graph from $\Gamma_{\{MN, HExt\}}(K_1)$ is homogeneously orderable. To prove the converse note that every h-extremal vertex v either has a maximum neighbour or contains a proper nontrivial homogeneous set H in its neighbourhood. After homogeneously reducing H to v_H vertex v has a maximum neighbour v_H. □

Lemma 4.8. *A graph G is homogeneously orderable iff each 2-connected component of G is homogeneously orderable.*

Proof. Let G be a homogeneously orderable graph with h-extremal ordering $\sigma = (v_1, \dots, v_n)$. Denote by $\sigma|_K$ the ordering of vertices of a 2-connected component K of G induced by σ. We show that $\sigma|_K = (v_{i_1}, \dots, v_{i_l})$ is a h-extremal ordering for K. Suppose the contrary and let $i_j, j < l$, be the smallest index such that v_{i_j} is not h-extremal in $K_{i_j} := K_{\{v_{i_j}, \dots, v_{i_l}\}}$. By Lemma 3.2 the graph K_{i_j} is an isometric subgraph of K and hence connected. Since v_{i_j} is h-extremal in G_{i_j} there is a proper homogeneous set H_{i_j} dominating $D_{G_{i_j}}(v_{i_j}, 2)$. Obviously, if $H' := H_{i_j} \cap K_{i_j}$ is nonempty then H' dominates $D_{K_{i_j}}(v_{i_j}, 2)$. Thus, H' is empty, i.e. $H_{i_j} \cap K = \emptyset$. This implies, together with $H_{i_j} \subseteq N(v_{i_j})$ and $D_{G_{i_j}}(v_{i_j}, 2) = D_{G_{i_j}}(H_{i_j}, 1)$, that $G_{i_j} \cap K = \{v_{i_j}\}$ and hence $j = l$, a contradiction.

In order to prove the converse consider the tree $T(G)$ defined by the 2-connected components of G. Let K be a leaf of $T(G)$ and v be the only cutvertex of G in K. Then by Corollary 3.11 we have a h-extremal ordering $\sigma_K = (v_{i_1}, \dots, v_{i_r}, v)$ of K with vertex v at the end. By induction hypothesis $G \backslash (K \backslash \{v\})$ possesses a h-extremal ordering $\tau = (v_1, \dots, v_l)$. Obviously, $\sigma := (v_{i_1}, \dots, v_{i_r}, v_1, \dots, v_l)$ is a h-extremal ordering of G. □

Remark that a graph G is homogeneous iff each 2-connected component of G is a homogeneous graph (cf. [10]). Thus we can prove

Corollary 4.9. *Homogeneous graphs are homogeneously orderable.*

Proof. By Lemma 4.8 and the remark it is sufficient to show that every 2-connected homogeneous graph is homogeneously orderable. If there is no nontrivial homogeneous set then G is a tree and we are done. Otherwise we proceed by induction. Let H be a nontrivial proper homogeneous set of a 2-connected homogeneous graph G. By the definition of homogeneous graphs $G' := H\,Red(G, H, v_H)$ is homogeneous too. Thus, by induction hypothesis G' is homogeneously orderable. Since $G = H\,Ext(G', v_H, H)$ the assertion follows from Corollary 4.6. □

As usual we denote by $Ext^*(\mathbf{G})$ the transitive closure of the graph class $\mathbf{G}$ with respect to homogeneous extensions.

Theorem 4.10. (1) *Ext^*(tree) $\subset$ Ext^*(dually chordal gr.) $\subset$ homogeneously orderable gr.*

(2) *Ext^*(tree) $\subset$ homogeneous gr.*

(3) *distance-hereditary gr. $\subset$ homogeneous gr. $\subset$ homogeneously orderable gr* (*see Fig.* 3).

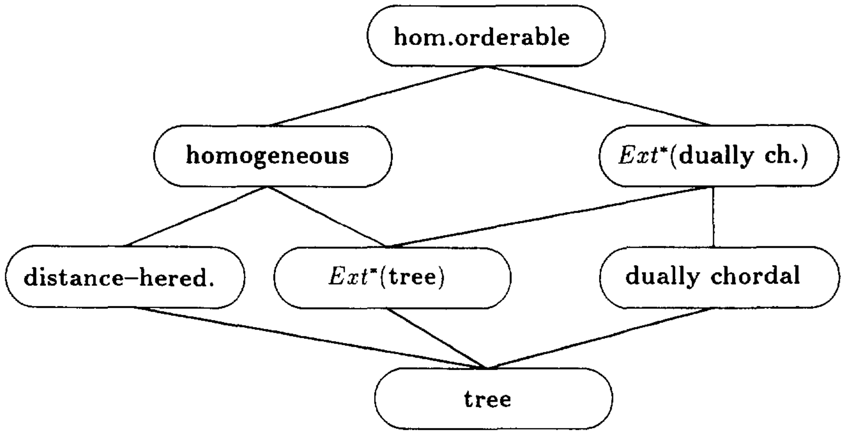

Fig. 3. Inclusion hierarchy of the considered graph classes.

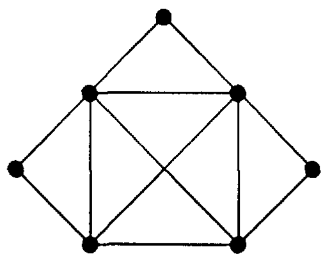

Fig. 4. A dually chordal graph which is not homogeneous.

Proof. The inclusions follow from above lemmata or are trivial. It remains to show that any of these inclusions is proper.

- A C_4 with a pendant vertex on each of its vertices is distance hereditary, hence homogeneous and homogeneously orderable but neither dually chordal nor in Ext^*(dually chordal graphs) not in Ext^*(tree).
- A C_k, $k \geqslant 5$, dominated by some vertex is in Ext^*(tree) and is dually chordal but not distance-hereditary.
- The C_4 is in Ext^*(tree) but not dually chordal.
- The graph shown in Fig. 4 is dually chordal but not homogeneous. □

5. Hereditary homogeneously orderable graphs

In this section we will characterize hereditary homogeneously orderable graphs (i.e. those graphs G for which each induced subgraph G' is also homogeneously orderable) in terms of forbidden subgraphs. Since distance-hereditary graphs are homogeneously

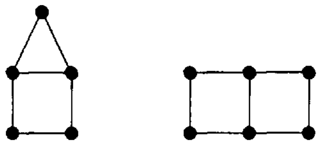

Fig. 5. The house and the domino.

orderable and have the property that their induced subgraphs are also distance-hereditary they are hereditary homogeneously orderable graphs. For our characterization house-hole-domino-free (HHD-free) graphs are important. A graph is *HHD-free* iff it does not contain an induced subgraph isomorphic to a k-cycle for $k \geqslant 5$ (the *holes*), the *house* and the *domino* (see Fig. 5).

HHD-free graphs are also characterized by an elimination ordering: A vertex v is called *semi-simplicial* iff v is not an inner point (midpoint) of any P_4 in G. Then, a *semi-simplicial ordering* is an ordering $\sigma = (v_1, \ldots, v_n)$ of the vertices of G such that for every index $i = 1, \ldots, n$ the vertex v_i is semi-simplicial in $G_i := G_{\{v_i, \ldots, v_n\}}$. In [22] the authors proved that a graph is HHD-free iff Lexicographic Breadth-First Search always generates a semi-simplicial ordering for every induced subgraph.

A class containing all HHD-free graphs is the class of pseudo-modular graphs (cf. [2]). A graph is *pseudo-modular* iff for any vertices v_1, v_2, v_3 there are vertices x_1, x_2, x_3 such that

$$d(v_i, v_j) = d(v_i, x_i) + d(x_i, x_j) + d(x_j, v_j) \quad \text{for all } i \neq j = 1, 2, 3,$$

and

$$d(x_1, x_2) = d(x_1, x_3) = d(x_2, x_3) \in \{0, 1\}.$$

We will prove that hereditary homogeneously orderable graphs are exactly the sun-free HHD-free graphs (in the sequel we call this call *HHDS-free*), where as usual a *k-sun* is a graph $S = (U \cup W, E)$ such that

1. $|U| = |W| = k$,
2. $U = \{u_1, \ldots, u_k\}$ is independent, $W = \{w_1, \ldots, w_k\}$ is a cycle (not necessary chordless),
3. $E = E(W) \cup \{u_i w_j : i = j \text{ or } i = j - 1 \bmod k, i, j \in \{1, \ldots, k\}\}$, where $E(W)$ is a set of edges only between W-vertices.

If W is complete then S is called *complete sun*, otherwise an *incomplete sun*.

Lemma 5.1. *Let G be a HHDS-free graph. Then every 2-graph D of G contains a proper homogeneous set dominating D.*

Proof. Let D be a 2-graph of G and v be semi-simplicial in D. In the following all neighbourhoods are restricted to D.

(1) For any two vertices u, w of $N^2(v)$ there is a common neighbour in $N(v)$: If not then let x, y be vertices of $N(v)$ such that $xu \in E$, $xw \notin E$, $yw \in E$ and $yu \notin E$. Since v is semi-simplicial $xy \in E$ holds. If u and w are adjacent we obtain a house, a contradiction. Hence, $uw \notin E$. But $u, w \in D$, i.e. $d_D(u, w) = 2$. Thus there is a vertex $z \in N^2(v)$ adjacent to both u and w. The following three cases can arise:

- $zx \notin E$ and $zy \notin E$ – we obtain a C_5,
- $zx \in E$ and $zy \notin E$ or vice versa – we obtain a house,
- $zx \in E$ and $zy \in E$ – we obtain a 3-sun.

In any of the above cases we get a forbidden induced subgraph, so u, w have a common neighbour in $N(v)$.

(2) Let H be the set of vertices of $N(v)$ dominating $N^2(v)$:

$$H = \{x \in N(v)\colon N^2(v) \subseteq N(x)\}.$$

We claim that H is nonempty.

The claim is shown by induction on the number k of vertices in $N^2(v)$. For $k = 1$ there is nothing to show and for $k = 2$ we are done by (1). So let $k \geqslant 3$, $N^2(v) = \{y_0, \dots, y_{k-1}\}$. By the induction hypothesis for any of the three sets $N^2(v) \backslash \{y_i\}$, $i = 0, 1, 2$, there is a vertex x_i dominating these sets. If for some $i \in \{0, 1, 2\}$ vertex x_i is adjacent to y_i we are done. So assume $x_i y_i \notin E$ for $i = 0, 1, 2$. Consider the path $x_i - v - x_{i+1} - y_i$ where addition is taken modulo 3. If $x_i x_{i+1} \notin E$ vertex v is not semi-simplicial, a contradiction. Thus $\{x_0, x_1, x_2\}$ is a clique. By considering the subgraph induced by the vertices $\{v, x_i, y_i, x_j, y_j\}$ for $i \neq j$ we conclude that $\{y_0, y_1, y_2\}$ must be independent, for otherwise we obtain a forbidden house. But now the vertices $\{x_i, y_i\colon i = 0, 1, 2\}$ form a 3-sun, a contradiction. Therefore, there is at least one vertex in $N(v)$ dominating $N^2(v)$.

(3) H is homogeneous and dominates D:
By the definition of H we have only to consider vertices of $N(v) \backslash H$. Suppose there are nonadjacent vertices $w \in N(v) \backslash H$ and $x \in H$. Since w is not in H there must be a vertex y in $N^2(v)$ which is not adjacent to w. But now, v is mid-point of the $P_4 y - x - v - w$, a contradiction. □

Lemma 5.2. *Let G be a HHDS-free graph. Then each clique in G^2 is contained in some 2-graph in G.*

Proof. Let C be a maximal clique in G^2. We will show that C is a 2-graph in G. Note that for any set $U \supset C$ we have $diam(G_U) \geqslant 3$ for otherwise U would be complete in G^2. Thus it suffices to show $diam_C(C) \leqslant 2$. Assume $diam_C(C) \geqslant 3$.

Note that for any pair c, c' of vertices of C we have $d_G(c, c') \leqslant 2$. Thus there must be a set $U \subseteq V \backslash C$ such that $diam_{G_{U \cup C}}(C) \leqslant 2$ and for each $U' \subset U$ $diam_{G_{U' \cup C}}(C) > 2$, i.e. U is minimal. Define $F := G_{U \cup C}$. Therefore for each $u \in U$ there are personal neighbours of u, i.e. nonadjacent vertices $u_1, u_2 \in F$ such that u is the only common neighbour of u_1, u_2 in F. Furthermore $diam_F(F) \geqslant 3$ since C is maximal in G^2.

Since F is an induced subgraph of G it must be HHDS-free. Hence there is a semi-simplicial vertex v in F.

Case 1: $v \in U$.

Let v_1, v_2 be personal neighbours of v. Note that $N_F^2(v) \neq \emptyset$ since otherwise $F = D_F(v, 1)$ is a clique in G^2. Let $x \in N_F^2(v)$ and $y \in N_F(v)$ be a neighbour of x. If y is one of v_1, v_2, say v_1, then $xv_2 \notin E$. Thus, v is midpoint of the P_4 $v_2 - v - v_1 - x$, a contradiction. So let y be distinct from v_1 and v_2. Note that neither y is adjacent to both v_1, v_2 nor x is. W.l.o.g. assume $v_1 y \notin E$. If $v_1 x \notin E$ v is midpoint of the P_4 $v_1 - v - y - x$. If $v_1 x \in E$ then $v_2 x \notin E$. But now, either $v_2 y \notin E$ implying the P_4 $v_2 - v - y - x$, or $v_2 y \in E$ yielding a house induced by $\{v_1, v_2, v, x, y\}$. In any case we obtain a contradiction.

Case 2: $v \in C$.

Then $C \subseteq D_F(v, 2)$. Since $diam_C(C) \geqslant 3$ there are vertices c_1, c_2 such that $d_C(c_1, c_2) \geqslant 3$. Thus not both vertices can be contained in $N(v)$. W.l.o.g. let $c_1 \in N_F^2(v)$ and let $x \in F$ be a neighbour of v and c_1.

Case 2.1: $c_2 \in N_F(v)$.

If $xc_2 \notin E$ we obtain the P_4 $c_2 - v - x - c_1$, a contradiction. Otherwise x must be a vertex of U. Since $C \cup \{x\}$ is not a clique in G^2 there must be a vertex $c_3 \in C$ such that $d_G(x, c_3) \geqslant 3$. Thus, $c_3 \in N^2(v)$ implying $d_G(x, c_3) = 3$. Let y be a common neighbour of v and c_3 in F. By distance requirements we obtain $xy \notin E$ and $xc_3 \notin E$. Therefore, v is mid-point of the P_4 $x - v - y - c_3$, a contradiction.

Case 2.2: $c_2 \in N_F^2(v)$.

If $x \in U$ we can proceed as in Case 2.1. So assume that c_1 has no neighbour in $U \cap N_F(v)$. But $d_F(c_1, c_2) = 2$. Thus there is a vertex $u \in U \setminus N_F(v)$ adjacent to both c_1, c_2. Let $y \in F$ be a common neighbour of v and c_2. Note that $c_1 y \notin E$. Thus, if $xy \notin E$ then v is mid-point of $c_1 - x - v - y$, a contradiction. So let $xy \in E$. Now, u must be in $N_F^2(v) \cup N_F^3(v)$. If $u \in N_F(v)$ then the vertices c_1, x, y, c_2, u induce either a C_5 (for ux, $uy \notin E$), a house (for $ux \in E$ and $uy \notin E$ or vice versa) or a 3-sun (for ux, $uy \in E$), contradicting that F is HHDS-free.

Otherwise, i.e. if $u \in N_F^3(v)$, then the vertices c_1, x, y, c_2, u induce a C_5, again a contradiction. So we are done. □

Corollary 5.3. *Let G be a HHDS-free graph. Then $\mathscr{DH}(G)$ is conformal.*

Proof. We have to show that every clique C of $2SEC(\mathscr{DH}(G))$ is contained in some hyperedge. By Lemma 3.1 we have $2SEC(\mathscr{DH}(G)) = G^2$. Thus the above lemma implies that C is contained in a 2-graph F of G. From Lemma 5.1 the existence of a homogeneous set H in F dominating F follows. Hence, C is contained in the hyperedge $D_F(H, 1)$ of $\mathscr{DH}(G)$. □

Lemma 5.4. *Let G be a house-free graph and v be a semi-simplicial vertex in G with $rad(G) \geqslant 2$. Then $G^2 \setminus \{v\} = (G \setminus \{v\})^2$.*

Proof. At first we show that v is not a cut vertex using $rad(G) \geqslant 2$ and the semi-simplicity of v. Assume to the contrary that there are at least two connected components K_1, K_2 of $G \backslash \{v\}$. Let $x \in K_1$ and $y \in K_2$. If one of these vertices is not in $D_G(v, 1)$ any shortest path connecting x and y induces a P_4 in $D(v, 2)$ such that v is midpoint. Hence, $\{x, y\} \subseteq N(v)$ and thus $e(v) = 1$, a contradiction.

Next, note that any edge in $(G \backslash \{v\})^2$ is an edge in $G^2 \backslash \{v\}$. Now suppose there are vertices x, y such that xy is an edge in $G^2 \backslash \{v\}$ but not in $(G \backslash \{v\})^2$. Thus $d_G(x, y) = 2$ and $N_G(x) \cap N_G(y) = \{v\}$. Since $e(v) \geqslant 2$ there must be a vertex w in $N^2(v)$. Let u be one of its neighbours in $N(v)$. If $u = x$ or $u = y$ the semi-simplicial vertex v is midpoint of the P_4 either $w - u - v - y$ or $w - u - v - x$. By similar arguments w must be adjacent to exactly one of the vertices x, y, say x. This forces $uy \in E$ and $xu \notin E$. But now we have an induced house. □

Lemma 5.5. *If G is a HHDS-free graph then G^2 is chordal.*

Proof. We proceed by induction on $|V|$. Let v be a semi-simplicial vertex of G and suppose G^2 is not chordal. By the induction hypothesis $(G \backslash \{v\})^2 = G^2 \backslash \{v\}$ is chordal. Hence, any chordless cycle of length $k \geqslant 4$ in G^2 must contain v. Let $C = v - v_1 - \cdots - v_{k-1} - v$ such a cycle. Since $d_G(v_1, v_{k-1}) \geqslant 3$ at least one of the distances $d_G(v, v_1)$ and $d_G(v, v_{k-1})$ must be 2. Suppose $d_G(v, v_{k-1}) = 1$ and let x be a vertex adjacent to both v and v_1 in G. Due to the distance requirements v_{k-1} cannot be adjacent to x or v_1 in G, thus v is midpoint of the P_4 $v_{k-1} - v - x - v_1$, a contradiction. Therefore, $d_G(v, v_1) = d_G(v, v_{k-1}) = 2$. Let w_{k-1} be adjacent to v and v_{k-1} in G. The semi-simplicity of v then implies $xw_{k-1} \in E$. Hence, $C' = x - v_1 - \cdots - v_{k-1} - x$ is a cycle in $G^2 \backslash v$. By the induction hypothesis that graph is chordal, thus x must be adjacent to each vertex v_i in G^2, $i = 1, \ldots, k-1$. Since for any $i = 2, \ldots, k-2$ we have $d_G(x, v_i) = d_G(x, v_{i+1}) = 2 \geqslant d_G(v_i, v_{i+1})$ the pseudo-modularity of HHD free graphs implies the existence of a neighbour w_i of x which is adjacent to both v_i and v_{i+1} in G. Obviously, for $i \neq j$ we have $w_i \neq w_j$.

At first consider the case $k = 4$. The subgraph induced by $\{v, x, w_2, v_3, w_3\}$ implies the edge w_2w_3, since G is house-free. If v_1 and v_2 are adjacent we obtain a house induced by $\{x, v_1, v_2, w_2, w_3\}$, a contradiction. Otherwise (i.e. $v_1v_2 \notin E$) by pseudo-modularity of G we have a vertex w_1 adjacent to x, v_1 and v_2. If $w_1w_2 \notin E$ we get a house induced by $\{v_1, x, w_2, v_2, w_1\}$. If $w_1w_2 \in E$ then we get a 3-sun induced by $\{v, v_3, w_1\} \cup \{x, w_3, w_2\}$.

For the sequel let $k \geqslant 5$. We consider the subgraph induced by the vertices $\{x, w_{i-1}, w_i, w_{i+1}, v_i, v_{i+1}\}$ for $i = 3, \ldots, k-2$. We will prove $w_{i-1}w_i \in E$ and $w_iw_{i+1} \in E$. First note that $xv_i \notin E$, $xv_{i+1} \notin E$ and $w_{i-1}v_{i+1} \notin E$, $w_{i+1}v_i \notin E$.

Suppose $v_iv_{i+1} \in E$. Since G does not contain a house the edges $w_{i-1}w_i$ and w_iw_{i+1} must exist. So the vertices $\{x, w_{i-1}, v_i, v_{i+1}, w_{i+1}\}$ induce either a C_5 or a house depending on whether w_{i-1} is adjacent to w_{i+1} or not, a contradiction in both cases. Hence, $v_iv_{i+1} \notin E$.

Now by assuming $w_{i-1}w_i \notin E$ we get the chordless 4-cycle $x - w_{i-1} - v_i - w_i$. The only possible edges are w_iw_{i+1} and $w_{i-1}w_{i+1}$. If both edges do not exist we obtain a domino, in all other cases we get a house.

Consequently, $w_{i-1}w_i \in E$, $w_iw_{i+1} \in E$ and $v_iv_{i+1} \notin E$ for $i = 3, \ldots, k-2$. Now consider v_1 and v_2. If these vertices are adjacent we obtain a house induced by $\{v_1, v_2, w_2, w_3, x\}$. If v_1 is not adjacent to v_2 then by pseudo-modularity of G there must be a vertex w_1 adjacent to $\{x, v_1, v_2\}$. Obviously, $v_1w_2 \notin E$. Thus, we obtain the edge w_1w_2 since the subgraph induced by $\{x, v_1, v_2, w_1, w_2\}$ must be house-free. If w_1 is not adjacent to w_3 we get a 3-sun induced by $\{v_1, w_3, v_2\} \cup \{x, w_1, w_2\}$. So $w_1w_3 \in E$. By assuming $v_2v_3 \in E$ we obtain a house induced by $\{x, w_3, v_3, v_2, w_1\}$. If v_2 is not adjacent to v_3 then the vertices $\{x, w_1, w_2, \ldots, w_{k-1}\} \cup \{v, v_1, v_2, v_3, \ldots, v_{k-1}\}$ induce a k-sun. This completes the proof. □

In preparing our main result of this section we finally use another graph class containing all HHD-free graphs. A graph is called *weakly chordal* iff it does not contain any induced cycles of length greater than four or their complements. Since each complement of a cycle of length greater than five contains an induced house HHD-free graphs are weakly chordal. These graphs were introduced in [19] and characterized in [20] in terms of 2-pairs. Hereby, a *2-pair* is a pair of nonadjacent vertices such that each induced path joining these vertices is of length 2. In [20] the authors proved, that a graph is weakly chordal iff each induced subgraph is either complete or contains a 2-pair. Using this characterization we show

Lemma 5.6. *In weakly chordal house-free graphs any incomplete sun contains a complete* 3-*sun.*

Proof. Let $\{w_0, \ldots, w_{k-1}\} \cup \{u_0, \ldots, u_{k-1}\}$ induce an incomplete k-sun in a weakly chordal house-free graph G. Since the sun is incomplete the connected subgraph induced by the cycle $w_0 - \cdots - w_{k-1}$ is not a clique and hence must contain a 2-pair $w_i, w_j, i < j, |i-j| \geqslant 2$. We conclude that any vertex $w_l, l \neq i,j, |l-i| = 1$ or $|l-j| = 1$ modulo k must be adjacent to both vertices w_i, w_j. Now consider the subgraph induced by $\{w_{i-1}, w_i, w_{i+1}, w_j, u_i\}$. Since G is house-free the vertices w_{i-1} and w_{i+1} must be adjacent. But now, $\{w_{i-1}, w_i, w_{i+1}\} \cup \{w_j, u_i, u_{i+1}\}$ induces a 3-sun. □

Now to the main result of this section:

Theorem 5.7. *A graph G is hereditary homogeneously orderable iff it does not contain a C_k for $k \geqslant 5$, a house, a domino, or a complete k-sun for $k \geqslant 3$ as an induced subgraph.*

Proof. To verify that the stated graphs are not homogeneously orderable is straightforward. For the converse we have to show by Theorem 3.9 that for any induced subgraph H of a HHDS-free graph $\mathcal{DH}(H)$ is a dual hypertree. Since any induced subgraph of a HHDS-free graph is HHDS-free again the assertion follows from the above lemmata. □

Corollary 5.8. *Hereditary homogeneously orderable graphs are hereditary pseudo-modular.*

6. The recognition algorithm

Our polynomial-time recognition algorithm of homogeneously orderable graphs bases on Lemma 2.10, Corollary 3.10 and Lemma 3.7. Assume that v is h-extremal with $e(v) \geqslant 2$ and let $\bar{G} = (V, \bar{E})$ be the complement of G. By Lemma 2.10 there is a proper homogeneous dominating set $H \subseteq N(v)$ dominating $D(v, 2)$. Thus, in $\bar{G}$ no vertex of H is adjacent to some vertex of $D_G(v, 2)\backslash H$. It suffices to consider the complement of the graph induced by the disk $D(v,2)$. Let C_v denote the connected component of this graph which contains v (and hence $N^2(v)$).

Lemma 6.1. *A vertex v such that $e(v) \geqslant 2$ is h-extremal iff $H := N(v)\backslash C_v$ is a homogeneous set.*

Proof. Let C_v be the connected component of the graph induced by $D(v, 2)$ such that $v \in C_v$. If v is h-extremal then by Lemma 2.10 there is a homogeneous set $H' \subseteq N(v)$ dominating $D(v, 2)$. Obviously, $H' \cap C_v = \emptyset$. Since $H' \subseteq H$ the set H is nonempty. So it remains to show that H is homogeneous, but this is trivial. The other direction is obvious. □

In the following algorithm all neighbourhoods are restricted to the rest graph $G_i := G_{\{v_i, \ldots, v_n\}}$.

Algorithm RecHom:

Input: A connected graph $G = (V, E)$.
Output: A h-extremal ordering $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$ or answer 'NO'.

(1) Compute G^2.
(2) **if** G^2 is not chordal **then** STOP. NO
(3) **else** let $\tau = (v_1, \ldots, v_n)$ be a perfect elimination ordering of G^2 and
    $k(\tau) := \min\{i : G^2_{\{v_i, \ldots, v_n\}}$ is complete$\}$;
(4) **for** $i := 1$ **to** $k(\tau) - 1$ **do**
(5)   BFS(v_i);
(6)   compute a connected component C_{v_i} generated by $D(v_i, 2)$ in $\bar{G}_i$;
(7)   determine $H := N(v_i)\backslash C_{v_i}$;
(8)   **if** $H = \emptyset$ then STOP. NO else $H_i := H$;
(9) $\sigma(i) := (v_i, H_i)$;
(10) **endfor**;

(11) BFS$(v_{k(\tau)})$;
(12) compute the connected component $C_{v_{k(\tau)}}$ generated by $D(v_{k(\tau)}, 2)$ in $\overline{G_{k(\tau)}}$;
(13) determine $H := N(v_{k(\tau)}) \backslash C_{v_{k(\tau)}}$;
(14) **if** $H = \emptyset$ **then** STOP.NO
(15) **else** let $H = \{u_1, \ldots, u_r\}$; $V(G_{k(\tau)}) \backslash H = \{w_1, \ldots, w_s\}$;
(16) **for** $i := 1$ **to** $r - 1$ **do** $\sigma(i + k(\tau) - 1) := (u_i, V(G_{k(\tau)}) \backslash H)$;
(17) **for** $i := 1$ **to** s **do** $\sigma(i + k(\tau) + r - 2) := (w_i, \{u_r\})$;
(18) endfor

Theorem 6.2. *The algorithm RecHom is correct and works within* $O(n^3)$ *steps.*

Proof. The correctness of the algorithm immediately follows from Theorem 3.9, Lemma 2.10, Corollary 3.10, Lemma 3.7 and Lemma 6.1.
Time bound: The square G^2 of a graph G can be computed in $O(n^2)$. The chordality can be tested in linear time in the size of G^2 and if the graph is chordal one gets a perfect elimination ordering τ in the same time. Thus, (1)–(3) are done in $O(n^2)$ steps. Lines (5)–(10) are iterated $k(\tau)$ times. BFS takes $O(|E|)$ steps and finding the connected components in the complement graph $\bar{G}$ takes $O(|\bar{E}|)$ steps. Thus, the total amount of time is bounded by $O(n^3)$. □

7. The Steiner tree problem

In this section, we present an algorithm solving the Steiner tree problem on homogeneously orderable graphs in time $O(|E(G^2)|)$ provided a h- extremal ordering is given. Recall that given a Steiner set $T \subset V$ we have to compute a minimal set $S \subseteq V$ such that $T \subseteq S$ and G_S is connected.

We may assume that G_T is disconnected for otherwise there is nothing to do, and connectedness can be tested in linear time using Depth First Search (DFS).

At first some technical lemmata.

Lemma 7.1. *Let v be a h-extremal vertex of a graph $G = (V, E)$ with homogeneous dominating set $H \subset N(v)$ and $u, w \in N(v)$. Define $G' := (V, E \cup \{uw\})$. Then only the distance between u and w is changed in G', i.e. for any vertex x of V and any vertex y of $V \backslash \{u, w\}$ we have $d_G(x, y) = d_{G'}(x, y)$.*

Proof. Follows immediately from the properties of H. □

Lemma 7.2. *Let G be a homogeneously orderable graph, $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$ be a h-extremal ordering of G, $u, w \in N(v_1) \backslash H_1$, G' defined as above with $v = v_1$. Then G' is homogeneously orderable and σ is a h-extremal ordering of G'.*

Proof. Since $G^2 = (G')^2$ by Corollary 3.10 it suffices to show that H_i remains homogeneous in G_i'. Suppose the contrary and let $i\,(i \geqslant 2)$ be the smallest index such that the set H_i is not homogeneous in G_i'. Then, u, w are G_i and, say, $u \in H_i$, $w \notin H_i$. Since H_i dominates $D_{G_i}(v_i, 2)$ we conclude $d_{G_i}(v_i, w) \geqslant 3$. Lemma 3.2 implies $d_G(v_i, w) \geqslant 3$. But then $d_{G_i'}(v_i, w) = 2$ contradicts Lemma 7.1. □

Now we are ready to formulate the algorithm. To make the algorithm clear we consider at first a homogeneously orderable graph G with a h-extremal vertex v and a homogeneous set $H \subseteq N(v)$ dominating $D(v, 2)$. Furthermore, let $T \subset V$ be given. For the sequel define $G' := G \backslash \{v\}$ and let S' be an optimal solution of the Steiner tree problem in G' with respect to a Steiner set T' defined in the different cases:

Case 1: $T \subseteq D(v, 1)$.

This is a trivial case. With $S := T \cup \{v\}$ we are done.

Case 2: $v \in T$ and $T \cap N(v) = \emptyset$ but $T \backslash D(v, 1) \neq \emptyset$.

Define $T' := (T \backslash \{v\}) \cup \{h\}$ for some vertex $h \in H$ and $S := S' \cup \{v\}$. We claim that S is optimal for G with respect to T.

Suppose to the contrary that there is a set F containing T such that G_F is connected and $|F| < |S|$. Since $v \in T$ we have $v \in F$. Define $F' := F \backslash \{v\}$. For $|F'| = |F| - 1 < |S| - 1 = |S'|$ and by the optimality of S' the set F' cannot be a solution in G' for T'. First assume that F' is not connected. Then v is a cutvertex in G_F. Let $x_1, \ldots, x_k$, $k \geqslant 2$, be the neighbours of v in F. Since H is homogeneous and a subset of $N(v)$ dominating the whole disk $D(v, 2)$ no x_i, $i = 1, \ldots, k$ can belong to H. But $T \cap N(v) = \emptyset$. Thus we can replace the vertices $x_1, \ldots, x_k$ by some vertex $h \in H$ obtaining a smaller set, a contradiction. Thus F' is connected and hence it cannot contain h. But $v \in F$ and $T \backslash D(v, 1) \neq \emptyset$. Thus, F' must contain some vertex from $N(v)$ which can be replaced by h, again a contradiction. Consequently, S is optimal.

Case 3: $v \in T$ and $T \cap N(v) \neq \emptyset$ and $T \backslash D(v, 1) \neq \emptyset$.

Case 3.1: $T \cap H \neq \emptyset$.

Here we have the problem that a solution S' of G' for $T' := T \backslash \{v\}$ may contain some vertex of H. We must decide whether this H-vertex is necessary or whether it can be replaced by v (see Fig. 6, the set T is formed by the filled circles).

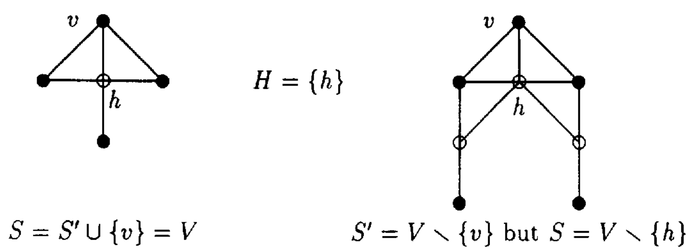

Fig. 6. An example to explain the Steiner tree algorithm.

Thus, we make $T \cap N(v)$ complete and define $G' := (V \setminus \{v\}, (E \setminus \{vx: x \in N(v)\}) \cup \{xy: x, y \in T \cap N(v)\})$ and $S := S' \cup \{v\}$. By Lemma 7.2 G' has the same h-extremal ordering $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$. To verify the correctness note that for each solution F in G for T the set $F \setminus \{v\}$ is a solution in G' for T' except the case $F \cap (N(v) \setminus T) \neq \emptyset$. But now we consider $F \setminus (D(v, 1) \setminus T) \cup \{h\}$ for some $h \in H$ which is a solution in G' for T'.

Case 3.2: $T \cap H \neq \emptyset$.

We define $T' := T \setminus \{v\}$ and $S := S' \cup \{v\}$. The correctness is trivial.

Case 4: $v \notin T$ and $T \setminus D(v, 1) \neq \emptyset$.

With $T' := T$ we define $S := S'$. Assume there is a set $F \subseteq V$ containing T such that G_F is connected and F has a smaller number of vertices than S. Since S' is optimal in G' and v is not in T we conclude $v \in F$. Thus, $F' := F \setminus \{v\}$ includes T and cannot be connected. We proceed as in Case 2.

Theorem 7.3. *The Steiner tree problem on homogeneously orderable graphs can be solved in time* $O(|E(G^2)|)$ *provided a h-extremal ordering is given.*

Proof. The algorithm steps through the given h-extremal ordering and computes an optimal solution S recursively as described above. The time bound follows from the fact that all added edges are edges of the square of G, see Lemma 7.1. □

8. Summary

In this paper we defined a new class of graphs which is a common generalization of distance-hereditary graphs, dually chordal graphs and homogeneous graphs. We presented a characterization of the new class in terms of a tree structure of the closed neighbourhoods of homogeneous sets in 2-graphs which is closely related to the defining h-extremal ordering.

Moreover, we characterized the hereditary homogeneously orderable graphs by forbidden induced subgraphs as the house-hole-domino-sun-free graphs.

Finally, we gave a polynomial time solution for the recognition and the Steiner tree problem on homogeneously orderable graphs. Thus we obtain:

Class	Recognition		Steiner Tree	
Tree	$O(n)$	Folk	$O(n)$	Folk
*Ext**(**tree**)	$O(m)$	[26]	$O(m)$	[26]
Distance-hereditary graphs	$O(m)$	[18]	$O(m)$	[5]
Dually chordal graphs	$O(m)$	[6, 11]	$O(\|E(G^2)\|)$	[7, 11]
Homogeneous graphs	Polynomial	[10]	Polynomial	[10]
Homogeneously orderable graphs	$O(n^3)$	Here	$O(\|E(G^2)\|)$	Here

We write $O(m)$ instead of $O(n + m)$ since any graph is connected in our paper.

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