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Incidence graphs of biacyclic hypergraphs

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Abstract

It is well-known that the incidence graphs of totally balanced hypergraphs are exactly chordal bipartite graphs. This paper examines the incidence graphs of biacyclic hypergraphs. We characterize these graphs as absolute bipartite retracts with forbidden isometric wheels, or alternatively via an elimination scheme.

Keywords: Acyclic hypergraph; Incidence graph; Absolute bipartite retract; Chordal bipartite graph; Maximum neighbourhood ordering.

The *incidence graph* of a hypergraph $H = (X, \mathscr{E})$ is a bipartite graph with vertex set $V = X \cup \mathscr{E}$, where two vertices $x \in X$ and $e \in \mathscr{E}$ are adjacent if and only if $x \in e$. It is well-known (see [1,10,9,7]) that the incidence graphs of totally balanced hypergraphs are exactly chordal bipartite graphs. In this paper we investigate the incidence graphs of biacyclic hypergraphs and characterize them as absolute bipartite retracts with forbidden isometric wheels, or alternatively via an elimination scheme. As a corollary the characterization of biacyclic hypergraphs is obtained.

For general terminology concerning graphs and hypergraphs, we refer to [6] . In this paper all graphs are finite, undirected and without multiple edges.

By a (hypergraphic) *cycle* we mean a finite alternating sequence of distinct vertices and distinct edges $x_1 e_1 \ldots x_k e_k x_1 (k \geqslant 3)$ such that $x_i, x_{i+1} \in e_i$ for $1 \leqslant i \leqslant k (\text{mod } k)$. For graphs, the (graphic) cycle will be identified with the sequence of its vertices $v_1 \ldots v_k v_1$; an edge of the form $v_i v_j$ with $|i - j| \neq 1$ is called a *chord* of the cycle. As usual C_k denotes the cycle (with no chords) on k vertices.

A hypergraph is *totally balanced* if every cycle has an edge containing at least three vertices of the cycle. A graph G is *chordal* if it has no cycle $C_k (k \geqslant 4)$, and G is *chordal bipartite* graph if it is bipartite and has no cycle $C_k (k \geqslant 6)$.

The *dual hypergraph* H^* of a hypergraph $H = (X, \mathscr{E})$ has $\mathscr{E}$ as its vertex set and $\{e \in \mathscr{E} | x \in e\} (x \in X)$ as its edges. The *underlying graph* of a hypergraph H has vertex set X and two distinct vertices are adjacent if and only if they lie in a common

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edge of H. A hypergraph is *conformal* if every clique (= maximal complete subgraph) of its underlying graph is contained in an edge of H. A hypergraph H is *acyclic* if H is conformal and its underlying graph is chordal.

We call a hypergraph H *biacyclic* if both H and H^* are acyclic. The class of biacyclic hypergraphs gives a natural extension of the class of totally balanced hypergraphs.

Further it is convenient to use the terminology of the papers [4] and [2].

If graph $G = (V, E)$ contains all loops vv $(v \in V)$, G is called *reflexive*; if G contains no loop it is called *irreflexive*. Note that a bipartite graph is by definition irreflexive. The *neighbourhood* of v in G, denoted as $N(v)$, consists of all the vertices u such that $vu \in E$; so, $v \in N(v)$ if and only if v is a loop of G. A *walk* in G is a sequence $v_0v_1 \ldots v_k$ of vertices of G, such that $v_{i-1}v_i \in E$ (for all $i = 1, \ldots, k$); the integer k is called the *length* of the walk. The *square* G^2 of a graph G is the graph with the same vertices as G, where two vertices u and v are adjacent exactly when they are connected in G by a walk (which may repeat vertices) of length 2. Note that if G is irreflexive then G^2 is the *2-step graph* (see [8]), whereas G^2 would denote the standard square only when G is assumed to be reflexive. The *kth iterated neighbourhood* of vertex v in graph G, denoted by $N^k(v)$, is defined recursively by $N^1(v) = N(v)$ and $N^{k+1}(v) = N(N^k(v))$, where $N(S) = \cup\{N(v) : v \in S\}$ for a set S of vertices. Equivalently, $N^k(v)$ consists of all vertices of G which admit a walk of length k from v.

A family of sets $S_i, i \in I$, has the *Helly property*, if any subfamily $S_j, j \in J \subseteq I$, in which any two members have a non-empty intersection, has itself a non-empty intersection. A graph G is *neighbourhood-Helly* if the family of neighbourhoods of G has the Helly property. An *intersection graph* of a family $S_i, i \in I$, is a graph with vertex set I, and two vertices $h, l \in I$ are adjacent if and only if $S_h \cap S_l \neq \emptyset$. It is evident that the neighbourhoods intersection graph of G coincides with G^2.

We shall always assume (if contrary is not stipulated) in this paper that our graphs are connected, i.e. that any two vertices are joined by a walk. The *distance* in G between vertices u and v, denoted by $d(u, v)$, is the minimum length of any walk that starts in u and ends in v. An *isometric subgraph* of a graph G is an induced subgraph in which the distance between any two vertices u, v equals the distance $d(u, v)$ taken in G.

A subgraph G of G' is a *retract* of G', if there is a mapping $f : V' \to V$ such that (a) f is edge-preserving, i.e. $f(u)f(v) \in E$ for each $uv \in E'$, and (b) f is fixed on each vertex of G, i.e. $f(v) = v$ for all $v \in V$; in such a case the mapping f is called a *retraction* of G' onto G. It is easy to see that if G is a retract of G' then it is an isometric subgraph of G'. An *absolute bipartite retract* is a bipartite graph G such that whenever G is an isometric subgraph of a bipartite graph G', then G is a retract of G'. As was shown in [3] the family of all iterated neighbourhoods in an irreflexive graph G has the Helly property if and only if G is an absolute bipartite retract. The reader interested in the latter subject is directed to consult the paper [4] .

Let $B_k = (X, Y, E)$ be such a bipartite graph, that $X = \{x_1, \ldots, x_k\}, Y = \{y_1, \ldots, y_k\}$, and x_i is adjacent to y_j if and only if $i \neq j$. The bipartite graph $\hat{B}_k$ results from

B_k by adding two vertices: vertex y_0 adjacent to $x_1,\dots,x_k$ and vertex x_0 adjacent to $y_0, y_1,\dots,y_k$ [3] .

We shall need the following lemmas.

Lemma 1. *Let G be an irreflexive graph. Then G is neighbourhood-Helly if and only if it is triangle-free and each of its induced subgraphs $B_k (k \geqslant 3)$ extends to $\hat{B}_k$ in G.*

Proof. Let v_1, v_2 and v_3 form a triangle in G and K be a maximal clique of G containing v_1, v_2, v_3. Any two neighbourhoods of vertices from K are intersecting. By the Helly property there exists a new vertex $u \notin K$ adjacent to all vertices of K. This is in contradiction with maximality of this clique. Hence graph G must be triangle-free.

Suppose that G contains an induced subgraph B_k with vertices $x_1, y_1, x_2, y_2,\dots,x_k, y_k$. Then for any x_i, x_j $N(x_i) \cap N(x_j) \neq \emptyset$. Hence as the system of neighbourhoods has the Helly property there exists a common neighbour y_0 of $x_1,\dots,x_k$. Also for any y_i, y_j

$$N(y_i) \cap N(y_j) \neq \emptyset, \quad N(y_i) \cap N(y_0) \neq \emptyset, \quad N(y_j) \cap N(y_0) \neq \emptyset$$

hold. So there must be some common neighbour x_0 of y_0 and the vertices y_i in G. Now the vertices x_i, y_i together with x_0, y_0 induce a bipartite subgraph $\hat{B}_k$.

Conversely, let $\{N(x_i)\colon i = 1,\dots,k\}$ be a system of neighbourhoods such that $N(x_i) \cap N(x_j) \neq \emptyset$ for all i,j. Assume that $\cap\{N(x_i)\colon i = 1,\dots,k\} = \emptyset$ and $k \geqslant 3$ is as small as possible. Then there exists vertex $y_i \in \cap\{N(x_j)\colon j = 1,2,\dots,k; i \neq j\}$ for all i. Now the vertices $x_1,\dots,x_k, y_1,\dots,y_k$ induce a subgraph B_k. But this subgraph does not extend to $\hat{B}_k$ in G because (by assumption) there does not exist a common neighbour of $x_1,\dots,x_k$. □

Lemma 2. *For any (reflexive or irreflexive) graph G the iterated neighbourhoods intersection graph is chordal if and only if G^2 is chordal.*

Proof. Among all maximum induced cycles of the iterated neighbourhoods intersection graph choose a cycle

$$C = N^{k_1}(v_1)\dots N^{k_n}(v_n)N^{k_1}(v_1)$$

with a minimal sum $s = k_1 + \cdots + k_n$. We claim that C is formed by neighbourhoods only, i.e. $k_1 = \cdots = k_n = 1$. Assume the contrary, and let for example $k_1 \geqslant 2$.

Pick arbitrary vertices $a \in N^{k_1}(v_1) \cap N^{k_2}(v_2)$ and $b \in N^{k_1}(v_1) \cap N^{k_n}(v_n)$ different from v_1. Now consider two neighbours v_1' and v_1'' of vertex v_1 such that $d(v_1', a) = d(v_1, a) - 1$ and $d(v_1'', b) = d(v_1, b) - 1$. If equalities

$$N^{k_1 - 1}(v_1') \cap N^{k_n}(v_n) = \emptyset, \quad N^{k_1 - 1}(v_1'') \cap N^{k_2}(v_2) = \emptyset$$

are fulfilled, then the iterated neighbourhoods

$$N^{k_1 - 1}(v_1'), N^{k_2}(v_2), \dots, N^{k_n}(v_n), N^{k_1 - 1}(v_1'')$$

form an induced cycle with $n + 1$ edges, contradicting to maximality of C.

So, assume for example that $N^{k_1-1}(v_1') \cap N^{k_n}(v_n) \neq \emptyset$. Then replacing the iterated neighbourhood $N^{k_1}(v_1)$ by $N^{k_1-1}(v_1')$ in the cycle C, we obtain an induced cycle with n edges and sum $s-1$. This again contradicts to the choice of C. Hence, C consists of the neighbourhoods only, i.e. C is an induced cycle of the graph G^2.

Thus, for graph G the iterated neighbourhoods intersection graph has no cycles C_n with $n \geqslant 4$ if and only if G^2 is so. □

Note that in reflexive case this lemma appeared first in [12] (see also [11]).

Now let $G=(V,E)$ be a graph with $V=\{v_1,\ldots,v_n\}$. Following [5,15] we say that a vertex v has a *maximum neighbour* u if $N(z) \subseteq N(u)$ holds for all $z \in N(v)$. If $N(v) \subseteq N(w)$ for some $w \neq v$ we say that v is *covered* (by w) in G. Obviously, every pendant vertex in a connected graph with at least three vertices has a maximum neighbour and is covered. A linear ordering $(v_1, v_2, \ldots, v_n)$ of V is a *maximum neighbourhood ordering* of G if for all $i \in \{1,\ldots,n-2\}$ the vertex v_i has a maximum neighbour and is covered in $G_i = G(v_i,\ldots,v_n)$ (G_i is a subgraph induced by vertices $v_i,\ldots,v_n$). Note that in reflexive graph every vertex v which has a maximum neighbour u is also covered by u, whereas in irreflexive graph maximum neighbour and covering vertex are distinct (are adjacent).

Lemma 3. *Let G be the incidence graph of a biacyclic hypergraph. Then G has a vertex that is covered and has a maximum neighbour.*

In order to prove the lemma let us recall some definitions and results provided in [14]. A matrix M is in *doubly lexical order* [14] if its rows and columns as vectors are in increasing order.

An ordered 0–1 matrix M is *doubly supported Γ* [14,15] if for every pair $r_1 < r_2$ of rows and pair $c_1 < c_2$ of columns which form a Γ (i.e. an ordered 0–1 valued 2×2 matrix with exactly one 0 in the bottom right corner: $[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}]$), there exists a row $r_3 > r_2$ with $M(r_3,c_1) = M(r_3,c_2) = 1$ and a column $c_3 > c_2$ with $M(r_1,c_3) = M(r_2,c_3) = 1$.

The incidence matrix of a hypergraph $H=(X,\mathscr{E})$ is a matrix whose (i,j) entry is 1 if $x_i \in e_j$ and 0 otherwise. From results of [14] one can derive the following corollaries for the incidence matrix of biacyclic hypergraph (see [15]):

The biacyclic hypergraphs are exactly the hypergraphs whose incidence matrices have doubly supported Γ ordering. Moreover, any doubly lexical ordering of the incidence matrix of a biacyclic hypergaph is doubly supported Γ.

We use this fact in our proof.

Proof of Lemma 3. Let M be a doubly lexical ordering of the incidence matrix of a biacyclic hypergraph H. Denote by x_i the vertex of H corresponding to row i of M and by e_j the edge of H corresponding to column j. Let i be the first row of M having 1 in column 1. Let also j and h be the maximum indices for which $M(i,j)=1$ and $M(h,1)=1$. If $h=i$ or $j=1$ then the incidence graph of H has a pendant vertex (e_1 or x_1), and the proof is done.

Suppose therefore that $h \neq i$ and $j \neq 1$. Let k be any row of M with $i \leqslant k \leqslant h$ such that $M(k,1) = 1$. Since M is doubly supported Γ, the equality $M(k,1) = 1$ implies $M(k,j) = 1$ and the equality $M(k,l) = 1$ implies $M(h,l) = 1$ for any l. It follows that in the incidence graph $N(e_1) \subseteq N(e_j)$ and $N(x_k) \subseteq N(x_h)$ for every vertex $x_k \in N(e_1)$, i.e. the vertex e_1 has maximum neighbour x_h and is covered by e_j. □

The graph theoretic proof of this lemma (without using the ordered matrices) the interested reader may find in the earlier version of our paper (see [13]).

By $W_k (k \geqslant 4)$ we denote the bipartite wheel consisting of the induced cycle $x_1 y_1 x_2 y_2 \ldots x_k y_k x_1$ of length $2k$ and vertex z adjacent to $x_1, \ldots, x_k$.

Theorem. *For an irreflexive graph G the following statements are equivalent:*
(i) *G is the incidence graph of a biacyclic hypergraph;*
(ii) *G is a neighbourhood-Helly graph such that G^2 is chordal;*
(iii) *the system of all iterated neighbourhoods of G has the Helly property and its intersection graph is chordal;*
(iv) *G is an absolute bipartite retract that has no isometric bipartite wheels W_k with $k \geqslant 4$;*
(v) *G has a maximum neighbourhood ordering.*

Proof. (i)$\Leftrightarrow$(ii): First we prove that if G is a neighbourhood-Helly graph such that G^2 is chordal then G is bipartite. Assume the contrary, and let $C = v_1 v_2 \ldots v_n v_1$ be an induced odd cycle of minimal length. Due to minimality, the neighbourhoods $N(v_1), \ldots, N(v_n)$ form an induced cycle in G^2. Since G^2 is chordal this is possible only if $n = 3$. But G as a neighbourhood-Helly graph must be triangle-free.

So, G is bipartite and its vertex set V can be partitioned as $X \cup Y$ in such a way that every edge of G is of the form xy with $x \in X$ and $y \in Y$. It is evident that G is the incidence graph of the hypergraph $H = (X, \{N(y): y \in Y\})$. Moreover, $H^* = (Y, \{N(x): x \in X\})$. Now the equivalence is immediately obtained from the following simple facts:

(a) G as the incidence graph of a hypergraph H is neighbourhood-Helly if and only if both H and H^* are conformal;

(b) G^2 as the 2-step graph of G consists of two components that are the underlying graphs of H and H^*.

(ii)$\Rightarrow$(v): Let G be a neighbourhood-Helly graph such that G^2 is chordal, i.e. G be the incidence graph of some biacyclic hypergraph. By Lemma 3 there exists a vertex v with maximum neighbour u and covered by $w \neq v$. We show that graph $G - v = G(V \setminus \{v\})$ fulfills (ii) also. Indeed, $N(v) \subseteq N(w)$. Then the neighbourhoods of G containing v contain the vertex w too. So the graph $G - v$ is neighbourhood-Helly. Furthermore, $G - v$ is an isometric subgraph of G. Hence $(G-v)^2 = G^2 - v$, and graph $G^2 - v$ is chordal as an induced subgraph of chordal graph G^2.

Now repeatedly applying these arguments we obtain a maximum neighbourhood ordering of G.

(v) $\Rightarrow$ (iv): It is evident that every irreflexive graph which has a maximum neighbourhood ordering is bipartite. First we prove by induction that graph G is neighbourhood-Helly and has no isometric bipartite wheels W_k with $k \geqslant 4$. Let v be the first vertex from maximum neighbourhood ordering of G, and y,z are respectively maximal neighbour and covering v vertex. By the induction assumption graph $G - v$ fulfills the necessary properties.

Since $N(v) \subseteq N(z)$, any neighbourhood of G containing v contains the vertex z. Therefore, the removal of vertex v from G does not change the neighbourhoods intersection. Then it is sufficient to consider a system of pairwise intersecting neighbourhoods of G which contains the neighbourhood $N(v)$. As the vertex v has maximal neighbour y, all neighbourhoods of this system contain the vertex y. So, G is neighbourhood-Helly.

Assume that G contains an isometric bipartite wheel W_k with vertices $v_1, u_1, v_2, u_2, \ldots, v_k, u_k (k \geqslant 4)$ and w as a center of this wheel. Without loss of generality it may be assumed that w is adjacent to $u_1, \ldots, u_k$, and vertex v belongs to W_k. Since W_k is isometric subgraph and v has a maximum neighbour, v is distinct from w and $v_i, 1 \leqslant i \leqslant k$. Let for example $v = u_1$. Then the vertices $v_1, z, v_2, u_2, \ldots, v_k, u_k (k \geqslant 4)$ and w form in $G - v$ an isometric wheel W_k'. Contradiction.

If in addition we show that every induced cycle of length at least 8 contains a pair of vertices that has a common neighbour in G but not on the cycle, then from Lemma 1 and Corollary 7.2 of [3] one immediately obtains (iv). Consider the cycle C_k, $k \geqslant 8$. Let v be the vertex of C_k having minimal index i according to maximum neighbourhood ordering. Then any other vertex of C_k has index greater than i. Hence a maximum neighbour u of v in graph $G(v_i, \ldots, v_n)$ is adjacent to two vertices of C_k that are at distance 2 from v. The proof of implication is complete.

(iv) $\Rightarrow$ (iii): As we mentioned above the system of all iterated neighbourhoods of G has the Helly property if and only if G is an absolute bipartite retract. According to Lemma 2 it remains to prove that the neighbourhoods intersection graph G^2 is chordal. We show this by induction on the vertex number of G. Every absolute bipartite retract has a vertex v covered by another vertex w (see [3]). It is easy to see that for graph $G - v$ the condition (iv) is also true. By the induction assumption the graph $(G - v)^2$ is chordal. Suppose that there exists in G^2 a cycle C_n with $n \geqslant 4$ which is induced by neighbourhoods $N(v_1), \ldots, N(v_n)$. Without loss of generality assume that $v_1 = v$. The neighbourhoods $N(w), N(v_2), \ldots, N(v_n)$ in graph $(G - v)^2$ form a cycle of length n. Since $(G - v)^2$ is chordal this cycle is divided by chords into triangles. Note that $N(w)$ is one end of these chords.

Thus $N(w) \cap N(v_i) \neq \emptyset$ for all i. Any three neighbourhoods like $N(w), N(v_i), N(v_{i+1})$ (if $i = n$ then $v_{i+1} = v_1$) pairwise intersect. Therefore there exists a vertex u_i belonging to their common intersection. Now the vertices $v_1, v_2, \ldots, v_n$ together with $u_1, u_2, \ldots, u_n, w$ induce a wheel in G. This wheel is isometric subgraph of G because (by assumption) the cycle of G^2 formed by neighbourhoods $N(v_1), \ldots, N(v_n)$ is induced. This contradiction proves the implication.

(iii) $\Rightarrow$ (ii) is evident. $\square$

The equivalence of conditions (ii), (iii) and (v) of the theorem is true also for reflexive graph G (see [12]).

Corollary 1. *For any irreflexive graph G the following conditions are equivalent:*
(i) *every induced subgraph of G has a maximum neighbourhood ordering;*
(ii) *G is chordal bipartite.*

Corollary 2. *For any irreflexive graph G the following conditions are equivalent:*
(i) *every isometric subgraph of G has a maximum neighbourhood ordering;*
(ii) *G is bipartite and the wheels $W_k, k \geqslant 4$, and cycles $C_k, k \geqslant 6$, are not isometric subgraphs of G.*

The proof follows from the theorem and the fact that the bipartite graph, for which every isometric cycle has length four, is absolute bipartite retract (see [3]).

Now let $H = (X, \mathscr{E})$ be a hypergraph. For any $x \in X$ and $e \in \mathscr{E}$ denote $\mathscr{E}(x) = \{e \in \mathscr{E}: x \in e\}, \Gamma(x) = \{z \in X: \mathscr{E}(x) \cap \mathscr{E}(z) \neq \emptyset\}, \Gamma(e) = \{e_i \in \mathscr{E}: e_i \cap e \neq \emptyset\}$.

We have as a corollary from the theorem the following.

Corollary 3. *H is a biacyclic hypergraph if and only if H may be dismantled up to an edge with one vertex by application of the following two operations:*

(a) *delete a vertex x for which there exist a vertex $y \neq x$ and an edge e such that $\mathscr{E}(x) \subseteq \mathscr{E}(y)$ and $\Gamma(x) = e$;*

(b) *delete an edge e for which there exist a vertex x and an edge $e' \neq e$ such that $e \subseteq e'$ and $\Gamma(e) = \mathscr{E}(x)$.*

In conclusion, we observe that the maximum neighbourhood ordering turns out to be very useful for domination-like problems. It permits to find in graph G a minimum set $D \subset V$ such that every vertex of V has a neighbour in D in linear time [12,5]. Now depending of the adjacency of a vertex to itself, D is a dominating set in a reflexive graph and a totally dominating set in an irreflexive graph.

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