Theory of Computation

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Before we go into details,

what are the two fundamental questions
in theoretical Computer Science?

1. **Can** a given problem **be solved** by computation?
2. **How efficiently** can a given problem be solved by computation?

We focus on *problems* rather than on specific *algorithms* for solving problems.

To answer both questions mathematically, we need to start by formalizing the notion of "computer" or "machine".

So, course outline breaks naturally into three parts:

1. Models of computation (*Automata theory*)
    - Finite automata
    - Push-down automata
    - Turing machines
2. What can we compute? (*Computability Theory*)
3. How efficient can we compute? (*Complexity Theory*)

We start with overview of the first part

Models of Computations or Automata Theory

First we will consider more restricted models of computation

- Finite State Automata
- Pushdown Automata

Then,

- (universal) Turing Machines

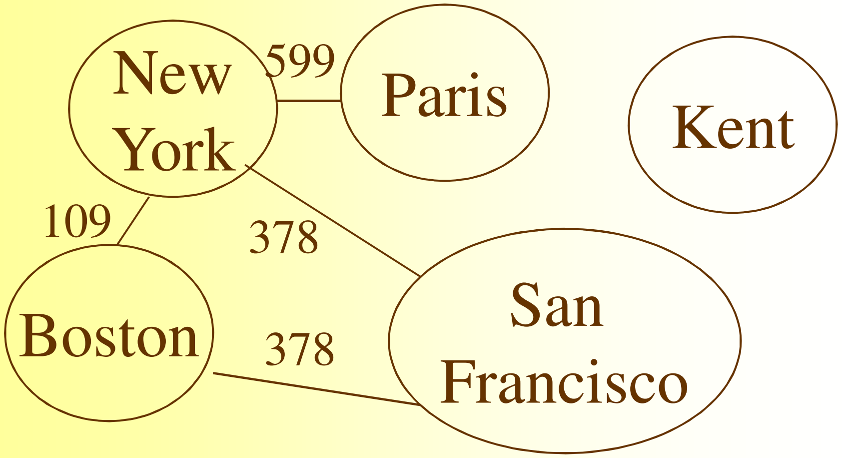
We will define "*regular expressions*" and "*context-free grammars*" and will show their close relation to Finite State Automata and to Pushdown Automata.

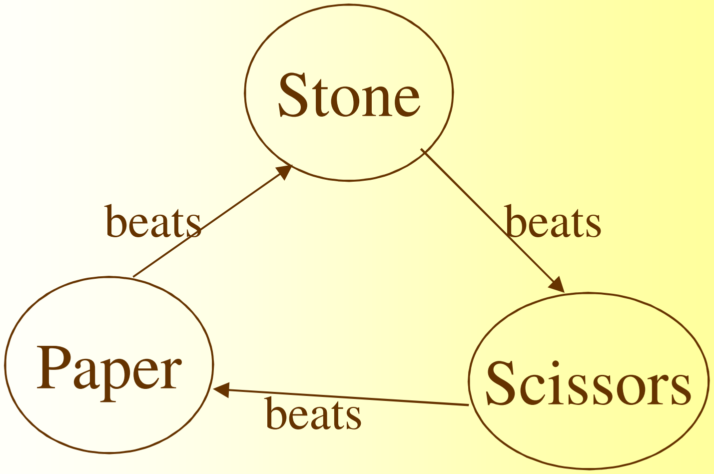
Used in compiler construction (lexical analysis)

Used in linguistic and in programming languages (syntax)

Graphs

- *G=(V,E)*
- vertices (*V*), edges (*E*)
- labeled graph, undirected graph, directed graph
- subgraph
- path, cycle, simple path, simple cycle, directed path
- connected graph, strongly connected digraph
- tree, root, leaves
- degree, outdegree, indegree





Strings and Languages

- *Alphabet* (any finite set of symbols)

$$\Sigma_1 = \{0,1\}$$
$$\Sigma_2 = \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z\}$$
$$\Gamma = \{0,1,x,y,z\}$$

- A *string* over an alphabet (a finite sequence of symbols from that alphabet)

$w_1 = 01001$ over Σ_1

$w_2 = abracadabra$ over Σ_2

$|w_1|$ is the length of string w_1 (=5)

ε is an empty string

the reverse of w_1 is $w_1^R = 10010$

substring, concatenation ($\overbrace{xx\cdots x}^{k} = x^k$)

Strings and Languages

• A *language* is a set of strings over a given alphabet (Σ_1).

$$L_1 = \{w: \ w \text{ contains 001 as a substring } \}$$

$$L_2 = \{w: \ |w| \text{ is even } \}$$

$\varepsilon \in L_2$

$|L_1|$ is the length of L_1 (this is an infinite language)

• Usual set operations as union and intersection can be applied to languages. $L_1 = \{11, 001\}$, $L_2 = \{0, 10\}$ ➔ $L_1 \cup L_2 = \{0, 10, 11, 001\}$, $L_1 \cap L_2 = \{ \}$

concatenation of two languages

$$L_1 L_2 = \{110, 1110, 0010, 00110\}$$

$$L_2 L_1 = \{011, 0001, 1011, 10001\}$$

Definitions, theorems, proofs

- ***Definitions*** describe the objects and notations
  - Defining some object we must make clear what constitutes that object and what does not.
- ***Mathematical statements*** about objects and notions
  - a statement which expresses that some object has a certain property
  - it may or may not be true, but must be precise
- A ***proof*** is a convincing logical argument that a statement is true
- A ***theorem*** is a mathematical statement proved true
  - this word is reserved for statements of special interest
  - statements that are interesting only because they assist in the proof of another, more significant statement, are called ***lemmas***
  - a theorem or its proof may allow us to conclude easily that another, related statements are true; these statements are called ***corollaries*** of the theorem

An example

- ***Definitions:***
  - A graph $G=(V,E)$, a node v, an edge (v,u), # of edges $|E|$,
  - incident, the degree $d(v)$ of a node v,
  - sum, even number.

- ***Theorem:*** For every graph G, the sum of the degrees of all the nodes in G is $2|E|$.
- ***Corollary:*** For every graph G, the sum of the degrees of all the nodes in G is an even number.

OR

- ***Lemma:*** For every graph G, the sum of the degrees of all the nodes in G is $2|E|$.
- ***Theorem:*** For every graph G, the sum of the degrees of all the nodes in G is an even number.
- ***Proof:*** (easy)

Types of proofs:

Proof by construction

• If theorem states that a particular type of object exists.

• A way to prove such a theorem is by demonstrating how to construct the object.

• A way to disprove a "theorem" is to construct an object that contradicts that statement (called a *counterexample*).

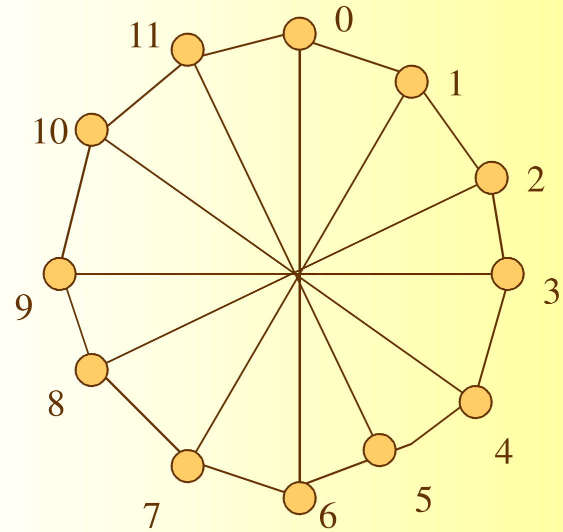
• ***Definition:*** A graph is ***k*-regular** if every node in the graph has degree ***k***

• ***Theorem:*** For each even number ***n*** greater than ***2***, there exists a ***3***-regular graph with ***n*** nodes.

• ***Proof:*** Construct a graph *G=(V,E)* as follows.

$$V = \{0,1,\ldots,n-1\}$$

$$E = \{\{i,i+1\} : \ for \ 0 \le i \le n-2\} \bigcup \{\{n-1,0\}\}$$
$$\bigcup \{\{i,i+n/2\} : \ for \ 0 \le i \le n/2-1\}$$



Proof by induction

• Prove a statement ***S(X)*** about a family of objects ***X*** (e.g., integers, trees) in two parts:

1. *Basis:* Prove for one or several small values of ***X*** directly.

2. *Inductive step:* Assume ***S(Y)*** for ***Y*** ``smaller than" ***X***;

   prove ***S(X)*** using that assumption.

Theorem: A binary tree with n leaves has $2n-1$ nodes.

Proof: • formally, $S(T)$: **if** T is a binary tree with n leaves, **then** T has $2n - 1$ nodes.

• induction is on the *size* = # of nodes in T.

Basis: if T has 1 node, it has 1 leaf. $1=2-1$, so OK

Induction: Assume $S(U)$ for trees with fewer nodes that T.

- T must be a root plus two subtrees U and V
- If U and V have u and v leaves, respectively, and T has t leaves, then $u + v = t$.
- By the induction hypothesis, U and V have $2u - 1$ and $2v - 1$ nodes, respectively.
- Then T has $1 + (2u - 1) + (2v - 1)$ nodes
  - $= 2 (u + v) - 1$
  - $= 2 t - 1$, proving inductive step.

If-And-Only-If Proofs

• Often, a statement we need to prove is of the form "***X* if and only if *Y***". We are often required to do two things:

1. Prove the *if-part:* Assume *Y* and prove *X*.
2. Prove the *only-if-part:* Assume *X* and prove *Y*.

Remember:

- the *if* and *only-if* parts are *converses* of each other.
- one part, say "if *X* then *Y*", says nothing about whether *Y* is true when *X* is false.
- an equivalent form to "if *X* then *Y*" is "if not *Y* then not *X*": the latter is the *contrapositive* of the former.

Equivalence of Sets

- many important facts in language theory are of the form that two sets of strings, described in two different ways, are really the same set.
- to prove sets *S* and *T* are the same, prove: *x* is in *S* if and only if *x* is in *T*. That is
    - Assume *x* is in *S*; prove *x* is in *T*.
    - Assume *x* is in *T*; prove *x* is in *S*.

Example: Balanced Parentheses

• Here are two ways that we can define ``*balanced parentheses*'':

1. *Grammatically*:

a) The empty string ε is balanced.

b) If *w* is balanced, then *(w)* is balanced.

c) If *w* and *x* are balanced, then so is *wx*.

2. *By Scanning :* *w* is balanced if and only if:

a) *w* has an equal number of left and right parentheses.

b) Every prefix of *w* has at least as many left as right parentheses.

• Call these **GB** and **SB** properties, respectively.

Theorem: A string of parentheses *w* is **GB** if and only if it is **SB**.

If part of the proof

- An induction on |*w*| (length of *w*). Assume *w* is **SB**; prove it is **GB**.

Basis: If $w = \varepsilon$ (length = 0), then *w* is **GB** by rule (a).

- Notice that we do not even have to address the question of whether ε is **SB** (it is, however).

Induction: Suppose the statement ``**SB** implies **GB**'' is true for strings shorter than *w*.

- *Case 1:* *w* is not ε, but has no nonempty prefix that has an equal number of (and).

  Then *w* must begin with (and end with) ; i.e., *w* = *(x).*

  - *x* must be **SB** (why?).
  - By the **IH**, *x* is **GB**.
  - By rule (b), *(x)* is **GB**; but *(x)* = *w*, so *w* is **GB**.

- *Case 2:* *w* = *xy*, where *x* is the shortest, nonempty prefix of *w* with an equal number of (and), and y is not ε .

  - *x* and *y* are both **SB** (why?).
  - By the **IH**, *x* and *y* are **GB**.
  - *w* is **GB** by rule (c).

Only-If part of the proof

- An induction on |*w*| (length of *w*). Assume *w* is **GB**; prove it is **SB**.

Basis: If *w* = ε (length = 0), then clearly *w* is **SB**.

Induction: Suppose the statement ``**GB** implies **SB**'' is true for strings shorter than *w,* and assume that *w* is not ε .

- *Case 1: w* is **GB** because of rule (b); i.e., *w = (x)* and *x* is **GB**.
    - by the **IH**, *x* is **SB**.
    - Since *x* has equal numbers of ('s and)'s, so does *(x).*
    - Since *x* has no prefix with more)'s than ('s, so does *(x).*
- *Case 2: w* is not ε and is **GB** because of rule (c); i.e., *w = xy*, and *x* and *y* are **GB**.
    - By the **IH**, *x* and *y* are **SB**.
    - (Aside) Trickier than it looks: we have to argue that neither *x* nor *y* could be ε , because if one were, the other would be *w*, and this rule application could not be the one that first shows *w* to be **GB**.
    - *xy* has equal numbers of ('s and)'s because *x* and *y* both do.
    - If *w* had a prefix with more)'s than ('s, that prefix would either be a prefix of *x* (contradicting the fact that *x* has no such prefix) or it would be *x* followed by a prefix of *y* (contradicting the fact that *y* also has no such prefix).
    - (Aside) Above is an example of *proof by contradiction*. We assumed our conclusion about *w* was false and showed it would imply something that we know is false.

Finite Automata

• An important way to describe certain simple, but highly useful languages called ``*regular languages*."

- A *graph* with a finite number of *nodes*, called *states*.
- *Arcs* are *labeled* with one or more *symbols* from some *alphabet*.
- One state is designated the *start state* or *initial state*.
- Some states are *final states* or *accepting states*.
- The *language of the FA* is the *set of strings* that label paths that go from the start state to some accepting state.

Example

• This FA scans HTML documents, looking for a list of what could be title-author pairs, perhaps in a reading list for some literature course.

• It accepts whenever it finds the end of a list item.

• In an application, the strings that matched the title (before ' **by** ') and author (after) would be stored in a table of title-author pairs being accumulated.