CHAPTER 3

The Church-Turing Thesis

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- Turing Machines
  - definitions, examples, Turing-recognizable and Turing-decidable languages
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- The definition of Algorithm
  - Hilbert's problems, terminology for describing Turing machines

Variants of Turing Machine (intro)

- There are alternative definitions of Turing machines, including versions with *multiple tapes* or with *non-determinism*.
- They are called ***variants*** of the Turing machine model.
- The original model and all its reasonable variants have the same power - they recognize the same class of languages.
- In this section we describe some of these variants and the proofs of equivalence in power.

Simplest equivalent "generalized" model

- In basic definition, the head can move to the left or right after each step: it cannot stay put.
- If we allow the head to stay put. The transition function would then have the form

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}.$$

- Does this make the model more powerful? Might this feature allow Turing machines to recognize additional languages?
- Of course not. We can replace each stay put transition with two transitions, one that moves to the right and the second back to the left.

Multi-tape Turing Machine

- A ***multi-tape TM*** is like an ordinary TM with several tapes.
- Each tape has its own head for reading and writing.
- Initially the input appears on tape 1, and others are blank.

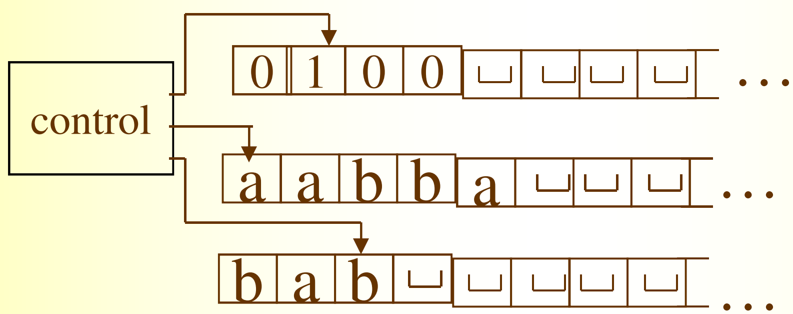
• The transition function is changed to allow for reading, writing, and moving the heads on all tapes simultaneously. Formally,

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k ,$$

where k is the number of tapes.

• The expression $\delta(q, a_1, \ldots, a_k) = (r, b_1, \ldots, b_k, L, R, \ldots, L)$

means that, if the machine is in state q and heads 1 through k are reading symbols a_1 trough a_k, the machine goes to state *r,* writes symbols b_1 through b_k, and moves each head to the left or right as specified.

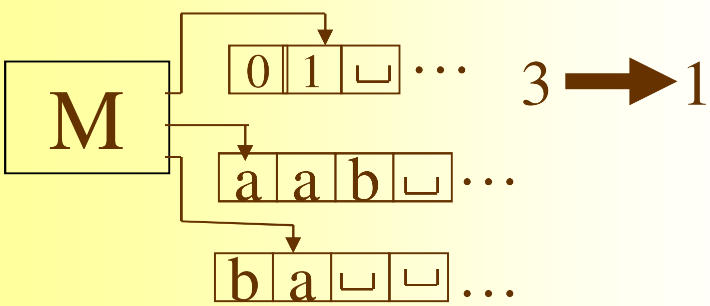


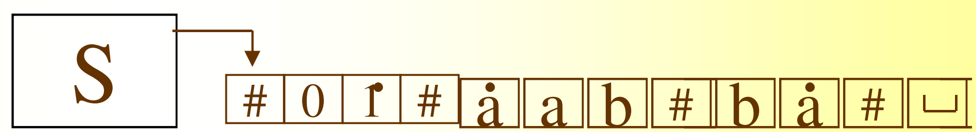
• Multi-tape TMs appear to be more powerful than ordinary TMs, but we will show that they are equivalent in power.

Multi-tape TMs vs. ordinary TMs

- ***Theorem:*** Every multi-tape Turing machine has an equivalent single tape Turing Machine.

- We show how to convert a multi-tape TM *M* to an equivalent single tape TM *S*.
- The key idea is to show how to simulate *M* with *S*.
- Let *M* has *k* tapes.
- Then *S* simulates the effect of *k* tapes by storing their information on its single tape.
- It uses new symbol # as a delimiter to separate the contents of the different tapes.
- *S* must also keep track of the locations of the heads.
- It does so by writing a tape symbol with a dot above it to mark the place where the head on that tape would be.
- Think of these as 'virtual' tapes and heads.





As before, the 'dotted' tape symbols are simply new symbols that have been added to the tape alphabet.

Multi-tape TMs vs. ordinary TMs (cont.)

S="On input $w = w_1 w_2 ... w_n$:

1. First *S* puts its tape into the format that represents all *k* tapes of *M*. The formatted tape contains

$$\# \dot{w}_1 w_2 ... w_n \# \dot{\sqcup} \# \dot{\sqcup} \# ... \#$$

2. To simulate a single move, *S* scans its tape from the first #, which marks the left-hand end, to the *(k+1)*st #, which marks the right-hand end, in order to determine the symbols under the virtual heads. Then *S* makes a second pass to update the tapes according to the way that *M*'s transition function dictates.

3. If at any point *S* moves one of the virtual heads to the right onto a #, this action signifies that *M* has moved the corresponding head onto the previously unread blank portion of that tape. So *S* writes a blank symbol on this tape cell and shifts the tape contents, from this sell until the rightmost #, one unit to the right. Then it continues the simulation as before.

Corollary: A language is Turing-recognizable if and only if some multi-tape Turing machine recognizes it.

Non-deterministic Turing Machine

- A ***non-deterministic TM*** is defined in the expected way: at any point of computation the machine may proceed according to several possibilities.
- The transition function for a non-deterministic TM has the form

$$\delta : Q \times \Gamma \to \mathbf{P}(Q \times \Gamma \times \{L, R\}).$$

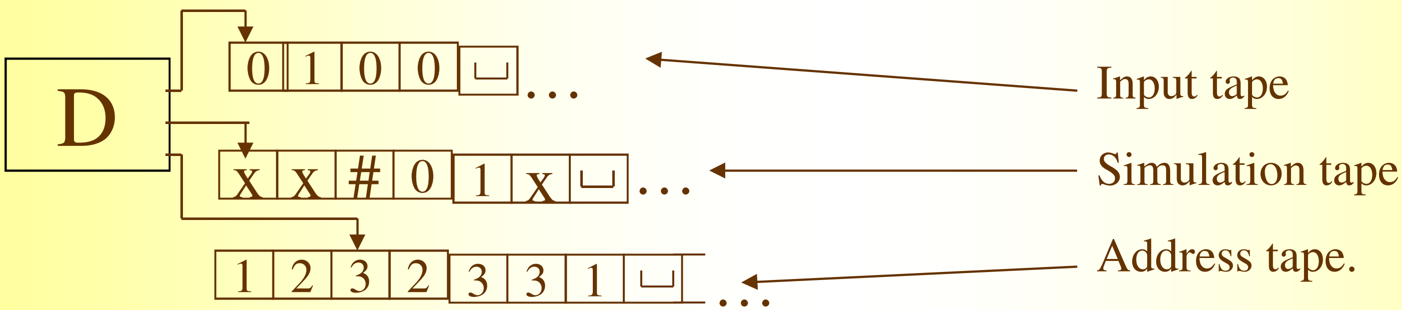
- The computation of a non-deterministic TM *N* is a tree whose branches correspond to different possibilities for the machine.
- Each node of the tree is a configuration of *N*. The root is the start configuration.
- If some branch of the computation leads to the accept state, the machine accepts the input.
- We will show that non-determinism does not affect the power of the Turing machine model.

- ***Theorem:*** Every non-deterministic Turing machine has an equivalent deterministic Turing Machine.

- We show that we can simulate any non-deterministic TM *N* with a deterministic TM *D*.
- The idea: *D* will try all possible branches of *N*'s non-deterministic computation.
- The TM *D* searches the tree for an accepting configuration. If *D* ever finds an accepting configuration, it accepts. Otherwise, *D*'s simulation will not terminate.

Non-deterministic TMs vs. ordinary TMs

- The simulating deterministic TM *D* has three tapes. By previous theorem this arrangement is equivalent to having a single tape.
  - Tape 1 always contains the input string and is never altered.
  - Tape 2 maintains a copy of *N*'s tape on some branch of its non-deterministic computation.
  - Tape 3 keeps track of *D*'s location in *N*'s non-deterministic computation tree.



- Every node in the tree can have at most *b* children, where *b* is the size of the largest set of possible choices given by *N*'s transition function.
- Tape 3 contains a string over $\Sigma_b = \{1,2,...,b\}^*$. Each symbol in the string tells us which choice to make next when simulating a step in one branch in *N*'s non-deterministic computation. This gives the address of a node in the tree.
- Sometimes a symbol may not correspond to any choice if too few choices are available for a configuration. In this case we say that the address is invalid, it does not correspond to any node.
- The empty string is the address of the root of the tree.

Non-deterministic TMs vs. ordinary TMs (cont.)

D="On input *w*:

1. Initially tape 1 contains the input *w*, and tapes 2 and 3 are empty.
2. Copy tape 1 to tape 2.
3. Use tape 2 to simulate *N* with input *w* on the branch of its non-deterministic computation. Before each step of *N* consult the next symbol on tape 3 to determine which choice to make among those allowed by *N*'s transition function. If no more symbols remain on tape 3 or if this non-deterministic choice is invalid, abort this branch by going to stage 4. Also go to stage 4 if a rejecting configuration is encountered. If an accepting configuration is encountered, *accept* the input.
4. Replace the string on tape 3 with the lexicographically next string. Simulate the next branch of *N*'s computation by going to stage 2."

Corollary 1: A language is Turing-recognizable if and only if some non-deterministic Turing machine recognizes it.

In a similar way one can show the following.

Corollary 2: A language is Turing-decidable if and only if some non-deterministic Turing machine decides it.

Equivalence with other models

- We have presented several variants of the Turing Machines and have proved them to be equivalent in power.
- Many other models of general purpose computation have been proposed in literature.
- Some of these models are very much like Turing machines, while others are quite different (e.g. λ -calculus).
- All share the essential feature of Turing machines, namely, ***unrestricted access to unlimited memory***, distinguishing them from weaker models such us finite automata and pushdown automata.
- All models with that feature turn out to be equivalent in power, so long as they satisfy certain reasonable requirements (e.g., the ability to perform only a finite amount of work in a single step).

More variants of Turing machine

- *k*-PDA, a PDA with *k* stacks.
- write-once Turing machines.
- Turing machines with doubly infinite tape.
- Turing machines with left reset $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{R, RESET\}$.
- Turing machines with stay put instead of left $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{R, S\}$.

- **If you missed a HW, try to give a complete answer to *one* of the problems 3.9, 3.11 – 3.14. Only *one* and *complete* answer will be accepted. Then you will get 10 points extra credit.**