The class **P**: polynomial time

- Theorems 1 and 2 illustrate an important distinction.
- On the one hand, we demonstrated at most a square or polynomial difference between the time complexity of problems measured on deterministic single tape and multi-tape Turing machines.
- On the other hand, we showed at most an exponential difference between the time complexity of the problems on deterministic and non-deterministic Turing machines.
- For our purpose, polynomial difference in running time are considered to be small, whereas exponential differences are considered to be large.
- Polynomial time algorithms are fast enough for many purposes, but exponential time algorithms rarely are useful. (For $n=1000$, n^3 = 1 billion (still manageable number), 2^n is much larger than the number of atoms in the universe.)
- All reasonable deterministic computational models are polynomially equivalent. Any one of them can simulate another with only a polynomial increase in running time.
- From here on we focus on aspects of time complexity theory that are unaffected by polynomial difference in running time. We consider such differences to be insignificant and ignore them.
- ***The Question is*** whether a given problem is polynomial or non-polynomial.
- So we came to an important definition in the complexity theory, **P** class.

The class **P**: definition

- ***Definition***: **P** is the lass of languages that are decidable in polynomial time on a deterministic single tape Turing machine. That is

$$P = \bigcup_k TIME(n^k).$$

- The class **P** plays an important role in our theory and is important because
  - **P** is invariant for all models of computation that are polynomially equivalent to the deterministic single tape TM, and
  - **P** roughly corresponds to the class of problems that are realistically solvable on a computer.
- When we analyze an algorithm to show that it runs in polynomial time, we need to do two things
  - First, give a polynomial upper bound (usually in big-O notation) on the number of stages that the algorithm uses when it runs on input of length *n*.
  - Then, examine the individual stages in the description of the algorithm to be sure that each can be implemented in polynomial time on a reasonable deterministic model.
- When both tasks have been done, we can conclude that it runs in polynomial time because we have demonstrated that it runs for a polynomial number of stages, each of which can be done in polynomial time, and the composition of polynomials is a polynomial.

Examples of problems in **P**

- We had: the problem whether w is a member of the language $A = \{0^k 1^k : k \geq 0\}$ is in **P.**
- Fortunately, there are many problems that are in **P.**
- The *PATH* problem is to determine whether a directed path exists from s to t.

$$PATH(G,s,t) = \{ <G,s,t>: G \text{ is a directed graph that has a directed path from } s \text{ to } t\}.$$

Theorem: $PATH \in P.$

- we use *breadth first search* and successively mark all nodes in G that are reachable from s by directed paths of length 1, then 2, then 3, through $m=|V|$.

M = "on *<G,s,t>:* where *G* is a directed graph with nodes *s* and *t*.

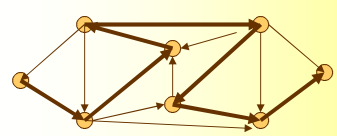
1. Place a mark on node *s.*
2. Repeat the following until no additional nodes get marked.
3. Scan all the edges of *G*. If an edge *(a,b)* is found going from marked node *a* to an unmarked node *b*, mark *b.*
4. If *t* is marked, *accept*; otherwise *reject*."

- ***Stages 1 ,4*** are executed only once. **Stage 3** runs at most $m=|V|$ times because each time except the last it marks an additional node in *G.* Hence, the **total number of stages** is $1+1+m$, giving a polynomial in the size of G.
- ***Stages 1,4*** easily implemented in polynomial time on any reasonable deterministic model. **Stage 3** involves a scan of the input and a test whether certain nodes are marked, which also is easily implemented in polynomial time.
- Hence, *M* is a polynomial time algorithm for *PATH*.

The class **NP**

- For some interesting and useful problems, polynomial time algorithms that solve them aren't known to exist.
- Why have we been unsuccessful in finding polynomial time algorithms for these problems? We don't know the answer to this important question.
- Perhaps these problems have, as yet undiscovered, polynomial time algorithms that rest on unknown principles.
- Or possibly some of these problems simply cannot be solved in polynomial time. They may be intrinsically difficult.
- One remarkable discovery concerning this question shows that the complexities of many problems are linked. The discovery of a polynomial time algorithm for one such problem can be used to solve an entire class of problems.
- A ***Hamiltonian path*** in a directed graph *G* is a directed path that goes through each node exactly once. Consider the problem of testing whether a directed graph contains a Hamiltonian path connecting two specified nodes.
- We can easily obtain an exponential time algorithm for the *HAMPATH* problem by brute-force approach which checks all possible permutations of nodes (n!).
- We need only add a check to verify that the potential path is Hamiltonian.
- No one knows whether *HAMPATH* is solvable in polynomial time.

HAMPATH={<G,s,t>: *G* is a directed graph with a Hamiltonian path from *s* to *t*}.



The class **NP**: definition

• Define the **non-deterministic time complexity class**

$NTIME(t(n)) = \{L : L \text{ is a language decided by an } O(t(n)) \text{ time Non-Deterministic Turing machine}\}.$

• ***Def:*** **NP** is the class of languages that are decidable in polynomial time on a non-deterministic Turing machine. That is

$$NP = \bigcup_k NTIME(n^k).$$

• The class **NP** is insensitive to the choice of reasonable non-deterministic computation model because all such models are polynomially equivalent.

Theorem: $HAMPATH \in NP.$

•The following is a non-deterministic Turing Machine (NTM) that decides the *HAMPATH* problem in non-deterministic polynomial time (we defined the time of a non-deterministic machine to be the time used by the longest computation branch).

N = "on <*G,s,t*>*:* where *G* is a directed graph with nodes *s* and *t*.

1. Write a list of *m* numbers $p_1, p_2, \ldots, p_m,$ where *m* is the number of nodes in *G*. Each number in the list is non-deterministically selected to be between *1* and *m*.
2. Check for repetitions in the list. If any are found, *reject*.
3. Check whether $s = p_1$ *and* $t = p_m$. If either fail, *reject*.
4. For each *i* between *1* and *m-1*, check whether (p_i, p_{i+1}) is an edge of *G*. If any are not, *reject*. Otherwise, *accept*."

• Clearly, this algorithms runs in non-deterministic polynomial time since all stages run in polynomial time.

Polynomial Time Verifiers

• The *HAMPATH* problem does have a feature called ***polynomial verifiability*** that is important for understanding its complexity.

• Even though we don't know of a fast (i,.e., polynomial time) way to determine whether a graph contains a Hamiltonian path, if such a path were discovered somehow (perhaps using the exponential time algorithm), we could easily convince someone else of its existence, simply by presenting it.

• In other words, *verifying* the existence of a Hamiltonian path may be much easier than *determining* its existence.

• We can give an equivalent definition of the **NP** class using the notion ***verifier.***

• A ***verifier*** for a language *A* is an algorithm *V*, where
A={*w: V* accepts <*w,c*> for some string *c*}.

• A verifier uses additional information, represented by the symbol *c* in definition. This information is called a ***certificate***, or ***proof***, of membership in *A*.

• Example: <*G,s,t*> belongs to *HAMPATH* if for some path *p*, *V* accepts <<*G,s,t*>,*p*> (that is, *V* says "yes, *p* is a Hamiltonian path from *s* to *t* of *G*). For the *HAMPATH* problem, a certificate for a string $<G,s,t> \in HAMPATH$ simply is the Hamiltonian path *p* from *s* to *t*.

• A ***polynomial time verifier*** is a verifier that runs in polynomial time in the length of *w*.

• A language *A* is ***polynomially verifiable*** if it has a polynomial time verifier.

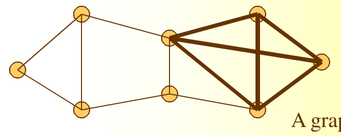
• ***Def:*** **NP** is the class of languages that have polynomial time verifiers.

•The verifier can check in polynomial time that the input is in the language when it is given the certificate.

CLIQUE is in **NP**

- A ***clique*** in an undirected graph *G* is a subgraph, wherein every two nodes are connected by an edge. A ***k-clique*** is a clique that contains *k* nodes.
- The ***clique problem*** is to determine whether a graph contains a clique of a specific size.

CLIQUE={<*G*,*k*>: *G* is an undirected graph with a *k*-clique}.



A graph with 4-clique.

Theorem: $CLIQUE \in NP$.

- ***Proof:*** The following is a verifier *V* for *CLIQUE*.

> *V* = "on input <<*G*,*k*>,*c*>:
> 1. Test whether *c* is a set of *k* nodes in *G*.
> 2. Test whether *G* contains all edges connecting nodes in *c*.
> 3. If both pass, *accept*; otherwise, *reject*."

- ***Alternative proof:*** If you prefer to think of **NP** in terms of non-deterministic polynomial Turing machine …

> *N* = "on <*G*,*k*>: where *G* is an undirected graph, *k* is an integer.
> 1. Non-deterministically select a subset *c* of *k* nodes in *G*.
> 2. Test whether *G* contains all edges connecting nodes in *c*.
> 3. If yes, *accept*; otherwise, *reject*."

SUBSET-SUM is in **NP**

- We have a collection of numbers, $x_1, x_2, ..., x_k$, and a target number *t*. We want to determine whether the collection contains a subcollection that adds up to *t*.

$$SUBSET-SUM = \{<S,t>: S = \{x_1, x_2, ..., x_k\} \ and$$
$$for \ some \ \{y_1, y_2, ..., y_l\} \subseteq \{x_1, x_2, ..., x_k\}, \ we \ have \ \sum y_i = t\}.$$

- For example *<{4,11,16,21,27},25>* is in *SUBSET-SUM* since 4+21=25.
- Note that $\{x_1, x_2, ..., x_k\}$ *and* $\{y_1, y_2, ..., y_l\}$ are multisets (we allow repetitions).

Theorem: $SUBSET-SUM \in NP$.

- ***Proof:*** The following is a verifier *V* for *SUBSET-SUM*.

> *V* = "on input <<*S*,*t*>,*c*>:
> 1. Test whether *c* is a collection of numbers that sum to *t*.
> 2. Test whether *S* contains all the numbers in *c*.
> 3. If both pass, *accept*; otherwise, *reject*."

- ***Alternative proof:*** If you prefer to think of **NP** in terms of non-deterministic polynomial Turing machine …

> *N* = "on <*S*,*t*>:
> 1. Non-deterministically select a subset *c* of the numbers in *S*.
> 2. Test whether *c* is a collection of numbers that sum to *t*.
> 3. If yes, *accept*; otherwise, *reject*."

The **P** versus **NP** question

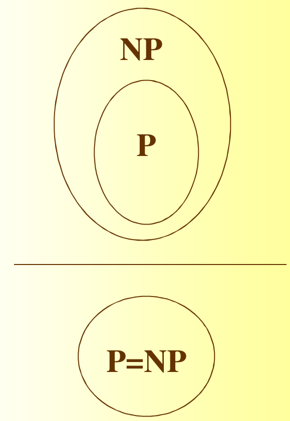
P = the class of languages that are decidable by polynomial time *deterministic* TMs.
NP = the class of languages that are decidable by polynomial time *non-deterministic* TMs.

OR EQUIVALENTLY

P = the class of languages where membership can be *decided* quickly (in pol. time).
NP = the class of languages where membership can be *verified* quickly (in pol. time).

- We presented examples of languages, such as *HAMPATH* and *CLIQUE*, that are members of **NP** but that are not known to be in **P**.
- No polynomial time algorithms are known for those problems.
- We are unable to *prove* the existence of a single language in **NP** that is not in **P**.
- The ***question*** of whether **P = NP** is one of the greatest unsolved problems in theoretical computer science.
- Most researchers believe that the two classes are not equal because people have invested enormous effort to find polynomial time algorithms for certain problems in **NP**, without success.
- The best method known for solving problems in **NP** deterministically uses exponential time. In other words, one can show that

$$\bigcup_k NTIME(n^k) = \boxed{NP \subseteq EXPTIME = \bigcup_k TIME(2^{n^k}).}$$



One of these two possibilities is correct.