Tree-like Structures in Graphs: a Metric Point of View

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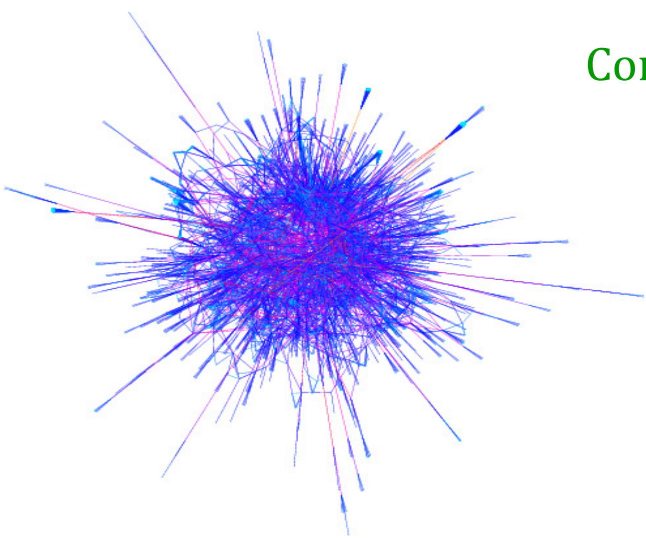
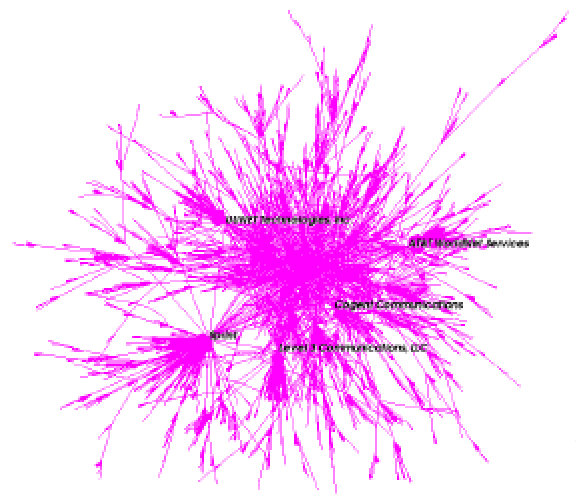




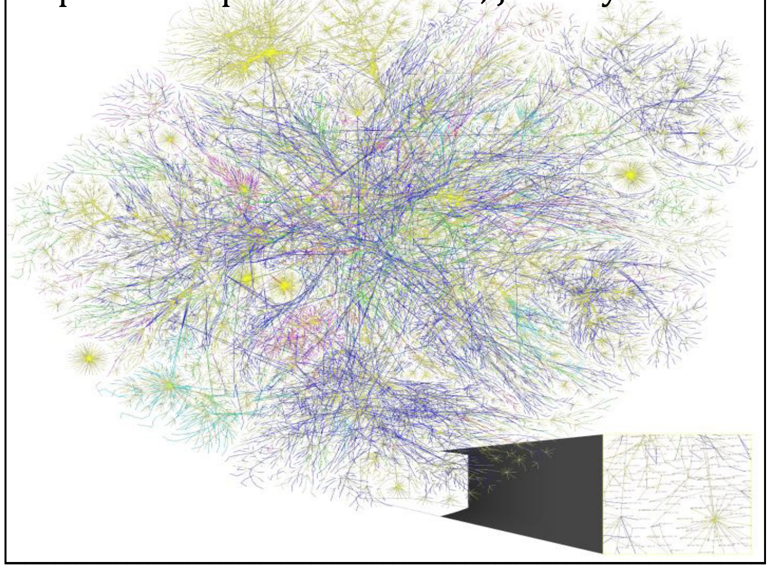
WG 2013

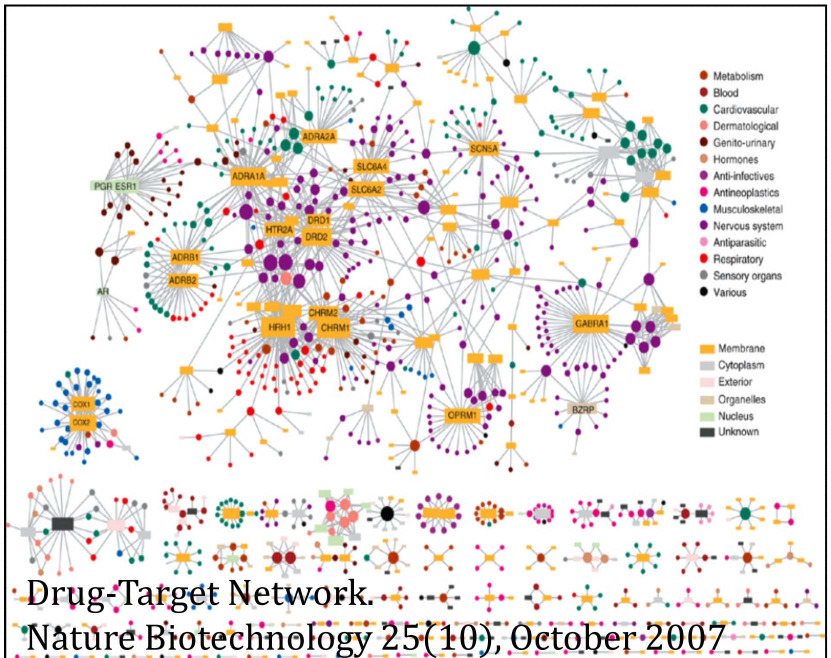
Pictures are taken from Blair D. Sullivan's presentation

Real-life networks and graphs

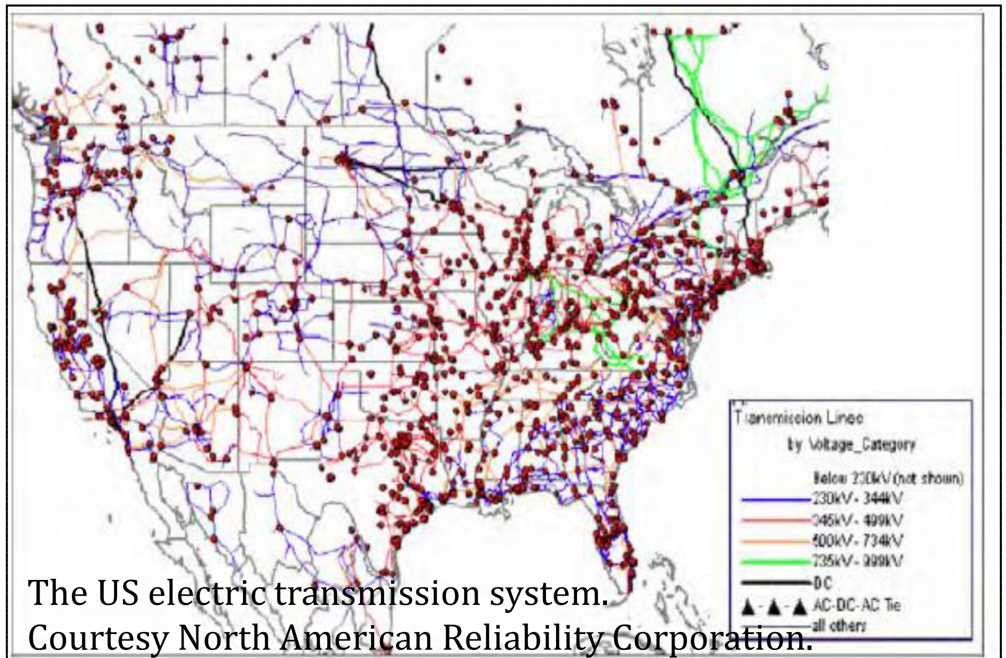
- Large networks are everywhere
- Can we understand their structure and exploit it?

A partial map of the Internet, January 15 2005





Drug-Target Network.
Nature Biotechnology 25(10), October 2007

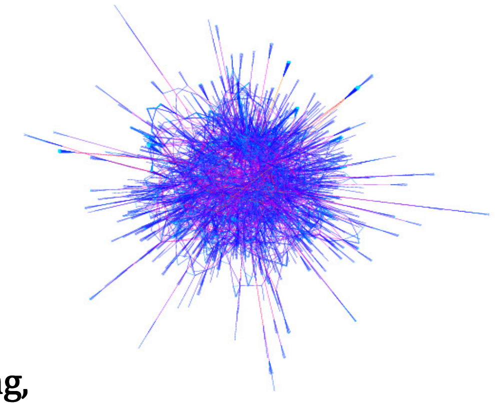


The US electric transmission system.
Courtesy North American Reliability Corporation.

Tree-like networks and graphs

Recent empirical and theoretical work has suggested that many real-life complex networks and graphs arising in Internet applications, in biological and social sciences, in chemistry and physics

have tree-like structures from a metric point of view.



Some prior empirical evidence

- **The Unreasonable Effectiveness of Tree-Based Theory for Networks with Clustering,** Melnik, Hackett, Porter, Mucha, Gleeson. Physical Review E, Vol. 83, No. 3 (2010).
- **Fast computation of empirically tight bounds for the diameter of massive graphs,** Magnien, Latapy, Habib. ACM J. of Experimental Algorithmics 13 (2008)
- "It was noted in recent years that **the Internet structure has a highly connected core and long stretched tendrils, and that most of the routing paths between nodes in the tendrils pass through the core.** Therefore, we suggest to embed the Internet distance metric in a hyperbolic space where routes are bent toward the center" Shavitt, Tankel. 2008. Hyperbolic embedding of internet graph for distance estimation and overlay construction. IEEE/ACM Trans. Netw. 16, 1 (2008).
- **Finding Hierarchy in Directed Online Social Networks**, Gupta, Shankar, Li, Muthukrishnan, Iftode. WWW2011.

Pictures are taken from Blair D. Sullivan's presentation

What do you mean, "tree-like" metrically?

Facebook: Caltech Network

Image credit: Traub, Kelsic, Mucha, Porter

Autonomous Systems

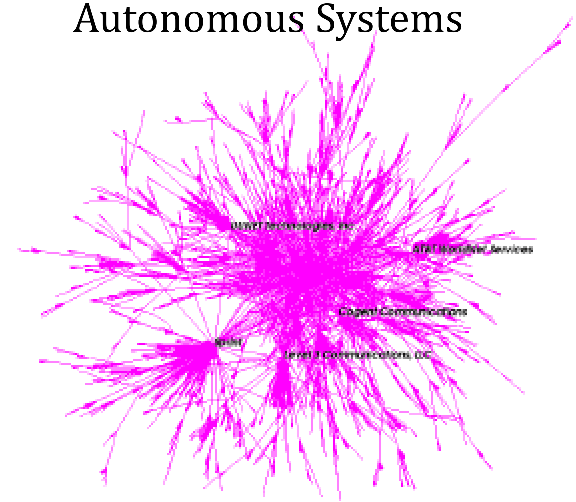


Image credit: Graphics@Illinois

Arxiv GR-QC collaboration

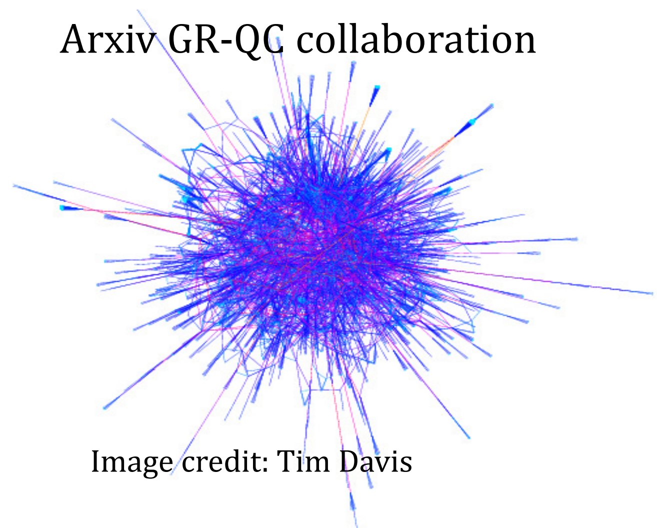


Image credit: Tim Davis

- no consensus has been reached on defining and measuring this tree-like structure

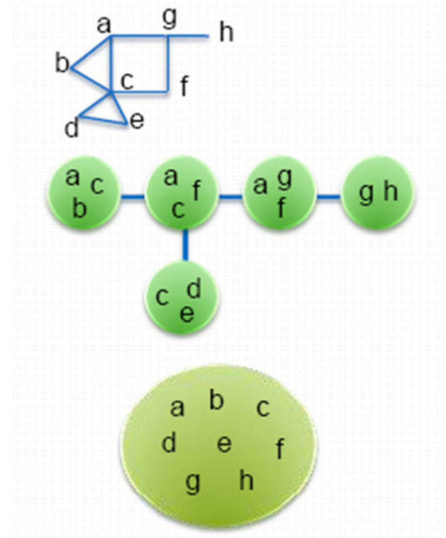
Graph parameters capturing “Tree-like”-ness

We consider here only unweighted and undirected graphs

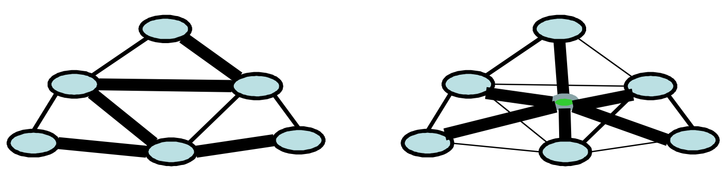
Although, some results extend to weighted graphs as well

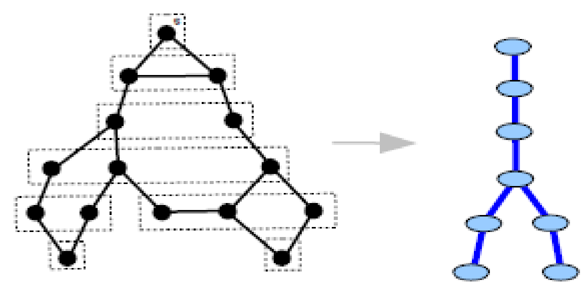
- Tree-width $tw(G)$ (combinatorial)

- Tree-length $tl(G)$
- Tree-breadth $tb(G)$
- Tree-stretch $ts(G)$
- Tree-distortion $td(G)$ (metric)
- Hyperbolicity $hb(G)$
- Cluster-diameter $\Delta_s(G)$ of a layering partition
- Cluster-radius $\rho_s(G)$ of a layering partition



All measuring tree-likeness - the smaller parameter, the closer graph to a tree





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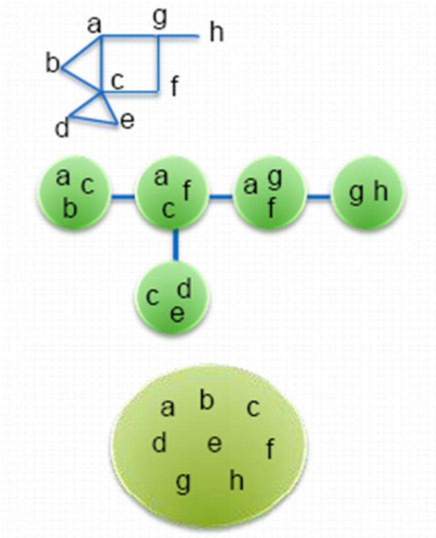
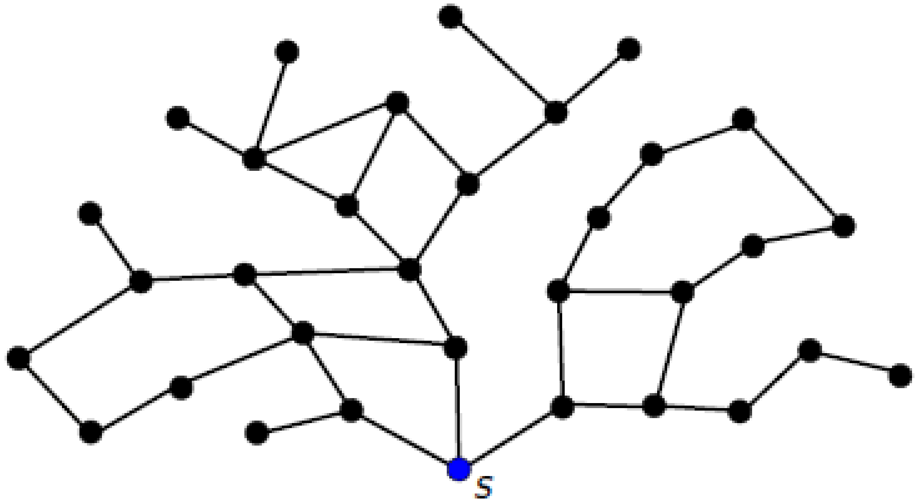


This talk:

- Discussion of these parameters
- Relations between them; their approximations
- Resulting approximation algorithms for optimization problems

Layering partition of a graph

[Brandstädt, Chepoi, Dragan: *J. Algorithms* (1999)]
[Chepoi, Dragan: *Eur. J. Combinatorics* (2000)]

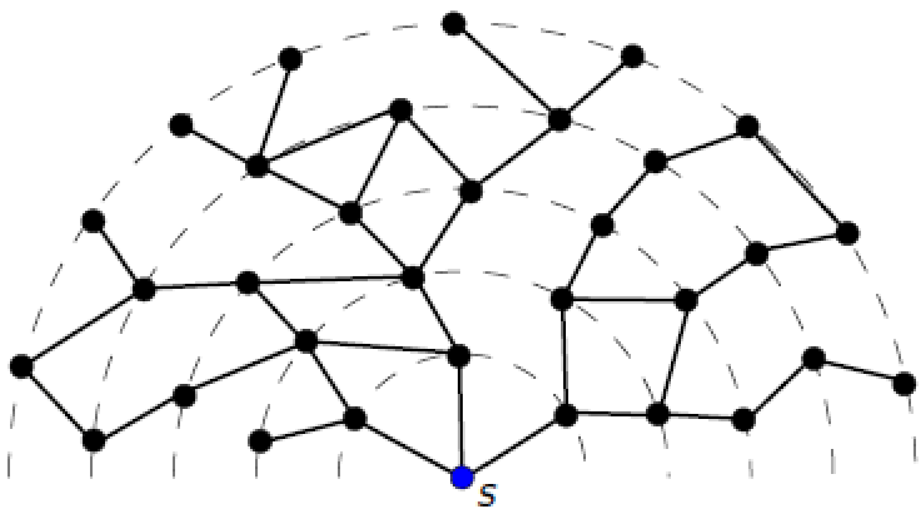


A layering of G is the partition of V into the concentric spheres

$$L^i = \{u \in V : d(s,u) = i\}, i = 0, 1, 2, \ldots .$$

A layering partition of G is a partition of each L^i into clusters $L^i_1, \ldots, L^i_{p_i}$: $u, v \in L^i$ belong to the same cluster L^i_j iff they can be connected by a path outside the ball $B_{i-1}(s)$ of radius $i-1$ centered at s.

Layering partition of a graph



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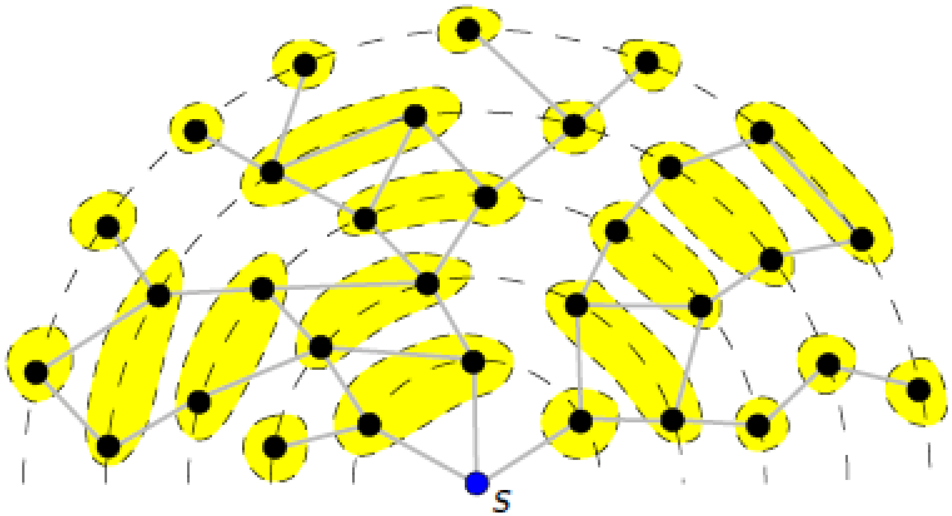

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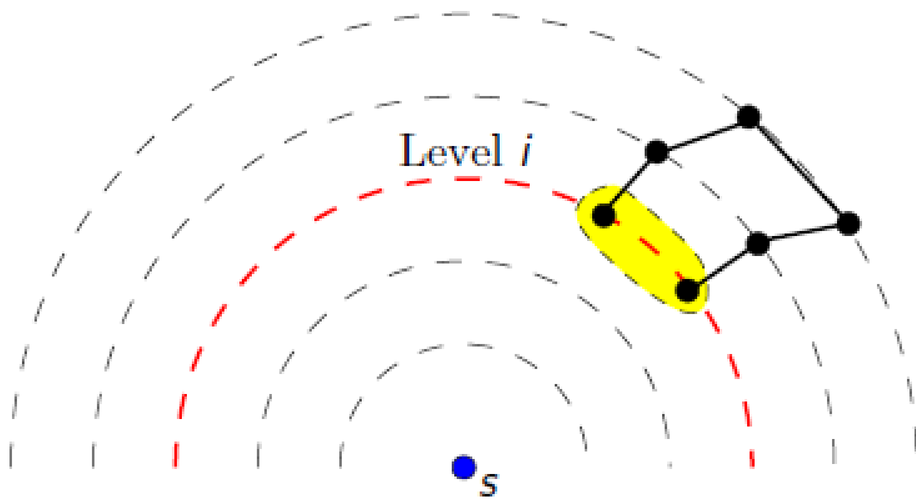

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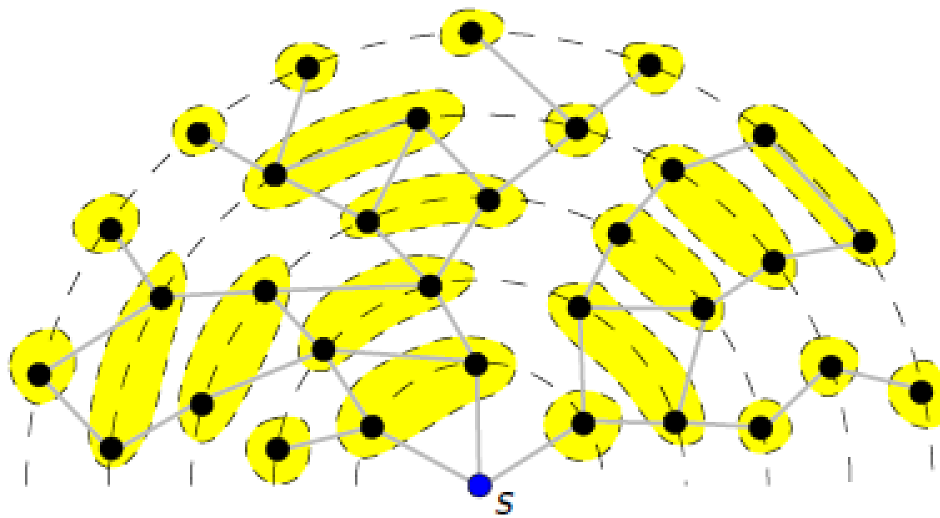
Layering partition of a graph

Can be constructed in *O(|E|)* time [Chepoi, Dragan: *Eur. J. Combinatorics* (2000)]



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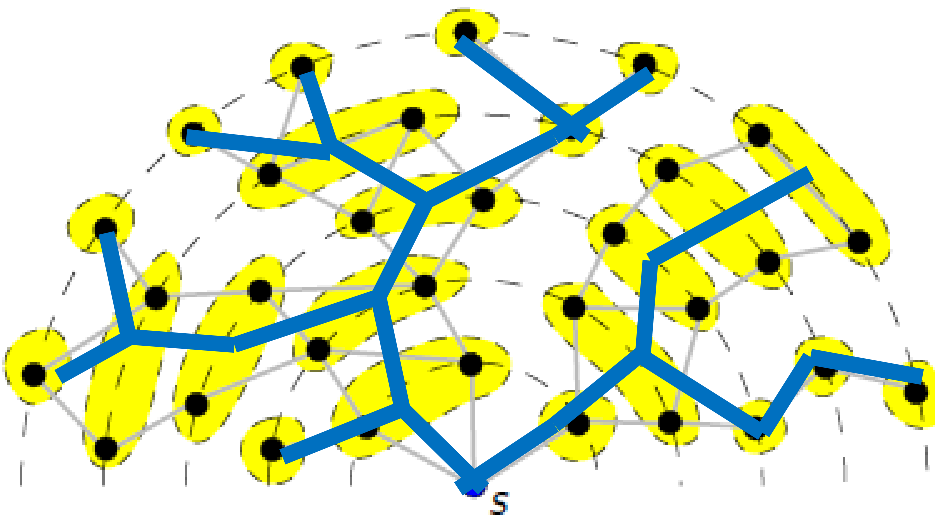
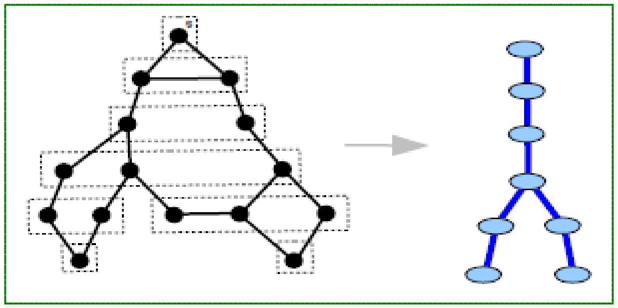
Γ-Tree of a layering partition

Can be constructed in *O(|E|)* time

[Chepoi, Dragan: *Eur. J. Combinatorics* (2000)]

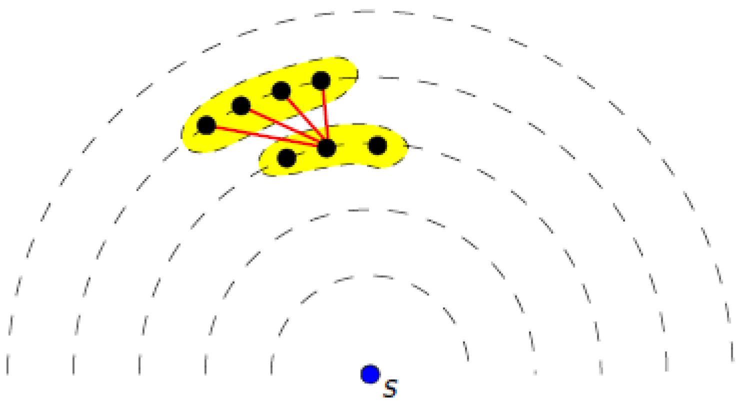


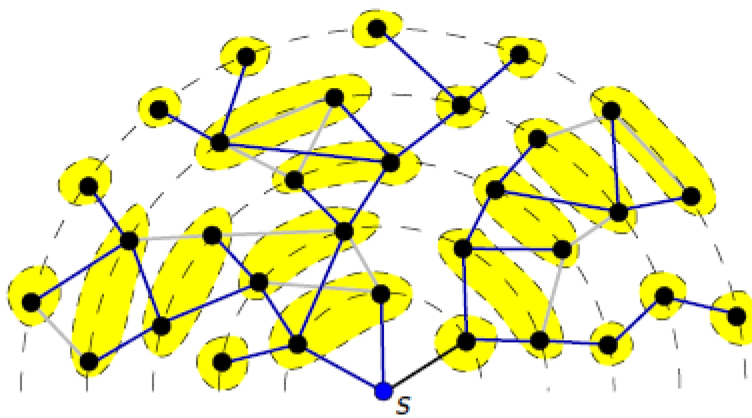


Distance approximating trees

Can be constructed in *O(|E|)* time

[Chepoi, Dragan: *Eur. J. Combinatorics* (2000)]





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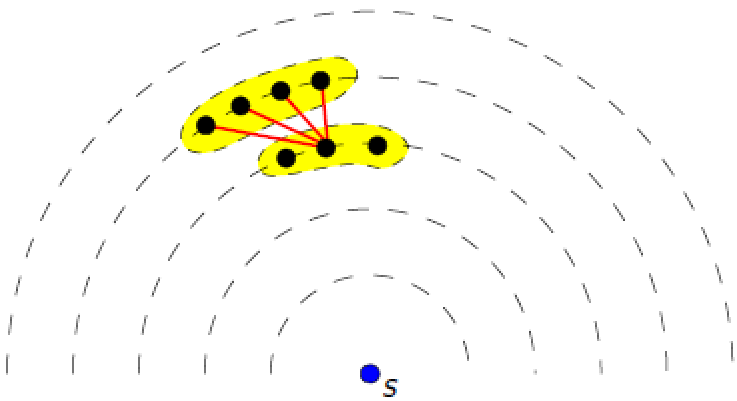
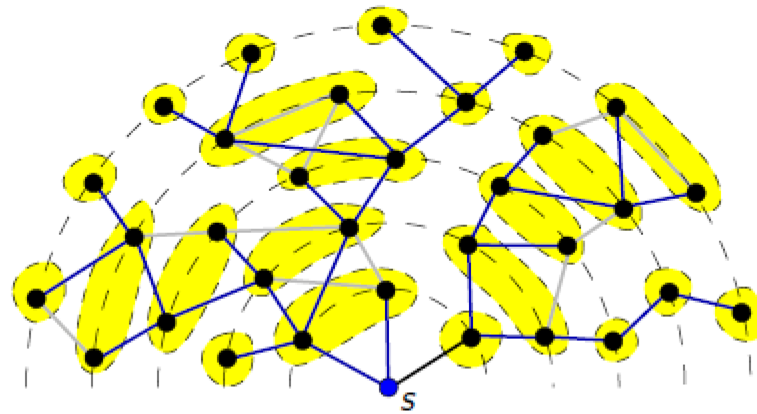
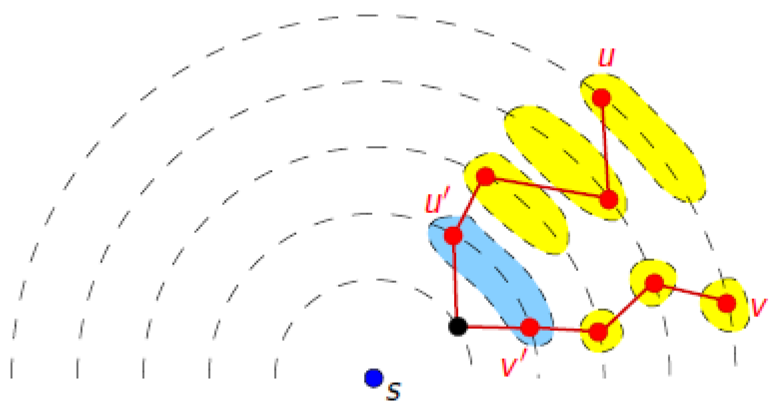






$$\forall\, u, v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + d_G(u',v')$$

Distance approximating trees

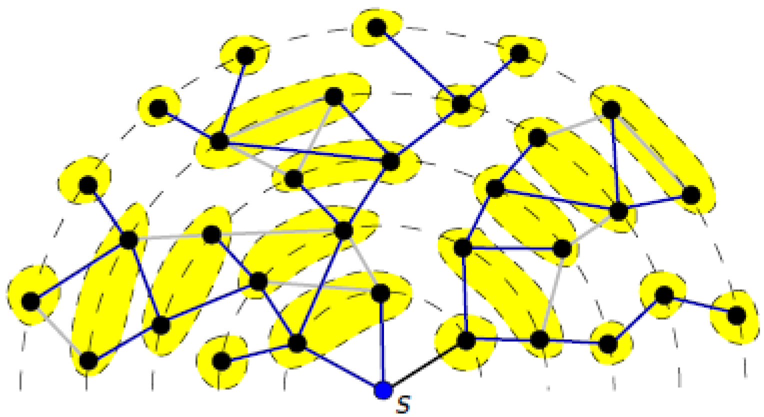
- Cluster-diameter $\boldsymbol{\Delta_s(G)}$ of a layering partition

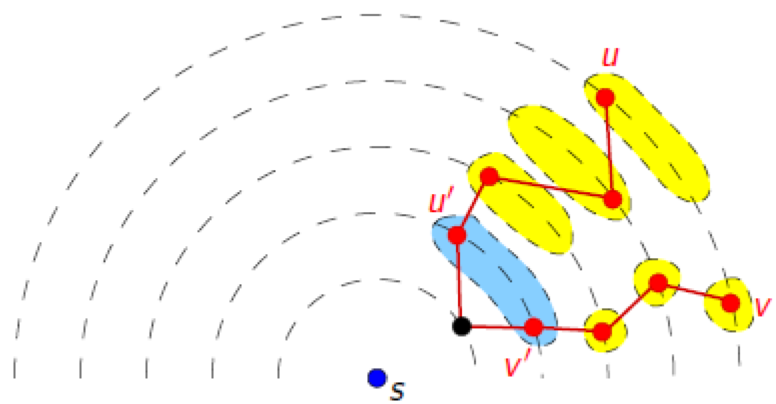
$$\boldsymbol{\Delta_s(G)} = \max\{d_G(u,v): u,v \text{ are in the same cluster}\}$$

- Cluster-radius $\boldsymbol{\rho_s(G)}$ of a layering partition

$$\boldsymbol{\rho_s(G)} = \min\{r: \forall\ cluster\ C_i\ \exists\ v_i \text{ with } C_i \subseteq B_r(v_i)\}$$

Parameters $\boldsymbol{\Delta_s(G)}, \boldsymbol{\rho_s(G)}$ can be computed in *O(n m)* time for any graph





$\Delta_s(G)$

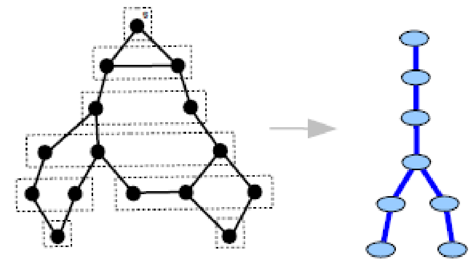
$$\forall\ G, s, \forall\ u,v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + d_G(u',v')$$

- $\forall\ G,s,\ \rho_s(G) \le \Delta_s(G) \le 2\rho_s(G)$ $\quad$ *as* $\forall\ S \subseteq V(G),\ rad_G(S) \le diam_G(S) \le 2rad_G(S)$

Particular graph classes

$\forall\, G, s, \exists T, \forall\, u, v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + \boldsymbol{\Delta_s(G)}$

the smaller parameter $\boldsymbol{\Delta_s(G)}$, the closer graph to a tree metric



- Chordal graphs: $\Delta_s(G) \le 3, \rho_s(G) \le 2\ \ (\forall\, G, s)$

  [Brandstädt, Chepoi, Dragan: *J. Algorithms* (1999)]

  $\forall\, u, v\ \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + 2$

  The length of largest induced cycles is 3

- k-Chordal graphs: $\Delta_s(G) \le {}^{k}/_{2}+2\ \ (\forall\, G, s)$

  [Chepoi, Dragan: *Eur. J. Combinatorics* (2000)]

  $\forall\, u, v\ \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + {}^{k}/_{2} + 2$

  The length of largest induced cycles is k

- More graph classes to come…

http://www.cs.kent.edu/~mabuata/graph_embed/unweighted_graph_tree_metric/index.htm

Real-life graphs / networks

$\Delta_s(G)$

$$\forall\, G, s, \exists T,\ \forall\, u, v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + d_G(u',v')$$

By Muad Abu-Ata, PhD student at Kent State University

Data set	|V|	|E|	diam(G)	# of clusters	$\Delta_s(G)$	Average cluster diam	% of ≤ 2
Yeast	2,224	6,609	11	1,037	6	0.119575699	98%
Homo Sapiens	16,711	115,406	10	6,817	5	0.03432595	99%
PPI	1,458	1,948	19	1,017	8	0.118977384	98%
DBLB-coauthors	317,080	1,049,866	22	99,828	11	0.45350002	98%
Amazon	334,863	925,872	44	72,278	21	0.489056144	95%
Dutch_Elite	3,621	4,311	22	2,934	10	0.070211316	99%
ITDK0304	190,914	607,610	26	89,856	11	0.270377048	97%
Aqualab 12/2007- 09/2008	31,845	143,383	9	16,287	6	0.05826733	99%
Dimes 3/2010	26,424	90,267	8	16,065	4	0.056582633	99%
Routeview	10,515	21,455	10	6,702	6	0.063264697	99%
AS_CAIDA	26,475	53,381	17	17,067	6	0.056424679	99%

Yeast has 981 clusters of diameter 0

18	1
23	2
6	3
5	4
2	5
2	6

PPI has 966 clusters of diameter 0

21	1
14	2
5	3
5	4
1	5
4	6
0	7
1	8

AS_CAIDA has 16459 clusters of diameter 0

361	1
174	2
46	3
21	4
4	5
2	6

Tree-Decomposition

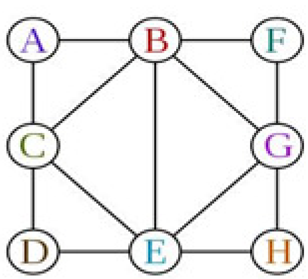
[Robertson, Seymour]

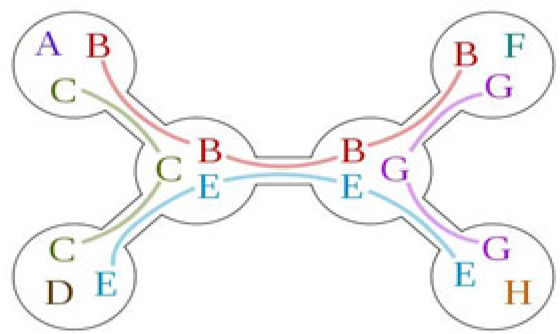
- Tree-decomposition $T(G)$ of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\}, T = (I, F))$ where $\{X_i : i \in I\}$ is a collection of subset of V (bags) and T is a tree whose nodes are the bags satisfying:

1) $\bigcup_{i \in I} X_i = V$

2) $\forall\, uv \in E, \exists\, i \in I \; s.t. \; u, v \in X_i$

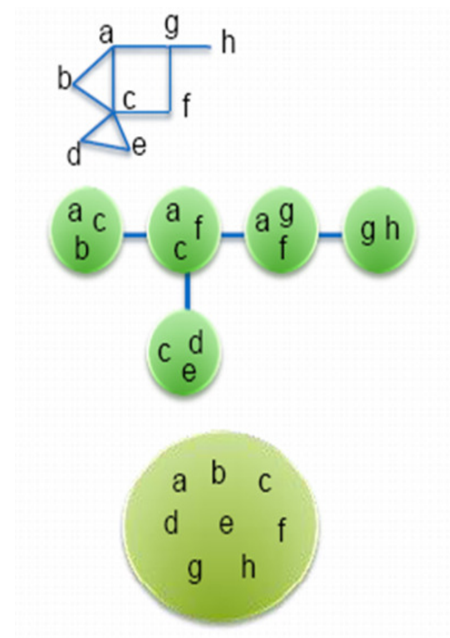
3) $\forall\, v \in V, \textit{the set of bags } \{i \in I, v \in X_i\} \textit{ form a subtree } T_v \textit{ of } T$





Tree-Decomposition and Graph Parameters

- Tree-width $\boldsymbol{tw(G)}$:
  - Width of $T(G)$ is $\max_{i \in I} |X_i| - 1$
  - $\boldsymbol{tw(G)}$: minimum width over all tree-decompositions
- Tree-length $\boldsymbol{tl(G)}$:
  - Length of $T(G)$ is $\max_{i \in I} \max_{u,v \in X_i} d_G(u,v)$
  - $\boldsymbol{tl(G)}$: minimum length over all tree-decompositions
- Tree-breadth $\boldsymbol{tb(G)}$:
  - Breadth is minimum r such that $\forall i \in I, \exists v_i$ with $X_i \subseteq D_r(v_i, G)$
  - $\boldsymbol{tb(G)}$: minimum breadth over all tree-decompositions



Tree-length was introduced in [Dourisboure, Gavoille: *DM* (2007)] and [Dragan,Lomonosov: *DAM* (2007)]

Tree-breadth was introduced in [Dragan,Lomonosov: *DAM* (2007)] and [Dragan, Köhler: *APPROX* (2011)]

↓

(R,D)-acyclic clustering

Tree-Decomposition and Graph Parameters

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- Tree-length $\boldsymbol{tl(G)}$:
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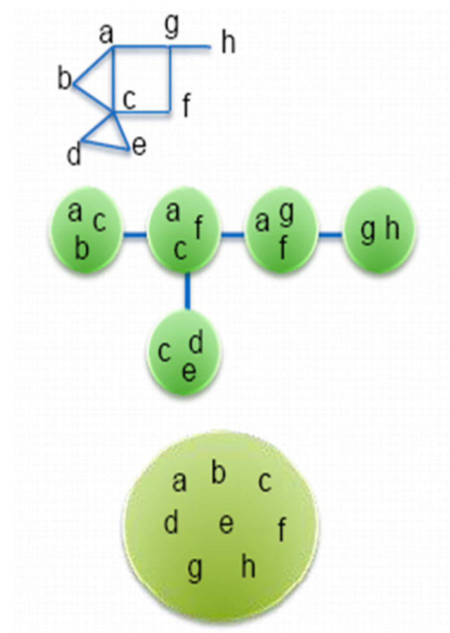

- $\forall\, G,\ tb(G) \leq tl(G) \leq 2tb(G)$ *as* $\forall\, S \subseteq V(G),\ rad_G(S) \leq diam_G(S) \leq 2rad_G(S)$
- $tw(G)$ and $tl(G)$ are not comparable (check cycles and cliques)

$$tw(C_{3k}) = 2, \qquad tl(C_{3k}) = k$$
$$tw(K_n) = n - 1, \qquad tl(K_n) = 1$$

Tree-Decomposition and Graph Parameters

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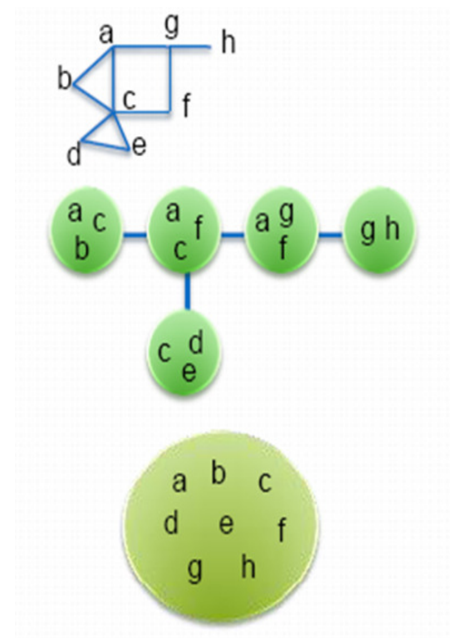

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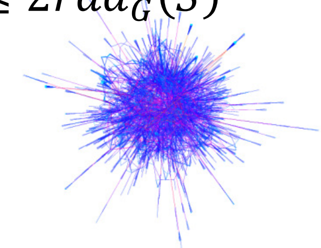


Many real-life networks (e.g., with a highly connected core) have a large tree-width but still exhibit a tree-like structure

Particular graph classes / networks

the smaller parameters $tl(G)$, $tb(G)$, the closer graph to a tree

- Chordal graphs: $tb(G) \leq tl(G) \leq 1$ (via clique tree) $\Delta_s(G) \leq 3$
- Chordal bipartite graphs: $tb(G) \leq 1$ [Dragan,Lomonosov: *DAM* (2007)]
- k-Chordal graphs: $tb(G) \leq tl(G) \leq k/2$ [Dourisboure, Gavoille: *DM* (2007)] $\Delta_s(G) \leq k/2+2$

From Michel Habib's presentation, June 2009

Real Data ? from CAIDA project

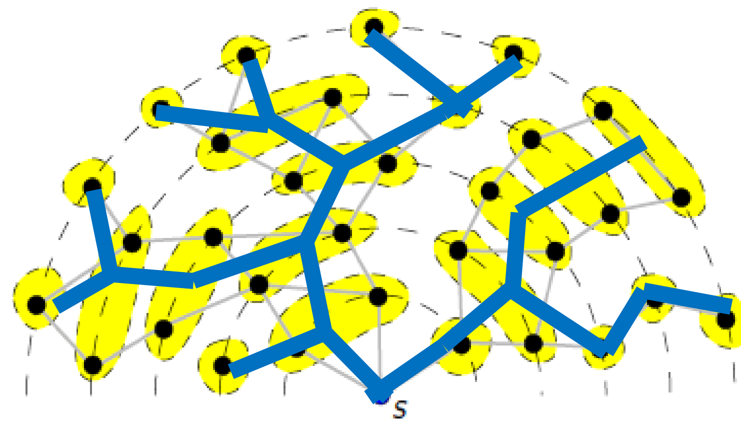
M. Soto, PhD student at Paris Diderot, has computed graph invariants on some real networks

2 graphs with normal graph distance

Internet Topology Data Kit (ITDK) graph of the routing machines
Treedwidth ≥ 234, Treelength≤ 10, Diameter$=19$, δ-hyperbolicity$=3$ (but for 96 % of the computed quadruplets the value is 1)

Autonomus System Internet Topology (AS-level) graph, a smaller graph
Treedwidth ≥ 82, Treelength ≤ 6, Diameter$=10$, δ-hyperbolicity$=2$ (but for 98 % of the computed quadruplets the value is 1)

Relationship between $tl(G), tb(G)$ and $\Delta_s(G), \rho_s(G)$



- Chordal graphs:
  $tb(G) \leq tl(G) \leq 1$ and $\Delta_s(G) \leq 3$
- k-Chordal graphs:
  $tb(G) \leq tl(G) \leq k/2$ and $\Delta_s(G) \leq k/2+2$

General graphs

- $\forall\, G, s,\ tl(G) - 1 \leq \Delta_s(G) \leq 3\, tl(G)$ [Dourisboure, Gavoille: *DM* (2007)]
- $\forall\, G, s,\ \rho_s(G) \leq 2\, tl(G)$ [Dourisboure, Dragan, Gavoille, Yan: *TCS* (2007)]
- $\forall\, G, s,\ tb(G) - 1 \leq \rho_s(G) \leq 3\, tb(G)$ [Dragan, Köhler: *APPROX* (2011)]

- To test if $tl(G) \leq \lambda$ is NP-complete for each $\lambda > 1$ [Lokshtanov: *DAM* (2010)]
- A tree-decomposition of length $\Delta_s(G) + 1 \leq 3\, tl(G) + 1$ can be obtained in linear time from the Γ-Tree of a layering partition. [Dourisboure, Gavoille: *DM* (2007)]

Consequences for bounded tree-length graphs

- For any graph G there is a tree T, constructible in linear time, such that $\forall\, u, v \in V,\; d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + \sout{\Delta_s(G)}$

  ↑ $3\; tl(G)$

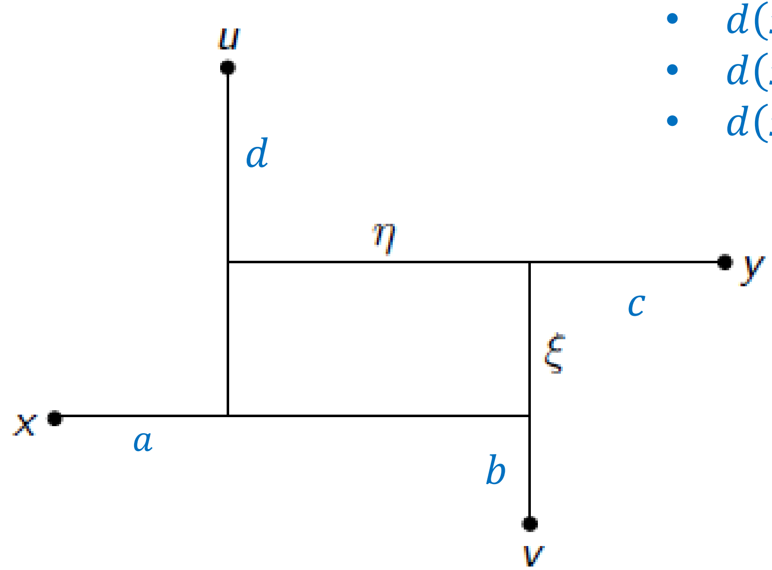
  the smaller parameter $tl(G)$ $(tb(G))$, the closer graph to a tree metric

- More results from [Dourisboure, Dragan, Gavoille, Yan: *TCS* (2007)] that employ inequalities $\Delta_s(G) \le 3\; tl(G)$ and $\rho_s(G) \le 2\; tl(G)$
  - Every n -vertex graph G has an additive $(4\; tl(G))$-spanner with at most $(2\; tl(G) + 1)(n - 1)$ edges constructible in polynomial time
  - Every n -vertex graph G has an additive $(2\; tl(G))$-spanner with at most $(tl(G) + logn)(n - 1)$ edges constructible in polynomial time
- More results from [Dragan, Köhler: *APPROX* (2011)] after few more slides

Hyperbolicity

δ-Hyperbolicity (M. Gromov, 1987)

for any four points u, v, w, x of a metric space (X, d), the two larger of the distance sums $d(u,v) + d(w,x)$, $d(u,w) + d(v,x)$, $d(u,x) + d(v,w)$ differ by at most 2δ.



- $d(x,y) + d(u,v) = a + b + d + c + 2\eta + 2\xi$
- $d(x,v) + d(u,y) = a + b + d + c + 2\eta$
- $d(x,u) + d(y,v) = a + b + d + c + 2\xi$

$$\min\{\eta, \xi\} \leq \delta$$

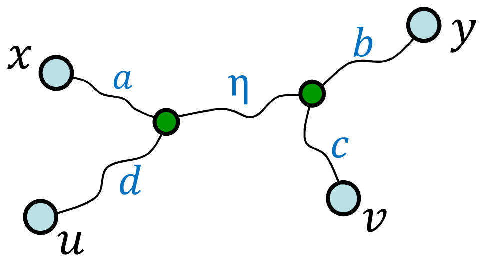
δ-Hyperbolicity measures the local deviation of a metric from a tree metric: *a metric is a tree metric iff it is 0-hyperbolic.*

Hyperbolicity of a graph

- The hyperbolicity ***hb*(*G*)** of a graph G is the smallest number δ such that $(V(G), d_G)$ is δ-hyperbolic.

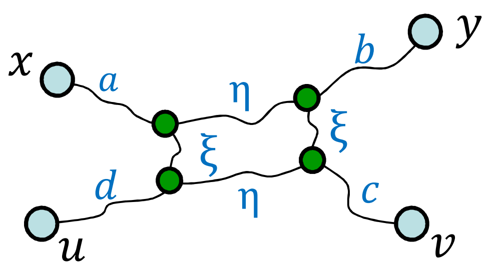
Subspace formed by four points

in tree metric



- $d_T(x,v) + d_T(u,y) = a + b + c + d + 2\eta$
- $d_T(x,y) + d_T(u,v) = a + b + c + d + 2\eta$
- $d_T(x,u) + d_T(y,v) = a + b + c + d$

in graph metric

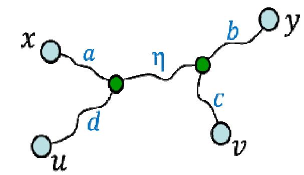


- $d_G(x,v) + d_G(u,y) = a + b + d + c + 2\eta + 2\xi$
- $d_G(x,y) + d_G(u,v) = a + b + d + c + 2\eta$
- $d_G(x,u) + d_G(y,v) = a + b + d + c + 2\xi$

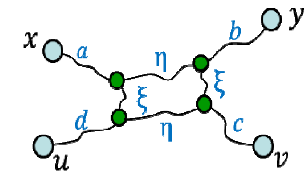
$\min\{\eta, \xi\} \leq \delta$

the smaller parameters δ, the closer graph to a tree metrically

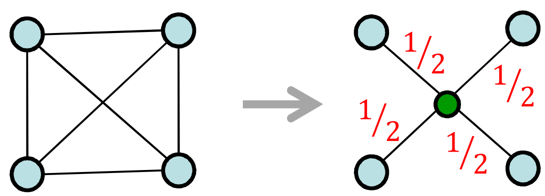
Particular graph classes



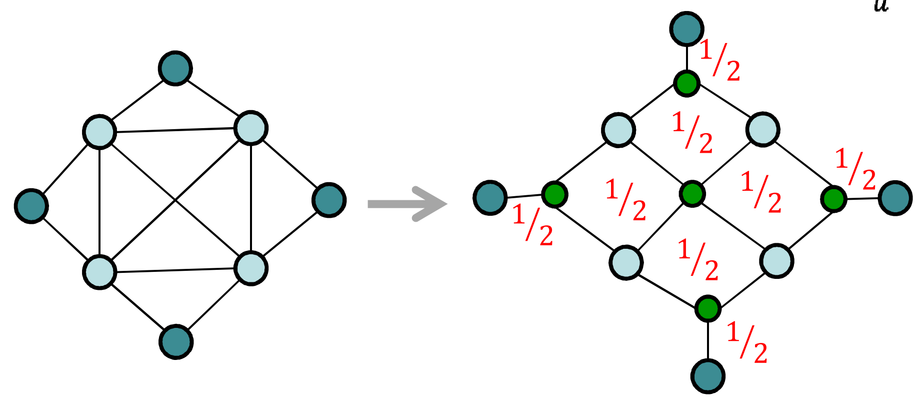
the smaller parameters $\boldsymbol{hb(G)}$, the closer graph to a tree metrically



- $\boldsymbol{hb(G)}$ can be computed naively in $O(n^4)$ time
- $\boldsymbol{hb(G)}$ is a half integer $({}^1/_2, 1, {}^3/_2, 2, \ldots)$ for unweighted graphs



$hb(K_n) = 0$ (is a tree metrically)



$hb(S_4) = 1$ (is not a tree metrically)

- $hb(G) = 0$ iff G is a block graph (metrically a tree)
- Chordal graphs: $hb(G) \leq 1$ $\qquad$ $\Delta_s(G) \leq 3$, $tb(G) \leq tl(G) \leq 1$
  [Brinkmann, Koolen, Moulton: *Annals of Combinatorics* (2001)]
- k-Chordal graphs (k>3): $hb(G) \leq {}^k/_4$ $\qquad$ $\Delta_s(G) \leq {}^k/_2+2$, $tb(G) \leq tl(G) \leq {}^k/_2$
  [Wu, Zhang: *E.J. on Combinatorics* (2011)]
- More graph classes to come…

http://www.cs.kent.edu/~mabuata/graph_embed/unweighted_graph_tree_metric/index.htm

Real-life graphs / networks

By Muad Abu-Ata, PhD student at Kent State University

Data set	|V|	|E|	diam(G)	$\Delta_s(G)$	hyperbolicity	% of ≤ 1
Yeast	2,224	6,609	11	6	2.5	99%
Homo Sapiens	16,711	115,406	10	5	-	
PPI	1,458	1,948	19	8	3.5	98%
DBLB-coauthors	317,080	1,049,866	22	11	-	
Amazon	334,863	925,872	44	21	-	
Dutch_Elite	3,621	4,311	22	10	4	96%
ITDK0304	190,914	607,610	26	11	-	
Aqualab 12/2007- 09/2008	31,845	143,383	9	6	-	
Dimes 3/2010	26,424	90,267	8	4	-	
Routeview	10,515	21,455	10	6	-	
AS_CAIDA	26,475	53,381	17	6	2.5	97%

PPI

hyperbolicity	relative frequency
0	0.4831
0.5	0.3634
1	0.1336
1.5	0.0179
2	0.0019
2.5	3.55E-05
3	1.65E-06
3.5	3.79E-09

Montgolfier, Soto, Viennot: *NCA* (2011)

Graph	Avg deg	Max deg	β	Hyp.	tw
CAIDA AS	6.31	1,815	2.19	2.0	$\in [82, 473]$
Erdös-Rényi	6.34	18	-	2.5	≥ 135
Barabási	6.00	283	2.92	2.0	≥ 130
AS degree dist.	6.31	1,815	2.19	1.5	≥ 110
Power Law	8.97	1,507	2.19	1.5	≥ 150

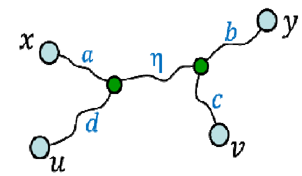
M. Soto (2009)

Internet Topology Data Kit (ITDK) graph of the routing machines
Treedwidth ≥ 234, Treelength≤ 10, Diameter$=19$,
δ-hyperbolicity$=3$ (but for 96 % of the computed quadruplets the value is 1)

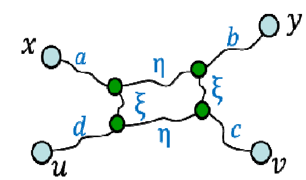
Autonomus System Internet Topology (AS-level) graph, a smaller graph
Treedwidth ≥ 82, Treelength ≤ 6, Diameter$=10$, δ-hyperbolicity$=2$ (but for 98 % of the computed quadruplets the value is 1)

Relationship between $tl(G), \Delta_s(G)$ and $hb(G)$

- Chordal graphs:
  $hb(G) \leq 1$ and $tb(G) \leq tl(G) \leq 1$ and $\Delta_s(G) \leq 3$



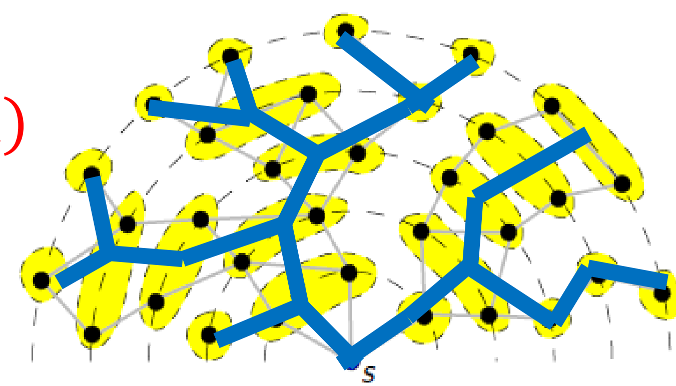
- k-Chordal graphs:
  $hb(G) \leq k/4$ and $tb(G) \leq tl(G) \leq k/2$ and $\Delta_s(G) \leq k/2+2$



General graphs

[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]
[Chepoi, Dragan, Estellon, Habib, Vaxes, Xiang: *Algorithmica* (2012)]

- $\forall\, G,\ hb(G) \leq tl(G) \leq O(hb(G) \log n)$
- $\forall\, G, s,\ hb(G) \leq \Delta_s(G) \leq O(hb(G) \log n)$



Recall:

- $\forall\, G, s,\ tl(G) - 1 \leq \Delta_s(G) \leq 3\, tl(G)$

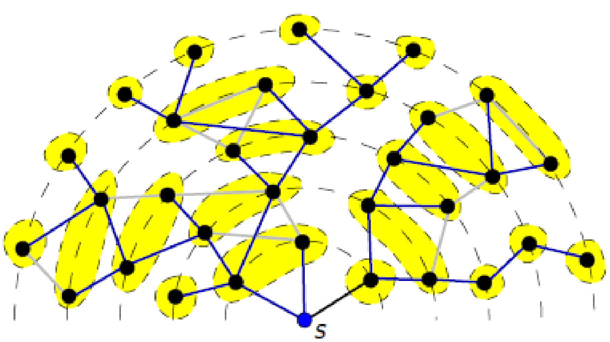
Distance approximating trees

[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]
[Chepoi, Dragan, Estellon, Habib, Vaxes, Xiang: *Algorithmica* (2012)]

- $\forall\, G, s,\ hb(G) \le \Delta_s(G) \le 4{+}12\ hb(G) + 8\ hb(G)\ log_2 \text{n}$

- For any graph G there is a tree T, constructible in linear time, such that $\forall\, u, v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + \sout{\Delta_s(G)}$ ← $O(hb(G) \log n)$

  equivalently,
  $\forall\, u, v \in V,\ d_G(u,v) - O(hb(G) \log n) \le d_T(u,v) \le d_G(u,v) + 2$
  (notice, T is unweighted and without Steiner points)



Distance approximating trees

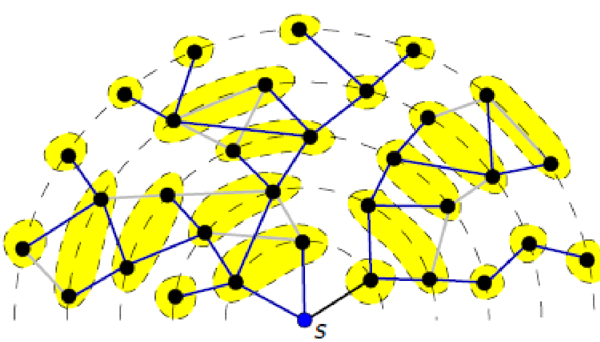
[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]
[Chepoi, Dragan, Estellon, Habib, Vaxes, Xiang: *Algorithmica* (2012)]

- $\forall\, G, s,\ hb(G) \leq \Delta_s(G) \leq 4 + 12\, hb(G) + 8\, hb(G)\, log_2 n$

- For any graph G there is a tree T, constructible in linear time, such that
  $\forall\, u, v \in V,\ d_T(u,v) - 2 \leq d_G(u,v) \leq d_T(u,v) +$ ~~$\Delta_s(G)$~~ ← $O(hb(G)\log n)$

  equivalently,
  $\forall\, u, v \in V,\ d_G(u,v) - O(hb(G)\log n) \leq d_T(u,v) \leq d_G(u,v) + 2$
  (notice, T is unweighted and without Steiner points)

Can be made non-expanding like in Gromov's case by allowing Steiner points and edge weights $\{0,1\}$ in T.



Theorem (Gromov, 1987)

For any δ-hyperbolic metric space (X,d) on n points and any fixed basepoint $s \in X$, there a tree T and a map $\varphi : X \to T$ such that

- $d_T(\varphi(s), \varphi(x)) = d(s,x)$ pour tout $x \in X$,
- $d(x,y) - 2\delta \log_2 n \leq d_T(\varphi(x), \varphi(y)) \leq d(x,y)$ for all $x, y \in X$.

The tree T can be constructed using $O(n^2)$ distance computations.

Easy to show:

- If for a graph G there is a tree T with $d_G(u,v) \leq d_T(u,v) \leq d_G(u,v) + r \quad \forall\, u,v \in V$ then G is r-hyperbolic
- If for a graph G there is a tree T with $d_T(u,v) \leq d_G(u,v) \leq d_T(u,v) + r \quad \forall\, u,v \in V$ then G is r-hyperbolic

More algorithmic results

Known algorithmic results about δ-hyperbolicity

The internet topology embeds with better accuracy into low-dimensional hyperbolic space than into Euclidian space of comparable dimension. PTAS for the Traveling Salesman Problem, efficient nearest neighbor search, distance labeling schemes and routing schemes, and approximation algorithms for covering and packing by balls.

Sparse additive spanners

[Chepoi, Dragan, Estellon, Habib, Vaxes, Xiang: *Algorithmica* (2012)]

Our results

(i) We show that approximating the diameter $diam(S)$, the radius $rad(S)$, and the center $C(S)$ of a subset S in a δ-hyperbolic geodesic space or graph with an $O(\delta)$-additive error can be done in the same way as for trees. This leads to very simple algorithms for fast approximating (and in some cases, for computing in linear time) of $diam(S)$, $rad(S)$, and $C(S)$.

(ii) We present a simple linear-time construction of distance approximating trees of δ-hyperbolic graphs with n vertices having the same additive distortion $O(\delta \log n)$ as Gromov's construction.

(iii) We establish that several classes of geometrically defined graphs have bounded hyperbolicity.

[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]

Recall:

- **The Unreasonable Effectiveness of Tree-Based Theory for Networks with Clustering,** Melnik, Hackett, Porter, Mucha, Gleeson. Physical Review E, Vol. 83, No. 3 (2010).
- **Fast computation of empirically tight bounds for the diameter of massive graphs,** Magnien, Latapy, Habib. ACM J. of Experimental Algorithmics 13 (2008)

Diameter, Radius, Center

Diameter

Let S be a finite set of points of a metric space (X, d).
Diameter: $diam(S) = \max\{d(u, v) : u, v \in S\}$.
Diametral pair: any pair of points $x, y \in S$ such that $d(x, y) = diam(S)$.

Furthest neighbors

The set $F(x)$ of furthest neighbors of a point $x \in X$ in S consists of all points of S at the maximum distance from x. The eccentricity $ecc(x)$ of $x \in X$ is the distance from x to any point of $F(x)$.

Center and radius

The center $C(S)$ of S is the set of points of X with minimum eccentricity. The radius $rad(S)$ of S is the eccentricity of central points, i.e., $rad(S)$ is the smallest radius of a ball of (X, d) enclosing all points of S (a ball $B(c, r) = \{x \in X : d(c, x) \leq r\}$ consists of all points $x \in X$ at distance at most r to c).

Fast computation of diameter, radius, and center

is a basic algorithmic problem in computational geometry and graph theory with applications in operation research, data clustering, location theory, and analysis of complex networks.

Tree-Folklore

C. Jordan (1869)

C. Jordan established that the center of a tree is a single point (and of a graphic tree is a vertex or an edge).

Diameter

The diameter $diam(S)$ of a set S in a tree T can be found in linear time by running the following folklore algorithm:

Algorithm 2FP

1. Pick an arbitrary point u of T
2. Find a furthest neighbor u of v in S
3. Find a furthest neighbor w of v in S
4. Return $d(v, w)$ as $diam(S)$ and v, w as a diametral pair of S

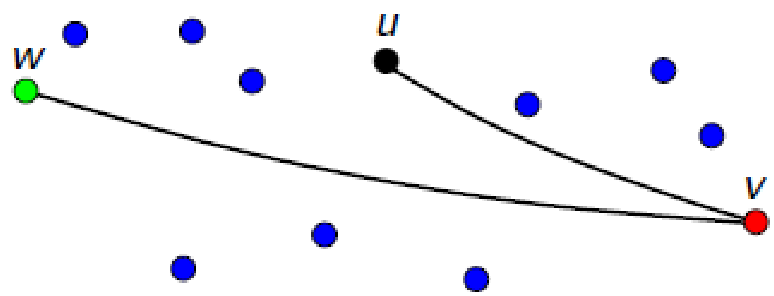
Center

To find the center of S it suffices to add the following step:

5. Return the midpoint c of the unique (v, w)-path of T

Diameter and Radius

[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]



Proposition 1

For a finite subset S of a δ-hyperbolic space (X, d) and any $u \in X$, if $v \in F(u)$ and $w \in F(v)$, then $d(v, w) \geq diam(S) - 2\delta$. The pair $\{v, w\}$ can be computed using $O(|S|)$ distance calculations.

Proposition 2

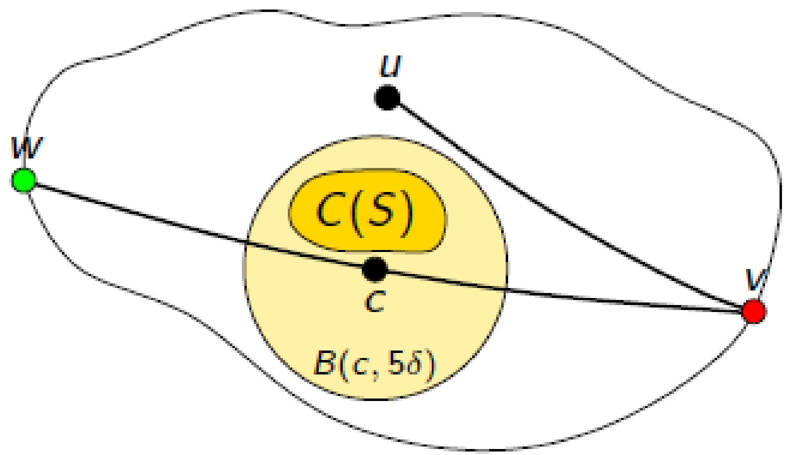
For a finite set S of a δ-hyperbolic geodesic space, $2rad(S) \geq diam(S) \geq 2rad(S) - 4\delta$.

Corollary 1

For a finite set S of a δ-hyperbolic geodesic space, $rad(S) \leq d(v, w)/2 + 3\delta$.

Center

[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]



Proposition 3

For a finite set S of a δ-hyperbolic geodesic space, $diam(C(S)) \leq 4\delta$.

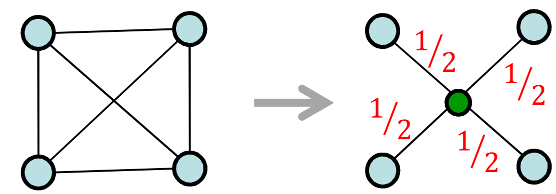
Let c be the middle of a geodesic $[v, w]$ between v and w.

Proposition 4

The inequality $ecc(c) \leq rad(S) + 5\delta$ holds for all δ-hyperbolic geodesic spaces and graphs. Moreover $C(S) \subseteq B(c, 5\delta)$ ($C(G) \subseteq B(c, 5\delta + 1)$ for δ-hyperbolic graphs).

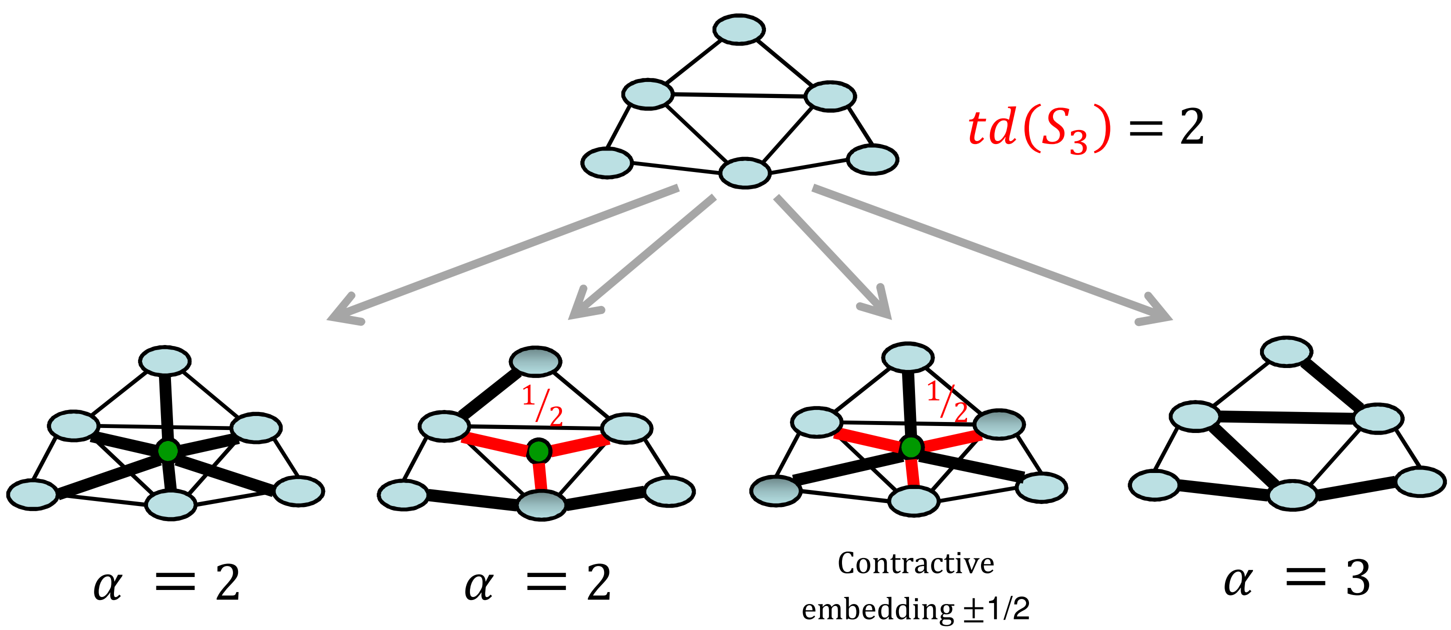
Tree-distortion $td(G)$

- Tree-distortion $td(G)$ of a graph $G = (V, E)$ is the smallest number α such that G admits a (not necessarily spanning, possibly weighted and having Steiter points) tree $T = (V \cup S,\ U)$ with
  $$\forall\, u, v \in V,\quad d_G(u,v) \le d_T(u,v) \le \alpha\, d_G(u,v).$$
- the smaller α, the closer graph to a tree



$td(K_n) = 1$ (is a tree metrically)

- The problem is known also as
  "non-contractive minimum distortion embedding into trees"
  (most popular among different embeddings into trees)



Variations of earlier results

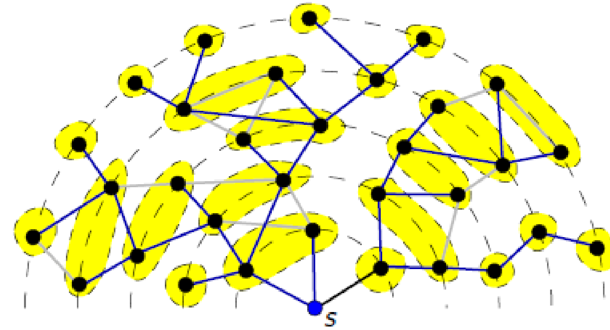
(to reach the form $\forall\, u,v \in V,\ \ d_G(u,v) \le d_T(u,v) \le \alpha\, d_G(u,v)$)

[Chepoi, Dragan, Newman, Rabinovich, Vaxes: *Discr.&Comput.Geom.* (2012)]

We had:

- For any graph G there is a tree T, constructible in linear time, such that $\forall\, u,v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + \Delta_s(G)$

  (notice, T is unweighted and without Steiner points)



- Assigning uniformly weight $\Delta_s(G)+1$ to all edges of T we get T_w with $\forall\, u,v \in V,\ \ d_G(u,v) \le d_{T_w}(u,v) \le (\Delta_s(G)+1)(d_G(u,v)+2)$

  (non-contractive)

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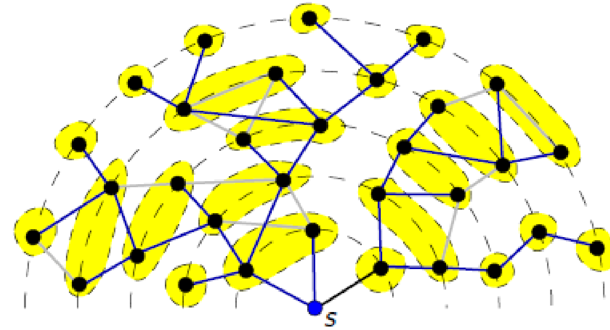

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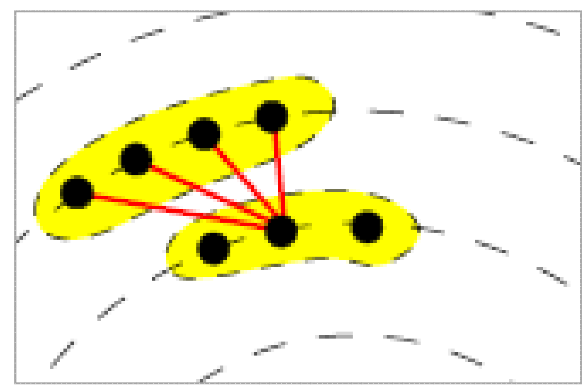
  (non-contractive)



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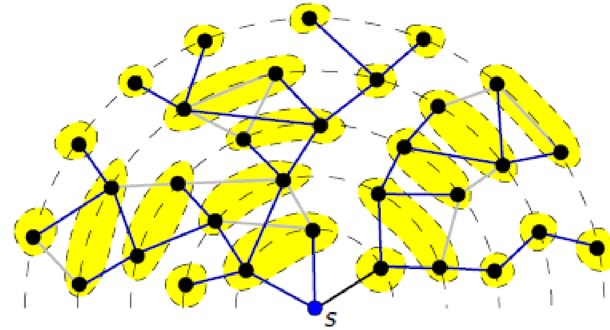

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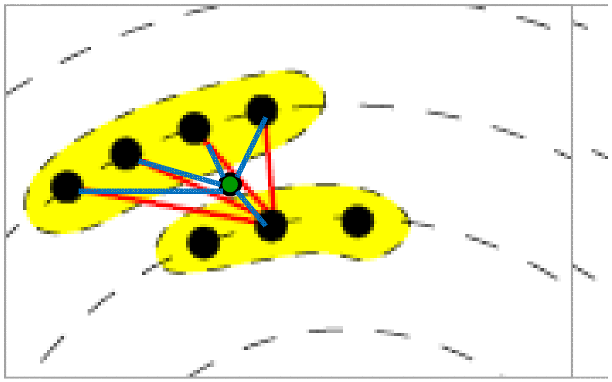
  (non-contractive)



- Introducing Steiner points and assigning uniformly weight $(\Delta_s(G)+1)/2$ to all edges of T we get T'_w with $\forall\, u, v \in V,\ d_G(u,v) \le d_{T'_w}(u,v) \le (\Delta_s(G)+1)(d_G(u,v)+1)$

Relations between $td(G)$ and $\Delta_s(G)$

- For any graph G there is a tree T, constructible in linear time, such that $\forall\, u,v \in V,\ d_T(u,v) - 2 \leq d_G(u,v) \leq d_T(u,v) + \Delta_s(G)$
- Assigning uniformly weight $\Delta_s(G) + 1$ to all edges of T we get T_w with $\forall\, u,v \in V,\ \ d_G(u,v) \leq d_{T_w}(u,v) \leq (\Delta_s(G) + 1)(d_G(u,v) + 2)$
- Introducing Steiner points and assigning uniformly weight $(\Delta_s(G) + 1)/2$ to all edges of T we get T'_w with $\forall\, u,v \in V,\ \ d_G(u,v) \leq d_{T'_w}(u,v) \leq (\Delta_s(G) + 1)(d_G(u,v) + 1)$

[Chepoi, Dragan, Newman, Rabinovich, Vaxes: *Discr.&Comput.Geom*. (2012)]

- $\forall\, G, s,\ \Delta_s(G)/3 \leq td(G) \leq 2\,\Delta_s(G) + 2$

Hence:

- For any graph G there is a tree T, constructible in linear time, such that $\forall\, u,v \in V,\ d_T(u,v) - 2 \leq d_G(u,v) \leq d_T(u,v) + 3td(G)$
- Assigning uniformly weight $\Delta_s(G) + 1$ to all edges of T we get T_w with $\forall\, u,v \in V,\ \ d_G(u,v) \leq d_{T_w}(u,v) \leq (3td(G) + 1)(d_G(u,v) + 2)$
- Introducing Steiner points and assigning uniformly weight $(\Delta_s(G) + 1)/2$ to all edges of T we get T'_w with $\forall\, u,v \in V,\ \ d_G(u,v) \leq d_{T'_w}(u,v) \leq (3td(G) + 1)(d_G(u,v) + 1)$

Consequences for minimum distortion embedding into trees

[Chepoi, Dragan, Newman, Rabinovich, Vaxes: *Discr.&Comput.Geom.* (2012)]

- $\forall\, G, s,\ \Delta_s(G)/3 \le td(G) \le 2\,\Delta_s(G) + 2$

If G admits a tree H with $\forall\, u, v \in V,\ \ d_G(u,v) \le d_H(u,v) \le \alpha\, d_G(u,v)$ then:

H may be weighted and may have Steiner points

- there is a tree T, constructible in linear time, such that $\forall\, u, v \in V,\ d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + 3\alpha$

(multiplicative distortion turned into an additive distortion; T is unweighted and no Steiner points)

- assigning uniformly weight $\Delta_s(G) + 1$ to all edges of T we get T_w with $\forall\, u, v \in V,\ \ d_G(u,v) \le d_{T_w}(u,v) \le (3\alpha + 1)(d_G(u,v) + 2) \le 12\alpha\, d_G(u,v)$

(a 12-approximation algorithm for minimum distortion embedding into trees)

- introducing Steiner points and assigning uniformly weight $(\Delta_s(G) + 1)/2$ to all edges of T we get T'_w with $\forall\, u, v \in V,\ d_G(u,v) \le d_{T'_w}(u,v) \le (3\alpha + 1)(d_G(u,v) + 1) \le 8\alpha\, d_G(u,v)$

(an 8-approximation algorithm for minimum distortion embedding into trees)

Previous approximation bounds and final 6-approximation

- The problem of minimum distortion embedding into trees is NP-hard
- 100 - approximation [Badoiu, Indyk, Sidiropoulos: *SODA* (2007)]
- 27 - approximation [Badoiu, Demaine, Hajiaghayi, Sidiropoulos, Zadimoghaddam: *APPROX* (2008)]

[Chepoi, Dragan, Newman, Rabinovich, Vaxes: *APPROX* (2010) and *Discr.&Comput.Geom.* (2012)]

- $\forall\, G, s, \Delta_s(G)/3 \leq td(G) \leq 2\,\Delta_s(G) + 2$

- 12 - approximation by a weighted tree without Steiner points
- 8 - approximation by a weighted tree with Steiner points

- $\forall\, G, s, \rho_s(G) \leq \max\{3td(G)\text{-}1, 2td(G) + 1\}$

- 9 - approximation by a weighted tree without Steiner points
- 6 - approximation by a weighted tree with Steiner points

$$\forall\, u, v \in V,\ d_G(u,v) \leq d_{T'_l}(u,v) \leq 3\alpha(d_G(u,v) + 1) \leq 6\alpha\, d_G(u,v)$$

(the larger the distance $d_G(u,v)$, the smaller the distortion)

$d_G(u,v) = 1 \Rightarrow$ distortion ≤ 6;
$d_G(u,v) = 2 \Rightarrow$ distortion ≤ 4.5;
$d_G(u,v) \geq 3 \Rightarrow$ distortion ≤ 4; ...

http://www.cs.kent.edu/~mabuata/graph_embed/unweighted_graph_tree_metric/index.htm

Real-life graphs / networks

- For any graph G there is a tree T, constructible in linear time, such that $\forall\, u, v \in V,\; d_T(u,v) - 2 \le d_G(u,v) \le d_T(u,v) + 3td(G)$

By Muad Abu-Ata, PhD student at Kent State University

Data set	|V|	Avg error left (d_G/d_T)	max error left (d_G/d_T)	% of left pairs $(d_G > d_T)$	Avg error right (d_T/d_G)	max error right (d_T/d_G)	% of right pairs $(d_T > d_G)$	Avg. relative error $(\lvert d_G - d_T\rvert / d_G)$	% of pairs $d_T = d_G$
Yeast	2,224	1.48714	5	56.3%	1.48714	3	12.2%	0.219268	31.5%
Homo Sapiens	16,711	1.533	4	2.8%	1.17564	3	25.2%	0.180092	72.0%
PPI	1,458	1.50159	7	70.5%	1.10486	3	9.1%	0.24669	20.4%
DBLB-coauthors	317,080	1.77416	9	95.8%	1.03535	3	0.6%	0.383101	3.6%
Amazon	334,863	2.48301	19	99.1%	1.04929	3	0.3%	0.536656	0.6%
Dutch_Elite	3,621	1.54045	7	73.0%	1.05818	3	3.9%	0.252341	23.1%
ITDK0304	190,914	1.60077	8	94.8%	1.02828	3	0.6%	0.331656	4.6%
Aqualab 12/2007- 09/2008	31,845	1.42269	4	31.7%	1.21947	3	35.8%	0.241815	32.5%
Dimes 3/2010	26,424	1.53666	3	5.7%	1.17552	3	44.4%	0.184767	49.9%
Routeview	10,515	1.40636	4	24.3%	1.18259	3	33.4%	0.205375	42.3%
AS_CAIDA	26,475	1.48085	4	21.4%	1.16106	3	35.4%	0.192302	43.2%

PPI

error	relative frequency
[7,6]	0.000030
(6,5]	0.000069
(5,4]	0.000214
(4,3]	0.007091
(3,2]	0.081369
(2,1)	0.616643
[1,2)	0.290158
[2,3]	0.004456

90%

Yeast

error	relative frequency
[5,4]	0.000127
(4,3]	0.000708
(3,2]	0.080864
(2,1)	0.481600
[1,2)	0.428692
[2,3]	0.008009

90%

AS_CAIDA

error	relative frequency
[4,3]	0.000015
(3,2]	0.029115
(2,1)	0.185172
[1,2)	0.771621
[2,3]	0.014077

95%

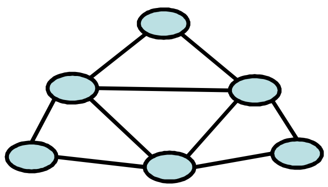
Homo Sapiens

error	relative frequency
[4,3]	0.000013
(3,2]	0.004992
(2,1)	0.023258
[1,2)	0.884653
[2,3]	0.087083

90%

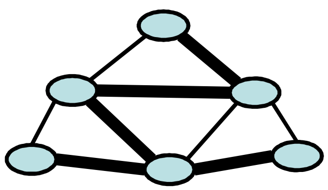
Tree-stretch $\boldsymbol{ts(G)}$

- Tree-stretch $\boldsymbol{ts(G)}$ of an unweighted undirected graph $G = (V, E)$ is the minimum number $\boldsymbol{t}$ such that G has a spanning tree $T = (V, E')$ with $d_T(u, v) \leq t$ for every edge $uv \in E$.
- Tree shape mimics graph shape: the smaller $\boldsymbol{t}$ the closer graph to a tree



G

$ts(G) = 3$



Corresponding decision problem: Tree $\boldsymbol{t}$-Spanner Problem

Given unweighted undirected graph $G = (V, E)$ and integer t.
Does G admit a spanning tree $T = (V, E')$ such that

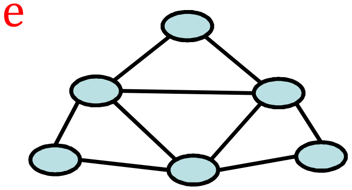
$$\forall\, u, v \in V, d_T(u, v) \leq t d_G(u, v)$$

(a multiplicative tree t-spanner of G)

Tree-stretch $\boldsymbol{ts(G)}$

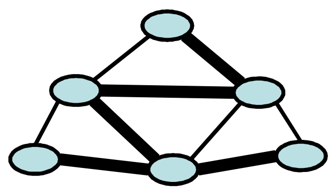
- Tree-stretch $\boldsymbol{ts(G)}$ of an unweighted undirected graph $G = (V, E)$ is the minimum number $\boldsymbol{t}$ such that G has a spanning tree $T = (V, E')$ with $d_T(u, v) \leq t$ for every edge $uv \in E$.

- Tree shape mimics graph shape: the smaller $\boldsymbol{t}$ the closer graph to a tree



G

$ts(G) = 3$



Corresponding decision problem: Tree $\boldsymbol{t}$-Spanner Problem

Given unweighted undirected graph $G = (V, E)$ and integers t, r.
Does G admit a spanning tree $T = (V, E')$ such that

$$\forall\, u, v \in V, d_T(u, v) \leq t d_G(u, v)$$

(a multiplicative tree t-spanner of G)

or

$$\forall\, u, v \in V, d_T(u, v) \leq d_G(u, v) + r$$

(an additive tree r-spanner of G)?

minimum $\boldsymbol{r}$ is called tree-surplus $\boldsymbol{tp(G)}$

Some previously known results

General unweighted graphs

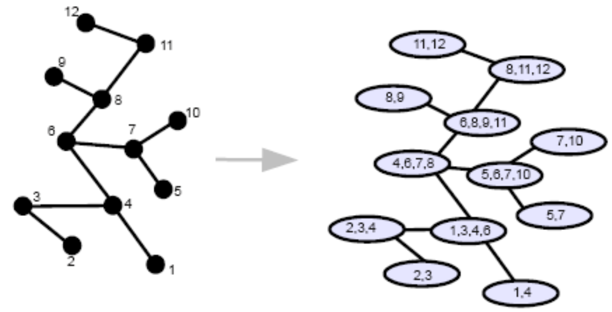
- NP-complete for $t > 3$, linear for $t = 1,2$, open for $t = 3$
  [Cai, Corneil: *SIAM J. Discrete Math*. (1995)]
- NP-hard to 2-approximate
  [Liebchen, Wünsch: *Discrete Appl. Math.* (2008)]
- $O(\log n)$-approximation
  [Emek, Peleg*: SIAM J. Comput*. (2008)]

Special graph classes

- Linear time for planar graphs and their generalizations for any fixed t
  [Dragan, Fomin, Golovach: *J. of Computer and System Sciences* (2011)]
- $ts(G)$ is constant for AT-free, strongly chordal, dually chordal, etc…
  [Kratsch, Le, Müller, Prisner, Wagner: *SIAM J. Discrete Math.* (2003)]
  [Brandstädt, Chepoi, Dragan: *J. Algorithms* (1999)] …
- $ts(G)$ is $\theta(\log n)$ for chordal graphs
  [Dragan, Köhler: *APPROX* (2011)]

Relations between $\boldsymbol{ts(G)}$ and $\boldsymbol{tb(G)}$ and consequences

[Dragan, Köhler: *APPROX* (2011)]



- $\forall\, G,\ \ tb(G) \le \lceil ts(G)/2 \rceil$ and $tl(G) \le ts(G)$
- Any connected n-vertex, m-edge graph G admits a tree $(2tb(G)\ log_2 n)$-spanner constructible in $O(nmlog^2 n)$ time from scratch.
- Any connected n-vertex, m-edge graph G admits a tree $(6tl(G)\ log_2 n)$-spanner constructible in $O(mlogn)$ time from scratch.
- $\forall\, G,\ \ ts(G) \le 2tb(G)\ log_2 n \le\ 2tl(G)\ log_2 n$

Hence, $\boldsymbol{O(log\, n)}$-approximation for $\boldsymbol{ts(G)}$

- One can construct from scratch for any graph G
  - a tree $(2\lceil ts(G)/2 \rceil\ log_2 n)$-spanner in $O(nmlog^2 n)$ time
  - a tree $(6ts(G)\ log_2 n)$-spanner in $O(mlogn)$ time

Compare with [Emek, Peleg: *SIAM J. Comput.* (2008)]

- One can construct from scratch for any graph G
  - a tree $(6ts(G)\ log_2 n)$-spanner in $O(nmlog^2 n)$ time

Relations between $tl(G), ts(G)$ and $td(G)$ and consequences

- $\forall\, G,\ \ tb(G) \leq \boldsymbol{tl(G)} \leq \boldsymbol{td(G)} \leq ts(G)$
- Any connected n-vertex, m-edge graph G admits a tree $(2tb(G)\ log_2 n)$-spanner constructible in $O(nmlog^2 n)$ time from scratch. [Dragan, Köhler: *APPROX* (2011)]
- $\forall\, G,\ \ ts(G) \leq 2td(G)\ log_2 n$

Hence, if a graph is embeddable into a tree with distortion α then it is embeddable to a spanning tree with stretch at most $2\alpha\ log_2 n$.

The bound is sharp (chordal graphs):

- $ts(G)$ is $\theta(\log n)$ [Dragan, Köhler: *APPROX* (2011)]
- $td(G)$ is $\theta(1)$ [Brandstädt, Chepoi, Dragan: *J. Algorithms* (1999)]

$\exists T, \forall\, u, v \in V,\ d_T(u,v) - 2 \leq d_G(u,v) \leq d_T(u,v) + 2$

Relations between $tl(G), ts(G)$ and $td(G)$ and consequences

- $\forall G,\ tb(G) \leq tl(G) \leq td(G) \leq ts(G)$
- Any connected n-vertex, m-edge graph G admits a tree $(2tb(G)\ log_2 n)$-spanner constructible in $O(nmlog^2 n)$ time from scratch. [Dragan, Köhler: *APPROX* (2011)]
- $\forall G,\ ts(G) \leq 2td(G)\ log_2 n$

Hence, if a graph is embeddable into a tree with distortion α then it is embeddable to a spanning tree with stretch at most $2\alpha\ log_2 n$.

Recall,

If G admits a tree H with $\forall\, u, v \in V,\ d_G(u,v) \leq d_H(u,v) \leq \alpha\, d_G(u,v)$ then

- there is a tree T, constructible in linear time, such that

$$\forall\, u, v \in V,\ d_T(u,v) - 2 \leq d_G(u,v) \leq d_T(u,v) + 3\alpha$$

- there is a easily constructible tree T'_l with $\forall\, u, v \in V,\ d_G(u,v) \leq d_{T'_l}(u,v) \leq 3\alpha(d_G(u,v) + 1) \leq 6\alpha\, d_G(u,v)$

Hence, if a graph admits a tree t-spanner then it is embeddable to a tree with distortion at most $6t$. Furthermore, tree t-spanner can be turned into additive distortion tree.

Relationships between parameters

- $\forall\, G, \quad tb(G) \leq tl(G) \leq 2tb(G)$ [folklore]
- $\forall\, G, s, \ \ \rho_s\,(G) \leq \Delta_s(G) \leq 2\rho_s(G)$ [folklore]
- $\forall\, G, s, \ \ tl(G) - 1 \leq \Delta_s(G) \leq 3\ tl(G)$ [Dourisboure, Gavoille: *DM* (2007)]
- $\forall\, G, s, \ \ \rho_s(G) \leq 2\ tl(G)$ [Dourisboure, Dragan, Gavoille, Yan: *TCS* (2007)]
- $\forall\, G, s, \ \ tb(G) - 1 \leq \ \rho_s(G) \leq 3\ tb(G)$ [Dragan, Köhler: *APPROX* (2011)]
- $\forall\, G, \ hb(G) \leq tl(G) \leq O(hb(G)\log n)$
- $\forall\, G, s, \ hb(G) \leq \Delta_s(G) \leq O(hb(G)\log n)$

[Chepoi, Dragan, Estellon, Habib, Vaxes: *SoCG* (2008)]

- $\forall\, G, s, \Delta_s(G)/3 \leq td(G) \leq 2\ \Delta_s(G) + 2$
- $\forall\, G, s, \ \ \rho_s\,(G) \leq \max\{3td(G)\text{-}1,\ 2td(G) + 1\}$

[Chepoi, Dragan, Newman, Rabinovich, Vaxes: *Discr.&Comput.Geom.* (2012)]

- $\forall\, G, \ \ tl(G) \leq td(G) \leq ts(G)$ and $tb(G) \leq \lceil ts(G)/2 \rceil$
- $\forall\, G, \ \ ts(G) \leq 2tb(G)\ log_2 n$
- $\forall\, G, \ \ ts(G) \leq 2td(G)\ log_2 n$

[Dragan, Köhler: *APPROX* (2011)]

$$hb(G) \leq tl(G) \leq td(G) \leq ts(G) \leq 2tb(G)\ log_2 n \leq O(hb(G) log^2 n)$$

Thank You

Special thanks to all organizers

Special thanks to
Andreas Brandstädt