SIAM J. COMPUT.
Vol. 30, No. 5, pp. 1662–1677

LINEAR TIME ALGORITHMS FOR HAMILTONIAN PROBLEMS ON (CLAW,NET)-FREE GRAPHS*

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Abstract. We prove that claw-free graphs, containing an induced dominating path, have a Hamiltonian path, and that 2-connected claw-free graphs, containing an induced doubly dominating cycle or a pair of vertices such that there exist two internally disjoint induced dominating paths connecting them, have a Hamiltonian cycle. As a consequence, we obtain linear time algorithms for both problems if the input is restricted to (claw,net)-free graphs. These graphs enjoy those interesting structural properties.

Key words. claw-free graphs, (claw,net)-free graphs, Hamiltonian path, Hamiltonian cycle, dominating pair, dominating path, linear time algorithms

AMS subject classifications. 05C38, 05C45, 05C85, 05C75, 68R10

PII. S0097539799357775

1. Introduction.

Hamiltonian properties of claw-free graphs have been studied extensively in the last couple of years. Different approaches have been made, and a couple of interesting properties of claw-free graphs have been established (see [1, 2, 3, 5, 6, 13, 14, 15, 16, 19, 22, 23, 25, 26]). The purpose of this work is to consider the algorithmic problem of finding a Hamiltonian path or a Hamiltonian cycle efficiently. It is not hard to show that both the Hamiltonian path problem and the Hamiltonian cycle problem are NP-complete, even when restricted to line graphs [28]. Hence, it is quite reasonable to ask whether one can find interesting subclasses of claw-free graphs for which efficient algorithms for the above problems exist.

Already in the eighties, Duffus, Jacobson, and Gould [12] defined the class of (claw,net)-free (CN-free) graphs, i.e., graphs that contain neither an induced claw nor an induced net (see Figure 1.1). Although this definition seems to be rather restrictive, the family of CN-free graphs contains a couple of graph families that are of interest in their own right. Examples of those families are unit interval graphs, claw-free asteroidal triple-free (AT-free) graphs, and proper circular arc graphs. In their paper [12], Duffus, Jacobson, and Gould showed that this class of graphs has the nice property that every connected CN-free graph contains a Hamiltonian path and every 2-connected CN-free graph contains a Hamiltonian cycle. Later, Shepherd [27] proved that there is an $O(n^6)$ algorithm for finding such a Hamiltonian path/cycle in CN-free graphs. Note also that CN-free graphs are exactly the Hamiltonian-hereditary

*Received by the editors June 23, 1999; accepted for publication (in revised form) January 13, 2000; published electronically November 28, 2000. An extended abstract of these results has been presented at the 25th International Workshop on Graph-Theoretic Concepts in Computer Science, Lecture Notes in Comput. Sci. 1665, Springer-Verlag, New York, 1999, pp. 364–376.

http://www.siam.org/journals/sicomp/30-5/35777.html

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graphs [10], i.e., the graphs for which every connected induced subgraph contains a Hamiltonian path.

In this paper we give a constructive existence proof and present linear time algorithms for the Hamiltonian path and Hamiltonian cycle problems on CN-free graphs. The important structural property that we exploit for this is the existence of an induced *dominating path* in every connected CN-free graph (Theorem 2.3). The concept of a dominating path was first used by Corneil, Olariu, and Stewart [8] in the context of AT-free graphs. They also developed a simple linear time algorithm for finding such a path in every AT-free graph [7]. As we show in Theorem 2.3, for the class of CN-free graphs, a linear time algorithm for finding an induced dominating path exists as well. This property is of interest for our considerations since we prove that all claw-free graphs that contain an induced dominating path have a Hamiltonian path (Theorem 3.1). The proof implies that, given a dominating path, one can construct a Hamiltonian path for a claw-free graph in linear time.

For 2-connected claw-free graphs, we show that the existence of a dominating pair is sufficient for the existence of a Hamiltonian cycle. (A *dominating pair* is a pair of vertices such that every induced path connecting them is a dominating path.) Again, given a dominating pair, one can construct a Hamiltonian cycle in linear time (Theorem 5.6). This already implies, for example, a linear time algorithm for finding a Hamiltonian cycle in claw-free AT-free graphs, since every AT-free graph contains a dominating pair and it can be found in linear time [9]. Unfortunately, CN-free graphs do not always have a dominating pair. For example, an induced cycle with more than six vertices is CN-free but does not have such a pair of vertices. Nevertheless, 2-connected CN-free graphs have another nice property: they have a good pair or an induced doubly dominating cycle. An *induced doubly dominating cycle* is an induced cycle such that every vertex of the graph is adjacent to at least two vertices of the cycle. A *good pair* is a pair of vertices, such that there exist two internally disjoint induced dominating paths connecting these vertices. We prove that the existence of an induced doubly dominating cycle or a good pair in a claw-free graph is sufficient for the existence of a Hamiltonian cycle (Theorems 5.1 and 5.5). Moreover, given an induced doubly dominating cycle or a good pair of a claw-free graph, a Hamiltonian cycle can be constructed in linear time. In section 4 we present an $O(m+n)$ time algorithm which, for a given 2-connected CN-free graph, finds either a good pair or an induced doubly dominating cycle.

For terms not defined here, we refer to [11, 17]. In this paper we consider finite connected undirected graphs $G = (V, E)$ without loops and multiple edges. The cardinality of the vertex set is denoted by n, whereas the cardinality of the edge set is denoted by m.

A *path* is a sequence of vertices $(v_0, \dots, v_l)$ such that all v_i are distinct and $v_i v_{i+1} \in E$ for $i = 0, \dots, l-1$; its *length* is l. An *induced path* is a path where $v_i v_j \in E$ if and only if $i = j-1$ and $j = 1, \dots, l$. A *cycle* (*k-cycle*) is a path $(v_0, \dots, v_k)$ $(k \geq 3)$ such that $v_0 = v_k$; its *length* is k. An *induced cycle* is a cycle where $v_i v_j \in E$ if and only if $|i-j| = 1$ (modulo k). A *hole* H_k is an induced cycle of length $k \geq 5$.

The *distance* $dist(v, u)$ between vertices v and u is the smallest number of edges in a path joining v and u. The *eccentricity* $ecc(v)$ of a vertex v is the maximum distance from v to any vertex in G. The *diameter* $diam(G)$ of G is the maximum eccentricity of a vertex in G. A pair v, u of vertices of G with $dist(v, u) = diam(G)$ is called a *diametral pair*.

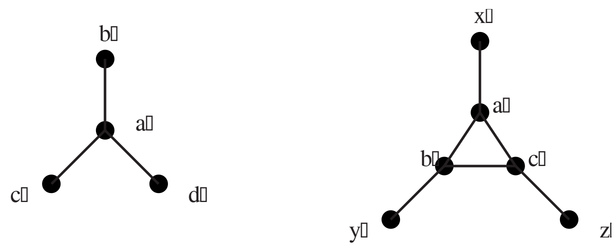

FIG. 1.1. *The claw* $K(\mathrm{a};\mathrm{b},\mathrm{c},\mathrm{d})$ *and the net* $N(\mathrm{a},\mathrm{b},\mathrm{c};\mathrm{x},\mathrm{y},\mathrm{z})$.

For every vertex we denote by $\mathrm{N}(v)$ the set of all neighbors of v, $\mathrm{N}(v) = \{u \in V : dist(u,v) = 1\}$. The *closed neighborhood* of v is defined by $\mathrm{N}[v] = \mathrm{N}(v) \cup \{v\}$. For a vertex v and a set of vertices $S \subseteq V$, the minimum distance between v and vertices of S is denoted by $dist(v,S)$. The *closed neighborhood* $\mathrm{N}[S]$ of a set $S \subseteq V$ is defined by $\mathrm{N}[S] = \{v \in V : dist(v,S) \leq 1\}$.

We say that a set $S \subseteq V$ *dominates* G if $\mathrm{N}[S] = V$, and S *doubly dominates* G if every vertex of G has at least two neighbors in S. An induced path of G which dominates G is called an *induced dominating path*. A shortest path of G which dominates G is called a *dominating shortest path*. Analogously one can define an *induced dominating cycle* of G. A *dominating pair* of G is a pair of vertices $v, u \in V$, such that every induced path between v and u dominates G. A *good pair* of G is a pair of vertices $v, u \in V$, such that there exist two internally disjoint induced dominating paths connecting v and u.

The *claw* is the induced complete bipartite graph $K_{1,3}$, and for simplicity, we refer to it by $K(a;b,c,d)$ (see Figure 1.1). The *net* is the induced six-vertex graph $N(a,b,c;x,y,z)$ shown in Figure 1.1. A graph is called *CN-free* or, equivalently, *(claw, net)-free* if it contains neither an induced claw nor an induced net. An *asteroidal triple* of G is a triple of pairwise nonadjacent vertices, such that for each pair of them there exists a path in G that does not contain any vertex in the neighborhood of the third one. A graph is called *AT-free* if it does not contain an asteroidal triple. Finally, a *Hamiltonian path* or *Hamiltonian cycle* of G is a path or cycle, respectively, containing all vertices of G.

2. Induced dominating path. In this section we give a constructive proof for the property that every connected CN-free graph contains an induced dominating path. In fact, we show that there is an algorithm that finds such a path in linear time. To prove the main theorem of this section we will need the following two lemmas.

LEMMA 2.1 (see [12]). *Let* $P = (x_1, \ldots, x_k)$ *be an induced path of a CN-free graph* G*, and let* v *be a vertex of* G *such that* $dist(v,P) = 2$*. Then any neighbor* y *of* v *with* $dist(y,P) = 1$ *is adjacent to* x_1 *or to* x_k*.*

LEMMA 2.2. *Let* P *be an induced path connecting vertices* v *and* u *of a CN-free graph* G*. Let also* s *be a vertex of* G *such that* $s \notin \mathrm{N}[P]$ *and* $dist(v,s) \leq dist(v,u)$*. Then*

1. *for every shortest path* P' *connecting* v *and* s*,* $P' \bigcap P = \{v\}$ *holds, and*
2. *if there is an edge* xy *of* G *such that* $x \in P \setminus \{v\}$ *and* $y \in P' \setminus \{v\}$*, then both* x *and* y *are neighbors of* v*.*

Proof. Let y be the vertex of $P' \setminus \{v\}$ which is closest to s and has a neighbor x on $P \setminus \{v\}$; clearly, $y \neq s$. Let s', v' be the neighbors of y on the subpaths of P' connecting y with s and y with v, respectively. Since $s' \notin \mathrm{N}[P]$, by Lemma 2.1, vertex

y must be adjacent to v or to u. If $yu \in E$, then $v'u \in E$, too (otherwise, we have a claw $K(y; s', v', u)$). But now $dist(v, u) \leq dist(v, v') + 1 = dist(v, y) < dist(v, s) \leq dist(v, u)$, and a contradiction arises. Therefore, y is adjacent to v, and since $y \notin P$, the paths P and P' have only the vertex v in common. Moreover, to avoid a claw $K(y; s', v, x)$, vertex x has to be adjacent to v. □

THEOREM 2.3. *Every connected CN-free graph G has an induced dominating path, and such a path can be found in $O(n + m)$ time.*

Proof. Let G be a connected CN-free graph. One can construct an induced dominating path in G as follows. Take an arbitrary vertex v of G. Using breadth first search (BFS), find a vertex u with the largest distance from v and a shortest path P connecting u with v. Check whether this path P dominates G. If so, we are done. Now, assume that the set $S = V \setminus \mathrm{N}[P]$ is not empty. Again, using BFS, find a vertex s in S with largest distance from v and a shortest path P' connecting v with s. Create a new path P^* by joining P and P' in the following way: $P^* = (P \setminus \{v\}, P' \setminus \{v\})$ if there is a chord xy between the paths P and P' (see Lemma 2.2), and $P^* = (P \setminus \{v\}, P')$, otherwise. By Lemma 2.2, the path P^* is induced. It remains to show that this path dominates G.

Assume there exists a vertex $t \in V \setminus \mathrm{N}[P^*]$. First, we claim that t is dominated neither by P nor by P'. Indeed, if $t \in (\mathrm{N}[P] \bigcup \mathrm{N}[P']) \setminus \mathrm{N}[P^*]$, then necessarily $tv \in E$ and $v \notin P^*$, i.e., neighbors $x \in P$ and $y \in P'$ of v are adjacent. Therefore, we get a net $N(v, y, x; t, s', u')$, where s' and u' are the vertices at distance two from v on paths P' and P, respectively. Note that vertices s', u' exist because $dist(v, s) \geq 2$.

Thus, t is dominated neither by P nor by P'. Moreover, from the choice of u and s we have $2 \leq dist(v, t) \leq dist(v, s) \leq dist(v, u)$. Now let P'' be a shortest path, connecting t with v, and let z be a neighbor of v on this path. Applying Lemma 2.2 twice (to P, P'' and to P', P''), we obtain a subgraph of G depicted in Figure 2.1. We have three shortest paths P, P', P'', each of length at least 2 and with only one common vertex v. These paths can have only chords of type zx, zy, xy. Any combination of them leads to a forbidden claw or net. This contradiction completes the proof of the theorem. Evidently, the method described above can be implemented to run in linear time. □

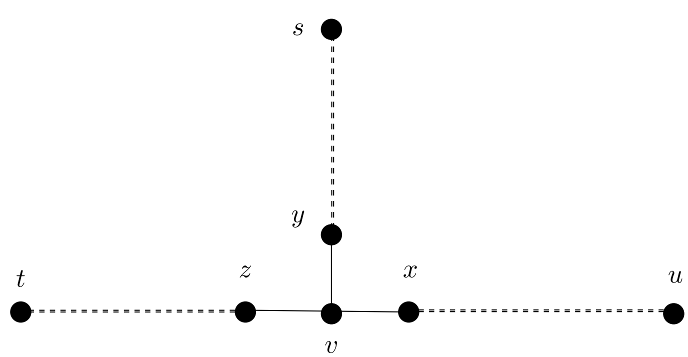

FIG. 2.1.

It is not clear whether CN-free graphs can be recognized efficiently. But, to apply our method for finding an induced dominating path in these graphs, we do not need to know in advance that a given graph G is CN-free. Actually, our method can be applied to any graph G. It either finds an induced dominating path or returns either a claw or a net of G, showing that G is not CN-free.

COROLLARY 2.4. *There is a linear time algorithm that for a given (arbitrary) connected graph G either finds an induced dominating path or outputs an induced claw or an induced net of G.*

Proof. Let G be a graph. For an arbitrary vertex v of G, we find a vertex u with the largest distance from v and a shortest path P connecting u with v. If P dominates G, then we are done. Else, we find a vertex $s \in V \setminus \mathrm{N}[P]$ with the largest distance from v and a shortest path P' connecting v with s. If there are vertices in $P' \setminus \{v\}$ which have a neighbor on $P \setminus \{v\}$, we take the vertex y that is closest to s and check whether y is adjacent to v and u. If it is adjacent neither to u nor to v, then G has a net or a claw (see the proof of Lemma 2.1). If $yu \in E$ or $yv \in E$ and a neighbor x of y on $P \setminus \{v\}$ is not adjacent to v, then G has a claw (see Lemma 2.2). Now, if we did not yet find a forbidden subgraph, then the only possible chord between the paths P and P' is xy with $xv, yv \in E$, and we can create an induced path P^* as described in the proof of Theorem 2.3. Hence, it remains to check whether P^* dominates G. If there exists a vertex $t \in V \setminus \mathrm{N}[P^*]$, then again we will find a net or a claw in G (see Theorem 2.3). It is easy to see that the total time bound of all these operations is linear. ∎

3. Hamiltonian path. In what follows we show that for claw-free graphs the existence of an induced dominating path is a sufficient condition for the existence of a Hamiltonian path. The proof for this result is constructive, implying that, given an induced dominating path, one can find a Hamiltonian path efficiently.

THEOREM 3.1. *Every connected claw-free graph G containing an induced dominating path has a Hamiltonian path. Moreover, given an induced dominating path, a Hamiltonian path of G can be constructed in linear time.*

Proof. Let $G = (V, E)$ be a connected claw-free graph and let $P = (x_1, \ldots, x_k)$ $(k \geq 1)$ be an induced dominating path of G. If $k = 1$, vertex x_1 dominates G and, since G is claw-free, there are no three independent vertices in $G - \{x_1\}$. (By $G - \{x_1\}$ we denote a subgraph of G induced by vertices $V \setminus \{x_1\}$.) If $G - \{x_1\}$ is not connected, then, again because G is claw-free, it consists of two cliques C_0, C_1 and a Hamiltonian path of G can easily be constructed. If $G - \{x_1\}$ is connected, we can construct a Hamiltonian path as follows. First, we construct a maximal path $P_1 = (y_1, \ldots, y_l)$, i.e., all vertices that are not in P_1 are neither connected to y_1 nor to y_l. Let R be the set of all remaining vertices. If $R = \emptyset$, we are done. If there is any vertex in R, it follows that $y_1 y_l \in E$ since otherwise there are three independent vertices in $G - \{x_1\}$. Furthermore, any two vertices of R are joined by an edge, since otherwise they would form an independent triple with y_1 (and with y_l as well). Hence, R induces a clique. Since $G - \{x_1\}$ is connected, there has to be an edge from a vertex $v_R \in R$ to some vertex $y_i \in P_1$ $(1 < i < l)$. Now we can construct a Hamiltonian path P of G: $P = (x_1, y_{i+1}, y_{i+2}, \ldots, y_l, y_1, y_2, \ldots, y_i, v_R, \tilde{R})$, where $\tilde{R}$ stands for an arbitrary permutation of the vertices of $R \setminus \{v_R\}$.

For $k \geq 2$ we first construct a Hamiltonian path P_2 for $G' = G(\mathrm{N}[x_1] \setminus \{x_2\})$ as described above, using x_1 as the dominating vertex. At least one endpoint of P_2 is adjacent to x_2 since if $G' - \{x_1\}$ is not connected, x_2 has to be adjacent to all vertices of either C_0 or C_1 (otherwise, there is a claw in G), and if $G' - \{x_1\}$ is connected, the construction gives a path ending in x_1 which is, of course, adjacent to x_2. To construct a Hamiltonian path for the rest of the graph we define for each vertex x_i $(i \geq 2)$ of P a set of vertices $C_i = \mathrm{N}(x_i) \setminus \bigcup_{j=1}^{i-1} \mathrm{N}[x_j]$. Each set C_i forms a clique of G since if two vertices $u, v \in C_i$ are not adjacent, then the set u, v, x_i, x_{i-1} induces a claw. Hence we can construct a path $P^* = (P_2, x_2, P_2^C, x_3, P_3^C, x_4, \ldots, x_{k-1}, P_{k-1}^C, x_k, P_k^C)$, where P_i^C stands for an arbitrary permutation of the vertices of $C_i \setminus \{x_{i+1}\}$. This path P^* is a Hamiltonian path of G because it obviously is a path, and, since P is a dominating path, each vertex of G has to be either on P, P_2, or in one of the sets C_i.

For the case $k = 1$ both finding the connected components of $G - \{x_1\}$ and constructing the path P_1 can easily be done in linear time. For $k \geq 2$ we just have to make sure that the construction of the sets C_i can be done in $O(n + m)$, and this can be realized easily within the required time bound. ∎

THEOREM 3.2. *Every connected CN-free graph G has a Hamiltonian path, and such a path can be found in $O(n + m)$ time.*

Proof. By Theorem 2.3, every connected CN-free graph has an induced dominating path P, and it can be found in linear time. Using the path P, by Theorem 3.1, one can construct a Hamiltonian path of G in linear time. ∎

Analogously to Corollary 2.4, we can state the following.

COROLLARY 3.3. *There is a linear time algorithm that for a given (arbitrary) connected graph G either finds a Hamiltonian path or outputs an induced claw or an induced net of G.*

Proof. The proof follows from Corollary 2.4 and the proof of Theorem 3.1. ∎

4. Induced dominating cycle, dominating shortest path, or good pair. In this section we show that every 2-connected CN-free graph G has an induced doubly dominating cycle or a good pair. Moreover, we present an efficient algorithm that, for a given 2-connected CN-free graph G, finds either a good pair or an induced doubly dominating cycle.

LEMMA 4.1. *Every hole of a connected CN-free graph G dominates G.*

COROLLARY 4.2. *Let H be a hole of a connected CN-free graph G. Every vertex of $V \setminus H$ is adjacent to at least two vertices of H.*

A *subgraph G' of G (doubly) dominates G* if the vertex set of G' (doubly) dominates G.

LEMMA 4.3. *Every induced subgraph of a connected CN-free graph G which is isomorphic to S_3 or S_3^- (see Figure 4.1) dominates G.*

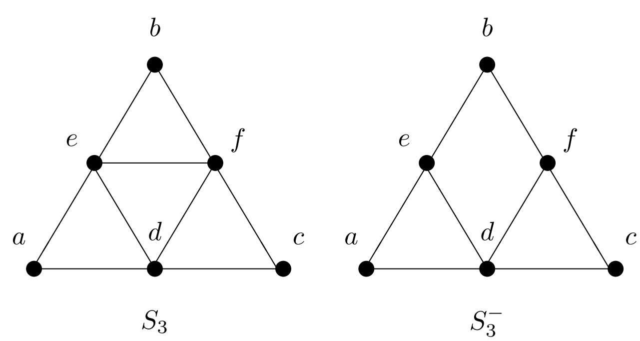

FIG. 4.1.

Proof. Let G contain an induced subgraph isomorphic to S_3^-, and assume that it does not dominate G. Then, there must be a vertex s such that $dist(s, S_3^-) = 2$. Let x be a neighbor of s from $\mathrm{N}[S_3^-]$. If x is adjacent neither to a, nor to b, nor to c (see Figure 4.1), then G contains a claw (e.g., if $xf \in E$, then a claw $K(f; b, c, x)$ arises). Thus, without loss of generality, x has to be adjacent to a or b.

If $xa \in E$, then x is adjacent neither to b nor to c, since otherwise we will get a claw ($K(x; a, b, s)$ or $K(x; a, c, s)$). To avoid a net $N(a, e, d; x, b, c)$ vertex x must be adjacent to e or d. But, if $ex \in E$, then $xd \in E$ too. (Otherwise, we will have a claw $K(e; b, d, x)$.) Analogously, if $xd \in E$, then also $xe \in E$. Hence, x is adjacent to both e and d, and a net $N(x, e, d; s, b, c)$ arises.

Now, we may assume that x is adjacent to b and not to a, c. To avoid a claw $K(b; x, e, f)$, x must be adjacent to e or f. But again, $xe \in E$ if and only if $xf \in E$. (Otherwise, we get a net $N(x, b, e; s, f, a)$ or $N(x, b, f; s, e, c)$.) Hence x is adjacent to both e and f and a claw $K(x; s, e, f)$ arises.

Consequently, S_3^- dominates G. Similarly, every induced S_3 (if it exists) dominates G. ∎

LEMMA 4.4. *Let P be an induced path connecting vertices v and u of a connected CN-free graph G. Let s be a vertex of G such that $s \notin \mathrm{N}[P]$ and $dist(v, s) \leq dist(v, u)$, $dist(u, s) \leq dist(v, u)$. Then G has an induced doubly dominating cycle, and such a cycle can be found in linear time.*

Proof. Let P_v and P_u be shortest paths connecting vertex s with v and u, respectively. Both these paths as well as the path P have lengths at least 2. Since $dist(v, s) \leq dist(v, u)$ and $dist(u, s) \leq dist(u, v)$, by Lemma 2.2, we have $P \bigcap P_v = \{v\}$ and $P \bigcap P_u = \{u\}$. Moreover, if there is a chord between P and P_v, then it is unique and both its endvertices are adjacent to v. The same holds for P and P_u; both endvertices of the chord (if it exists) are adjacent to u.

Now, without loss of generality, we suppose that $dist(s, u) \leq dist(s, v)$. Then, from $u \notin \mathrm{N}[P_v]$ and Lemma 2.2 we deduce that $P_u \bigcap P_v = \{s\}$ and between paths P_v and P_u at most one chord is possible, namely, the one with both endvertices adjacent to s. Consequently, we have constructed an induced subgraph of G shown in Figure 4.2 (only chords $s's''$, $v'v''$ and $u'u''$ are possible).

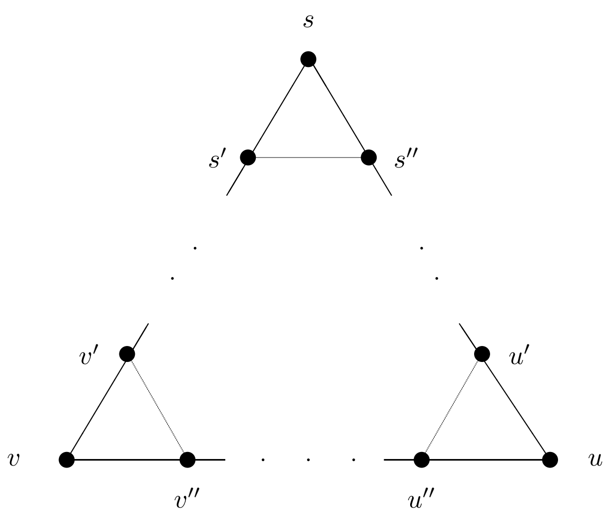

FIG. 4.2.

If the lengths of all three paths P, P_v, P_u are at least 3, then it is easy to see that G has a hole H_k ($k \geq 6$). Furthermore, if at least one of these paths has length greater than or equal to 4, or two of them have lengths 3, then G must contain a hole H_k ($k \geq 5$). It remains to consider two cases: lengths of both P_v and P_u are 2 and the length of P is 3 or 2. Clearly, in both of these cases the graph G contains either a hole H_k ($k \in \{5,6,7\}$) or an induced subgraph isomorphic to S_3^- or S_3. By Corollary 4.2, every hole of G doubly dominates G.

Let G contain an S_3^- with vertex labeling shown in Figure 4.1. We claim that the induced cycle (e,b,f,d,e) dominates G or G contains a hole H_6. Indeed, if a vertex s of G does not belong to S_3^-, then, by Lemma 4.3, it is adjacent to a vertex of S_3^-. Suppose that s is adjacent to none of e,b,f,d. Then, without loss of generality, $sa \in E$, and we obtain an induced subgraph of G isomorphic either to a net $N(e,a,d;b,s,c)$ or to $H_6 = (s,a,e,b,f,c,s)$, depending on whether vertices s and c are adjacent. Hence, we may assume that (e,b,f,d,e) dominates G, and since G is claw-free, this cycle is doubly dominating.

Now let G contain an S_3 with vertex labeling shown in Figure 4.1. We will show that every vertex of G is adjacent to at least two vertices of the cycle (e,f,d,e) or G contains a hole H_5. Suppose vertex s of G is adjacent to none of e,d. Then, by Lemma 4.3, s is adjacent to at least one of a,b,c,f. Let $sf \in E$. To avoid a claw, vertex s is adjacent to both b and c. But then a hole $H_5 = (s,b,e,d,c,s)$ arises. Assume that $sf \notin E$ and, without loss of generality, $sa \in E$. To avoid a net $N(a,e,d;s,b,c)$, s must be adjacent to b or c. In both cases a hole H_5 occurs.

Clearly, the construction of an induced doubly dominating cycle of G given above takes linear time. ◻

THEOREM 4.5. *There is a linear time algorithm that, for a given connected CN-free graph G, either finds an induced doubly dominating cycle or gives a dominating shortest path of G.*

Proof. Let G be a connected CN-free graph. One can construct an induced doubly dominating cycle or a dominating shortest path of G as follows (compare with the proof of Theorem 2.3). Take an arbitrary vertex v of G. Find a vertex u with the largest distance from v and a shortest path P connecting u with v. Check whether P dominates G. If so, we are done; P is a dominating shortest path of G. Assume now that the set $S = V \setminus \mathrm{N}[P]$ is not empty. Find a vertex s in S with the largest distance from v and a shortest path P_v connecting v with s. Create again a new path P^* by "joining" shortest paths P and P_v as in the proof of Theorem 2.3. We have proven there that P^* dominates G. Now let P_u be a shortest path between s and u. If $dist(s,u) \le dist(v,u)$ or both $dist(s,u) > dist(v,u)$ and $v \notin \mathrm{N}[P_u]$, then Lemma 4.4 can be applied to get an induced doubly dominating cycle of G in linear time. Therefore, we may assume that $dist(s,u) > dist(v,u) \ge dist(v,s)$ and $v \in \mathrm{N}[P_u]$. Now we show that the shortest path P_u dominates G. If v lies on the path P_u, then $P^* = P_u$ and we are done. Otherwise, let x be a neighbor of v in P_u. Note that $dist(v,s) > 1$ and so $x \ne s,u$. Since G is claw-free, v is adjacent to a neighbor $y \in P_u$ of x. Assume, without loss of generality, that x is closer to s than y. If we show that $dist(v,s) = 1 + dist(x,s)$ and $dist(v,u) = 1 + dist(y,u)$, then again, by the proof of Theorem 2.3, the path P_u will dominate G (as a path obtained by "joining" two shortest paths that connect v with u and v with s, respectively). By the triangle condition, we have $dist(u,s) < dist(v,u) + dist(v,s)$ (strict inequality because $v \notin P_u$) and $dist(v,s) \le 1 + dist(x,s)$, $dist(u,v) \le 1 + dist(y,u)$. Consequently, $dist(v,u) + dist(v,s) > dist(u,s) = dist(u,y) + 1 + dist(x,s) \ge dist(v,u) - 1 + 1 + dist(v,s) - 1 = dist(v,u) + dist(v,s) - 1$. That is, $dist(u,s) = dist(v,u) + dist(v,s) - 1$ and $dist(v,s) = 1 + dist(x,s)$, $dist(u,v) = 1 + dist(y,u)$. ◻

Since all our proofs were constructive, we can conclude the following.

COROLLARY 4.6. *There is a linear time algorithm that, for a given (arbitrary) connected graph G, either finds an induced doubly dominating cycle, or gives a dominating shortest path, or outputs an induced claw or an induced net of G.*

LEMMA 4.7. *Let $P = (v, x_2, \ldots, x_{k-1}, u)$ be a dominating shortest path of a graph G. Then* $\max\{ecc(v), ecc(u)\} \geq diam(G) - 1$.

Proof. Let x, y be a diametral pair of vertices of G; that is, $diam(G) = dist(x, y)$. If both x and y are on P, then necessarily $\{x, y\} = \{v, u\}$ and therefore $dist(v, u) = diam(G) = ecc(v) = ecc(u)$. If $x \in P$ and $y \in \mathrm{N}[P] \setminus P$, then either $x \neq v, u$ and $diam(G) = dist(x, y) = dist(v, u)$ holds or, without loss of generality, $v = x$ and $ecc(v) = diam(G)$. Finally, if both x and y are in $\mathrm{N}[P] \setminus P$ and $dist(v, u) < dist(x, y)$, then we may assume that at least one of x, y belongs to $\mathrm{N}(v)$, say, x. Hence, $dist(x, y) \leq 1 + dist(v, y) \leq 1 + ecc(v)$; that is, $ecc(v) \geq diam(G) - 1$. ∎

A pair of vertices u, v of G with $dist(u, v) = ecc(u) = ecc(v)$ is called a *pair of mutually furthest vertices.*

COROLLARY 4.8. *For a graph G with a given dominating shortest path, a pair of mutually furthest vertices can be found in linear time.*

Proof. Let $P = (v, x_2, \ldots, x_{k-1}, u)$ be a dominating shortest path of G with $ecc(v) \geq ecc(u)$. Then, by Lemma 4.7, $ecc(v) \geq diam(G) - 1$ holds. Denote by x a vertex of G such that $dist(v, x) = ecc(v)$. Note that both the eccentricity of v and a vertex furthest from v can be found in linear time by BFS. Now, if $ecc(x) = ecc(v)$, then v, x are mutually furthest vertices of G. Else, $ecc(x) > ecc(v) \geq diam(G) - 1$ must hold and vertices x and y, where y is a vertex with $dist(x, y) = ecc(x)$, form a diametral pair of G; $dist(x, y) = ecc(x) = ecc(y) = diam(G)$. ∎

In what follows we will use the fact that in a 2-connected graph every pair of vertices is joined by two internally disjoint paths. In order to actually find such a pair of paths, one can use Tarjan's linear time depth first search- (DFS)-algorithm for finding the blocks of a given graph. For the proof of Lemma 4.9, we refer to [21].

LEMMA 4.9. *Let G be a 2-connected graph, and let x, y be two different nonadjacent vertices of G. Then one can construct in linear time two induced, internally disjoint paths, both joining x and y.*

THEOREM 4.10. *There is a linear time algorithm that, for a given 2-connected CN-free graph G, either finds an induced doubly dominating cycle or gives a good pair of G.*

Proof. By Theorem 4.5, we get either an induced doubly dominating cycle or a dominating shortest path of G in linear time. We show that, having a dominating shortest path of a 2-connected graph G, one can find in linear time a good pair or an induced doubly dominating cycle. By Corollary 4.8, we may assume that a pair x, y of mutually furthest vertices of G is given. Let also P_1, P_2 be two induced internally disjoint paths connecting x and y in G. They exist and can be found in linear time by Lemma 4.9 (clearly, we may assume that $xy \notin E$, because otherwise $\mathrm{N}[x] = V = \mathrm{N}[y]$ and x, y together with a vertex $z \in V \setminus \{x, y\}$ will form a doubly dominating triangle). If one of these paths, say, P_1, is not dominating, then there must be a vertex $s \in V \setminus \mathrm{N}[P_1]$. Since x, y are mutually furthest vertices of G, we have $dist(s, x) \leq dist(x, y)$, $dist(s, y) \leq dist(x, y)$. Hence, we are in the conditions of Lemma 4.4 and can find an induced doubly dominating cycle of G in linear time. ∎

COROLLARY 4.11. *There is a linear time algorithm that, for a given (arbitrary) 2-connected graph G, either finds an induced doubly dominating cycle, or gives a good pair, or outputs an induced claw or an induced net of G.*

5. Hamiltonian cycle. In this section we prove that, for claw-free graphs, the existence of an induced doubly dominating cycle or a good pair is sufficient for the existence of a Hamiltonian cycle. The proofs are also constructive and imply linear time algorithms for finding a Hamiltonian cycle.

THEOREM 5.1. *Every claw-free graph G that contains an induced doubly dominating cycle has a Hamiltonian cycle. Moreover, given an induced doubly dominating cycle, a Hamiltonian cycle of G can be constructed in linear time.*

Proof. Let $DC = (x_1, \ldots, x_k, x_1)$ $(k \geq 3)$ be an induced doubly dominating cycle of G. As before, we define $C_i = \mathrm{N}(x_i) \setminus \bigcup_{j=1}^{i-1} \mathrm{N}[x_j]$ $(2 \leq i \leq k)$. Each set C_i forms a clique of G; otherwise, we would have a claw. Furthermore, $C_k = \emptyset$ holds, and the sets $\mathrm{N}[x_1], C_2, \ldots, C_{k-1}$ form a partition of the vertex set of G. Note that any vertex adjacent to x_k and not to x_j $(1 < j < k)$ belongs to $\mathrm{N}[x_1]$, since the cycle DC is doubly dominating. Let $G' = G(\mathrm{N}[x_1] \setminus \{x_2, x_k\})$ be the subgraph of G induced by $\mathrm{N}[x_1] \setminus \{x_2, x_k\}$. If we show that there is a Hamiltonian path P in G' starting at a neighbor of x_k and ending at a neighbor of x_2, then we are done; the cycle $(x_k, P, x_2, P_2^C, x_3, P_3^C, x_4, \ldots, x_{k-1}, P_{k-1}^C, x_k)$, is a Hamiltonian cycle of G (recall that P_i^C stands for an arbitrary permutation of the vertices of $C_i \setminus \{x_{i+1}\}$).

Since G' is a connected graph, by Theorem 3.1, there exists a Hamiltonian path $P' = (s, y_1, \ldots, y_l, t)$ of G'. Assume that $x_k s, x_k t \notin E$. Then, to avoid a claw $K(x_1; x_k, s, t)$, vertices s and t have to be adjacent, giving a new Hamiltonian path P'' of G' starting at x_1 and ending at a vertex y. If y is adjacent neither to x_k nor to x_2, then a claw $K(x_1; x_k, x_2, y)$ occurs. (Note that in case $k = 3$, i.e., $x_k x_2 \in E$, y is adjacent to at least one of x_k, x_2 because the cycle $DC = (x_1, x_2, x_3, x_1)$ is doubly dominating.) Without loss of generality, $y x_2 \in E$ and the path P'' is a desired path of G'.

So, we may assume that x_k is adjacent to t or s. Analogously, x_2 is adjacent to one of t, s. If x_k, x_2 are adjacent to different vertices, then we are done; the path P' starts at a neighbor of x_k and ends at a neighbor of x_2. Otherwise, let both x_k and x_2 be adjacent to t and not to s. Then a claw $K(x_1; x_k, x_2, s)$ arises when $k > 3$, or we get a contradiction with the property of $DC = (x_1, x_2, x_3, x_1)$ to be a doubly dominating cycle. ∎

COROLLARY 5.2. *Every claw-free graph, containing an induced dominating cycle of length at least* 4, *has a Hamiltonian cycle, and, given that induced dominating cycle, one can construct a Hamiltonian cycle in linear time.*

Let $G = (V, E)$ be a graph, and let $P = (x_1, \ldots, x_k)$ be an induced dominating path of G. P is called an *enlargeable* path if there is some vertex v in $V \setminus P$ that is either adjacent to x_1 or to x_k but not to both of them and, additionally, to no other vertex in P. Consequently, an induced dominating path P is called *nonenlargeable* if such a vertex does not exist. Obviously, every graph G that has an induced dominating path has a nonenlargeable induced dominating path as well. Furthermore, given an induced dominating path P, one can find in linear time a nonenlargeable induced dominating path P' by simply scanning the neighborhood of both x_1 and x_k. For the next theorem we will need an auxiliary result.

LEMMA 5.3. *Let G be a claw-free graph, and let $P = (x_1, x_2, \ldots, x_k)$ $(k > 2)$ be an induced nonenlargeable dominating path of G such that there is no vertex y in G with $\mathrm{N}(y) \cap P = \{x_1, x_k\}$. Then there is a Hamiltonian path in G that starts in x_1 and ends in x_k and, given the path P, one can construct this Hamiltonian path in linear time.*

Proof. Let $C_i = \mathrm{N}(x_i) \setminus \bigcup_{j=1}^{i-1} \mathrm{N}[x_j]$ $(i \geq 2)$. Since P is nonenlargeable, C_k is empty. Using the method described in the proof of Theorem 3.1, we can easily construct a path, starting in x_1 and ending in x_k, that contains all vertices of $C_2, \ldots, C_{k-1}$. This implies that we have to worry only about how to insert the vertices of the neighborhood of x_1 into this path. We have to consider two cases.

Case 1. $H = G(\mathrm{N}(x_1) \setminus \{x_2\})$ consists of two connected components C_0, C_1.

Since G is claw-free, both C_0 and C_1 induce cliques in G. Furthermore, x_2 is adjacent to all vertices of at least one of C_0 and C_1, say, C_1, because otherwise we have a claw in G.

Let y be an arbitrary vertex of C_0. Since P is nonenlargeable, y has at least one neighbor on $P \setminus \{x_1\}$, and let x_j be the one with smallest index. By the preconditions of our lemma, $j \neq k$. If $j > 2$, then y has to be adjacent to x_{j+1} as well, since otherwise $K(x_j; y, x_{j-1}, x_{j+1})$ is a claw. Furthermore, y is adjacent to all vertices $c_j \in C_j$, since otherwise $K(x_j; y, x_{j-1}, c_j)$ is a claw. Hence, when constructing the Hamiltonian path, we can simply add y to C_j.

Now we consider the set Y of all vertices y of C_0 with $yx_2 \in E$. Suppose there is a vertex c_2 in C_2 with $c_2 \neq x_3$. If there is a vertex $c_1 \in C_1$ that is nonadjacent to vertex c_2, then there is an edge from every vertex $c_0 \in Y$ to c_2; otherwise, $K(x_2; c_0, c_1, c_2)$ is a claw of G. This implies that we can construct a Hamiltonian path with the required properties. If, on the other hand, all vertices of C_1 are adjacent to all vertices of C_2, we can construct such a path by starting in x_1, traversing through Y, x_2, C_1, C_2, and proceeding as before. Now suppose that there is no vertex $c_2 \in C_2$ with $c_2 \neq x_3$. In this case either all vertices $c_0 \in Y$ or all vertices $c_1 \in C_1$ have to be adjacent to x_3, because otherwise $K(x_2; c_0, c_1, x_3)$ is a claw. Suppose, without loss of generality, that all vertices of Y are adjacent to x_3. Then we construct the path by starting in x_1, traversing through C_1, x_2, Y, x_3, and proceeding as before.

Case 2. $H = G(\mathrm{N}(x_1) \setminus \{x_2\})$ induces a connected graph.

If x_2 is not adjacent to any of the vertices in H, then H has to be a clique and we can apply the method described in case 1.

Suppose now that x_2 is adjacent to some vertex in H. First, we construct a Hamiltonian path $P' = (y_1, \ldots, y_l)$ in H, which is done as in the proof of Theorem 3.1, since there is no independent triple in H. Now we claim that either x_2 is adjacent to one of y_1 or y_l, or P' does in fact induce a Hamiltonian cycle of H implying again the existence of a path with an end-vertex adjacent to x_2. Indeed, suppose x_2 is not adjacent to any of the endvertices of P'. Then, since G is claw-free, y_1 has to be adjacent to y_l, because otherwise $K(x_1; y_1, y_l, x_2)$ would induce a claw in G. Hence P' induces a Hamiltonian cycle in H.

Using P', we can easily construct a Hamiltonian path in $\mathrm{N}[x_1]$ starting in x_1 and ending in x_2. The rest of the Hamiltonian path of G can be constructed as before. ∎

In fact, we can prove a slightly stronger result. Let $A = \mathrm{N}(x_1) \setminus \bigcup_{j=2}^{k} \mathrm{N}[x_j]$, $B = \mathrm{N}(x_k) \setminus \bigcup_{j=1}^{k-1} \mathrm{N}[x_j]$, and, as usual, $C_i = \mathrm{N}(x_i) \setminus \bigcup_{j=1}^{i-1} \mathrm{N}[x_j]$ $(i \geq 2)$. Each of these sets forms a clique of G.

LEMMA 5.4. *Let G be a claw-free graph, and let $P = (x_1, x_2, \ldots, x_k)$ $(k > 2)$ be an induced dominating path of G such that there is no vertex y in G with $\mathrm{N}(y) \cap P = \{x_1, x_k\}$. Let also P be enlargeable but only to one end, e.g., $A = \emptyset$, $B \neq \emptyset$, and assume that there exists an edge zb with $z \in C_{k-1} \setminus \{x_k\}$ and $b \in B$. Then there is a Hamiltonian path in G that starts in x_1 and ends in x_k and, given the path P, one can construct this Hamiltonian path in linear time.*

Proof. First, we can easily construct a path, starting in x_1 and ending in x_{k-1}, that contains all vertices of $C_2, \ldots, C_{k-2}$. Then we attach to this path a path which starts at x_{k-1}, goes through C_{k-1}, B using all their vertices, and ends in x_k. Finally, we insert the vertices of the neighborhood of x_1 into the obtained path as we have done in the proof of Lemma 5.3. □

THEOREM 5.5. *Let G be a 2-connected claw-free graph with a good pair u, v. Then G has a Hamiltonian cycle and, given the corresponding induced dominating paths, one can construct a Hamiltonian cycle in linear time.*

Proof. Let $P_1 = (u = x_1, \ldots, v = x_k)$, $P_2 = (u = y_1, \ldots, v = y_l)$ be the induced dominating paths, corresponding to the good pair u, v. By the definition of a good pair both k and l are greater than 2. We may also assume that, for any induced dominating path $P = (a_1, \ldots, a_s)$ of G with $s > 2$, no vertex $y \in V \setminus P$ exists such that $\mathrm{N}(y) \bigcap P = \{a_1, a_s\}$. Otherwise, P together with y would form an induced dominating cycle of length at least 4, and we can apply Corollary 5.2 to construct a Hamiltonian cycle of G in linear time.

Let A_1 be the set of vertices a_1 that are adjacent to x_1 but to no other vertex of P_1; let B_1 be the set of vertices b_1 that are adjacent to x_k but to no other vertex of P_1. A_2 and B_2 are defined accordingly for P_2. Of course, each of the sets A_1, A_2, B_1, B_2 forms a clique of G.

First we assume that one of these paths, say, P_1, is nonenlargeable, i.e., $A_1 = \emptyset$, $B_1 = \emptyset$. In this case we do the following. We remove the inner vertices of P_2 from G and get the graph $G - (P_2)$, where (P_2) denotes the inner vertices of P_2. Then, using P_1, we create a Hamiltonian path in $G - (P_2)$ that starts at u and ends at v (Lemma 5.3), and we add (P_2) to this path to create a Hamiltonian cycle of G.

We can use this method for creating a Hamiltonian cycle of G whenever we have two internally disjoint paths P, P' of G both connecting u with v such that one of them is an induced dominating and nonenlargeable path of the graph obtained from G by removing the inner vertices of the other path.

Now we suppose that both paths P_1, P_2 are enlargeable. Because of symmetry we have to consider the following three cases.

Case 1. There exist a vertex $a_1 \in A_1 \setminus A_2$ and a vertex $b_1 \in B_1 \setminus B_2$.

In this case there must be edges from a_1, b_1 to inner vertices y_i, y_j of P_2. Consequently, we can form a new path P_2' by starting in u and traversing through A_1, $y_i, \ldots, y_j$, B_1, v, where $(y_i, \ldots, y_j)$ is the subpath of P_2 between y_i and y_j. Evidently, P_2' contains all vertices of B_1, A_1 and is internally disjoint from P_1, which is nonenlargeable in $G - (P_2')$.

Case 2. $B_1 = B_2$ and either $A_1 = A_2$ or there exists a vertex $a_1 \in A_1 \setminus A_2$.

In this case none of the vertices of $B := B_1 = B_2$ (if $B \neq \emptyset$) has a neighbor in $P_1 \bigcup P_2$ other than v. As G is 2-connected, for some vertex $b \in B$ there has to be a vertex $z \in V \setminus (P_1 \bigcup P_2 \bigcup B)$ with $zb \in E$. Since P_2 dominates G and $z \notin B$, vertex z must be adjacent to a vertex $y \in P_2 \setminus \{v\}$. If z is only adjacent to $y_1 = u$ but to no other vertex of P_2, then z necessarily belongs to A_2 and we can form a new path P_1' by starting in u, using all vertices of A_2, B and ending in v. Again, P_1' is internally disjoint from P_2 and P_2 is nonenlargeable in $G - (P_1')$. If $\mathrm{N}(z) \bigcap P_2 = \{u, v\}$, then we can apply Corollary 5.2.

Therefore, we may assume that z is adjacent to an inner vertex y of P_2. Now, if there exists a vertex $a_1 \in A_1 \setminus A_2$, then a_1 is adjacent to some vertex y' of (P_2) and we can construct a new path P_2' by using u, A_1, $y', \ldots, y$, z, B, v. (If B was empty, then P_2' ends at $\ldots, y', \ldots, y_{l-1}, v$.) This path is internally disjoint from P_1, which is nonenlargeable in $G - (P_2')$. If $A_1 = A_2$, then from the discussion above we may assume that either $A := A_1 = A_2$ is empty or there is a vertex $z' \in V \setminus (P_1 \bigcup P_2 \bigcup A)$ which is adjacent to a vertex of A and has a neighbor y' in (P_2). Hence, we can construct a path P_2' by using u, A, z', y', $\ldots$, y, z, B, v, which is internally disjoint from P_1. (If $z' = z$, then P_2' is constructed by using u, A, z, B, v.)

Case 3. A_2 is strictly contained in A_1, and B_1 is strictly contained in B_2.

Consider vertices $b \in B_1$, $z \in B_2 \setminus B_1$, and $z' \in A_1 \setminus A_2$ and cliques $C_i = \mathrm{N}(x_i) \setminus \bigcup_{j=1}^{i-1} \mathrm{N}[x_j]$ $(i \geq 2)$. If $zz' \in E$, then we can construct a new path P_2' by using u, A_1, z, B_1, v. This path is internally disjoint from P_1, which is nonenlargeable in $G - (P_2')$.

Since $z' \notin A_2$, there must be a neighbor $y' \in (P_2)$ of z'. If vertex b is adjacent to some vertex in $C_{k-1} \setminus \{v\}$, then we construct a new path P_2' by using $u, A_1, y', \ldots, v$. It will be internally disjoint from P_1, which is enlargeable only to one end (at $x_k = v$) in $G - (P_2')$. We are now in the conditions of Lemma 5.4 and can construct a Hamiltonian path of $G - (P_2')$ that starts in u and ends in v. Adding (P_2') to this path, we obtain a Hamiltonian cycle of G.

So, we may assume that $zz' \notin E$ for any vertex $z' \in A_1 \setminus A_2$ and that vertex b is not adjacent to any vertex of $C_{k-1} \setminus \{v\}$. From this we conclude also that $z \notin C_{k-1}$. But since $z \notin B_1$, there must be a neighbor $x_j \in (P_1)$ of z. We choose vertex $x_j \in (P_1)$ with the smallest j. Clearly, $1 < j < k-1$ and $z \in C_j$.

First we define a new induced path $P_1' := (P_1 \setminus \{x_{j+1}, \ldots x_{k-1}\}) \cup \{z\}$ and cliques $A_1' := \mathrm{N}(u) \setminus \bigcup_{x \in P_1' \setminus \{u\}} \mathrm{N}[x]$, $B_1' := \mathrm{N}(v) \setminus \bigcup_{x \in P_1' \setminus \{v\}} \mathrm{N}[x]$. We have $z' \in A_1'$, since otherwise from the construction of P_1', z' would be adjacent to z, and that is impossible.

Note that vertex x_{j+1} is dominated by the path P_2. If it is adjacent to only vertex v from P_2, then $j + 1 = k - 1$ and a claw $K(v; x_{k-1}, y_{l-1}, b)$ arises. Therefore, x_{j+1} must be adjacent to an inner vertex y of P_2. Now we define a new path P_2' by using $u, A_1', y', \ldots, y, x_{j+1}, C_{j+1}, x_{j+2}, \ldots, C_{k-1}, v$. It is internally disjoint from P_1' and contains all vertices of A_1' and C_i $(j + 1 \leq i \leq k - 1)$. It is clear from the construction that the path P_1' dominates the graph $G - (P_2')$. (Every vertex which was not dominated by the path P_1' in G belongs to some set C_i $(j + 1 \leq i \leq k - 2)$.)

It remains to show that the path P_1' is nonenlargeable in $G - (P_2')$. Assume by way of contradiction that it is enlargeable. Since $A_1' \subset (P_2')$, this is possible only if $B_1' \neq \emptyset$. Let p be a vertex of B_1'. Then p does not belong to B_1, since otherwise it should be adjacent to z, which is contained in (P_1'). (Recall that B_1, B_2 are cliques, $B_1 \subset B_2$, and $z \in B_2 \setminus B_1$.) Now, from $p \in B_1' \setminus B_1$ we conclude that the neighbors of p in $P_1 \setminus \{v\}$ are only vertices from $\{x_{j+1}, \ldots, x_{k-1}\}$, i.e., p belongs to a set C_s for some $s \geq j + 1$. Consequently, a contradiction to $C_s \subset (P_2')$ arises.

It is not hard to see that the above method can be implemented to run in linear time. ∎

THEOREM 5.6. *Every* 2*-connected claw-free graph G that contains a dominating pair has a Hamiltonian cycle, and, given a dominating pair, a Hamiltonian cycle can be constructed in linear time.*

Proof. Let v, u be a dominating pair of a 2-connected graph G. If $vu \notin E$, then by Lemma 4.9, there exist two internally disjoint induced paths connecting v and u. Both these paths dominate G, and, therefore, u, v is a good pair of G. Thus, the statement holds by Theorem 5.5.

Now let $vu \in E$. Define sets $A := \mathrm{N}(u) \setminus \mathrm{N}[v]$, $B := \mathrm{N}(v) \setminus \mathrm{N}[u]$, and $S := \mathrm{N}(v) \bigcap \mathrm{N}(u)$. Since G is claw-free, the sets A and B are cliques of G. Notice also that sets A, B, S, and $\{v, u\}$ form a partition of the vertex set of G.

If there is an edge ab in G such that $a \in A$ and $b \in B$, then vertices a, u, v, b induce a 4-cycle which dominates G. Hence, we can apply Corollary 5.2 to get a Hamiltonian cycle of G. Therefore, assume that no such edge exists. But since G is 2-connected, there must be edges ax, by with $x, y \in S$, $a \in A$, and $b \in B$. We distinguish between two cases. Let G_S denote the subgraph of G induced by S.

Case 1. G_S is disconnected.

Then, it consists of two cliques S_1 and S_2. Now, if vertices x, y are in different components of G_S, say, $x \in S_1$ and $y \in S_2$, then $(u, P^{A\setminus\{a\}}, a, x, P^{S_1\setminus\{x\}}, v, P^{B\setminus\{b\}}, b, y, P^{S_2\setminus\{y\}}, u)$ is a Hamiltonian cycle of G. (P^M stands for an arbitrary permutation of the vertices of a set M.) If x, y are in one component, say, S_1, then $(u, P^{A\setminus\{a\}}, a, x, P^{S_1\setminus\{x,y\}}, y, b, P^{B\setminus\{b\}}, v, P^{S_2}, u)$ is a Hamiltonian cycle of G.

Case 2. G_S is connected.

Then, by Theorem 3.1, there exists a Hamiltonian path $P = (s, y_1, \ldots, y_l, t)$ of G_S. Assume that $as, at \notin E$. Then, to avoid a claw $K(u; a, s, t)$, vertices s and t have to be adjacent, giving a Hamiltonian cycle $HC := (s, y_1, \ldots, y_l, t, s)$ of G_S. Vertices x and y split this cycle into two paths $P_1 = (x, \ldots, y)$ and $P_2 = HC \setminus P_1$. Hence, a cycle $(u, P^{A\setminus\{a\}}, a, P_1, b, P^{B\setminus\{b\}}, v, P_2, u)$ is a Hamiltonian cycle of G.

Now, we may assume that a is adjacent to s or t. Analogously, b is adjacent to one of t, s. If a, b are adjacent to different vertices, say, $as, bt \in E$, then $(u, P^{A\setminus\{a\}}, a, P, b, P^{B\setminus\{b\}}, v, u)$ is a Hamiltonian cycle of G. Finally, if a, b are adjacent only to s (similarly, to t), then $(u, P \setminus \{s\}, v, P^{B\setminus\{b\}}, b, s, a, P^{A\setminus\{a\}}, u)$ is a Hamiltonian cycle of G. ∎

THEOREM 5.7. *Every* 2*-connected CN-free graph G has a Hamiltonian cycle, and such a cycle can be found in $O(n + m)$ time.*

Proof. The proof follows from Theorems 4.10, 5.1, and 5.5. ∎

COROLLARY 5.8. *There is a linear time algorithm that for a given (arbitrary)* 2*-connected graph G either finds a Hamiltonian cycle or outputs an induced claw or an induced net of G.*

COROLLARY 5.9. *A Hamiltonian cycle of a* 2*-connected (claw,AT)-free graph can be found in $O(n + m)$ time.*

Remark. Corollary 5.8 implies that every 2-connected unit interval graph has a Hamiltonian cycle, which is, of course, well known (see [24, 20]). The interesting difference of the above algorithm compared to the existing algorithms for this problem on unit interval graphs is that it does not require the creation of an interval model. It also follows from Corollaries 3.3 and 5.8 that both the Hamiltonian path problem and the Hamiltonian cycle problem are linear time solvable on proper circular arc graphs. Note that previously known algorithms for these problems had time bounds $O(m + nlogn)$ [18].

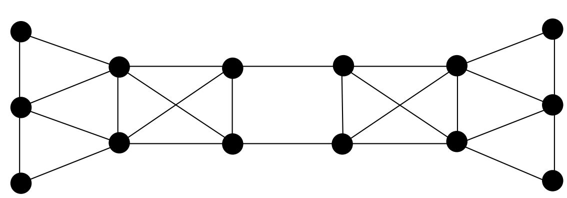

FIG. 5.1. *Claw-free graph, containing a dominating pair and a net.*

It should also be mentioned that Theorems 3.1 and 5.5 do cover a class of graphs that is not contained in the class of CN-free graphs. Figure 5.1 shows a graph that is claw-free, does contain a dominating/good pair and, consequently, a dominating path, but, obviously, it is neither AT-free nor net-free.

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