A Linear-Time Algorithm for Connected r-Domination and Steiner Tree on Distance-Hereditary Graphs

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Abstract: A *distance-hereditary graph* is a connected graph in which every induced path is isometric, i.e., the distance of any two vertices in an induced path equals their distance in the graph. We present a linear time labeling algorithm for the minimum cardinality connected r-dominating set and Steiner tree problems on distance-hereditary graphs. © 1998 John Wiley & Sons, Inc. Networks 31: 177–182, 1998

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1. INTRODUCTION

A *distance-hereditary graph* is a connected graph in which every induced path is isometric, i.e., the distance of any two vertices in an induced path equals their distance in the graph. These graphs were introduced by E. Howorka [11], who gave the first characterizations of distance-hereditary graphs. For instance, a connected graph G is distance-hereditary if and only if every circuit in G of length at least 5 has a pair of chords that cross each other.

A *dominating set* D of a graph $G = (V, E)$ is defined as a set of vertices $D \subseteq V$ such that every vertex in V is either in D or is adjacent to some vertex in D. D is a *connected dominating set* of G iff D dominates G and the subgraph induced by D is connected. For a given graph G and a set $S \subseteq V$ (of *terminal vertices*), a *Steiner tree* is a tree which spans all vertices of S. The *Steiner tree problem* asks for a minimum cardinality Steiner tree.

There are many papers that investigated the Steiner tree problem and the problem of finding minimum dominating sets in graphs with (and without) additional requirements to the dominating sets. The problems are, in general, *NP*-complete. For more special graphs, the situation is sometimes better (for a bibliography on domination, cf. [10]; for a recent survey on special graph classes, cf. [2]). In [5], D'Atri and Moscarini proposed $O(|V||E|)$ algorithms to solve the minimum cardinality connected dominating set and Steiner tree problems on distance-hereditary graphs.

Here, we study the following generalized domination (r-*domination*) problem: Let $(r(v_1), \ldots, r(v_n))$ be a sequence of nonnegative integers which is given together with the input graph. For any two vertices u, v denote by $dist(u, v)$ the length (i.e., number of edges) of a shortest path between u and v in G. A subset $D \subseteq V$ is an r-*dominating set* in G iff for every $v \in V$ there is a $u \in D$ with $dist(u, v) \le r(v)$. D is a *connected r-dominating set* of G iff D r-dominates G and the subgraph induced

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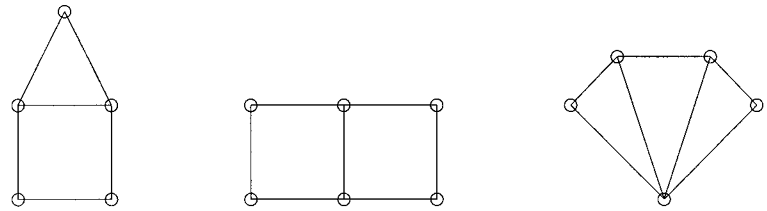

Fig. 1. Forbidden induced subgraphs in a distance-hereditary graph.

by D is connected. The *connected r-domination problem* consists of finding a minimum cardinality connected r-dominating set of G. It is easy to see that the Steiner tree problem is a particular instance of the connected r-domination problem when $r(v) = 0$ for any terminal vertex and $r(v) = \infty$ for all other vertices.

In this paper, we present a linear-time labeling algorithm for the connected r-domination and Steiner tree problems on distance-hereditary graphs. The obtained result not only generalizes but also improves the corresponding result of D'Atri and Moscarini. Recall that efficient algorithms for the connected r-domination problem have been found also for strongly chordal graphs [4] and dually chordal graphs [6] (see also [3]).

2. TERMINOLOGY AND BASIC PROPERTIES

We shall consider finite, simple loopless, undirected, and connected graphs $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the vertex set and E is the edge set of G, and we shall use more-or-less standard terminology from graph theory [8].

Let v be a vertex of G. We denote the *neighborhood* of v, consisting of all vertices adjacent to v, by $N(v)$, and the *closed neighborhood* of v, the set $N(v) \cup \{v\}$, by $N[v]$. The *k-th neighborhood* of v, denoted by $N^k(v)$, is defined as the set of all vertices of distance k to v, i.e., $N^k(v) = \{u \in V : dist(u, v) = k\}$.

A vertex v of G is a *leaf* if $|N(v)| = 1$. Two vertices v and u are *twins* if they have the same neighborhood $[N(v) = N(u)]$ or the same closed neighborhood $(N[v] = N[u])$. *True twins* are adjacent; *false twins* are not. We denote by $\langle S \rangle$ the subgraph of G induced by the vertices of $S \subset V$.

Several interesting characterizations of distance-hereditary graphs in terms of the existence of particular kinds of vertices (leaves, twins) and in terms of metric and neighborhood properties and forbidden configurations were provided by Bandelt and Mulder [1] and by D'Atri and Moscarini [5]. Some algorithmic aspects are considered in [5, 7, 9, 12, 13]. The following propositions list the basic information on distance-hereditary graphs that is needed in the sequel.

Proposition 1 [1, 5]. *For a graph G, the following conditions are equivalent:*

(1) *G is distance-hereditary.*
(2) *The house, domino, fan (see Fig. 1) and the cycles C_k of length $k \geq 5$ are not induced subgraphs of G.*
(3) *Every induced subgraph of G contains a leaf or a pair of twins.*
(4) *For arbitrary vertex x of G and every pair of vertices $v, u \in N^k(x)$, that are in the same connected component of the graph $\langle V \setminus N^{k-1}(x) \rangle$, we have*

$$N(v) \cap N^{k-1}(x) = N(u) \cap N^{k-1}(x).$$

Proposition 2 ([1]). *For any vertex v of distance-hereditary graph G if u, w are vertices in different components of $\langle N^k(v) \rangle$, then $N(u) \cap N^{k-1}(v)$ and $N(w) \cap N^{k-1}(v)$ are either disjoint or one of the two sets is contained in the other.*

3. PRELIMINARY RESULTS

As we already mentioned, every induced subgraph of a distance-hereditary graph contains a leaf or a pair of twins. In the subsequent linear-time algorithm for the minimum connected r-dominating set problem on distance-hereditary graphs, these kinds of vertices turn out to be important.

For the next two lemmas, let x be a leaf in graph G and let y be its neighbor. Let also $G - x = \langle V \setminus \{x\} \rangle$.

Lemma 1 [4, 6]. *Assume that $r(x) > 1$. A subset $D \subseteq V \setminus \{x\}$ is a minimum connected r-dominating set of graph G if D is a minimum connected r'-dominating set of graph $G - x$ with $r'(v) = r(v)$ when $v \neq y$ and $r'(y) = min\{r(y), r(x) - 1\}$.*

Lemma 2 [4, 6]. *Assume that $r(x) = 0$ and $r(u) = 0$ for some vertex $u \in V\setminus\{x\}$. A set D is a minimum connected r-dominating set of graph G if $D = D' \cup \{x\}$, where D' is a minimum connected r'-dominating set of graph $G - x$ with $r'(v) = r(v)$ when $v \neq y$ and $r'(y) = 0$.*

Instead of twins, next we consider some of their generalizations. A vertex set $A \subseteq V$ of a graph $G = (V, E)$ is *homogeneous* iff every vertex in $V\setminus A$ is adjacent to either all or none of the vertices of A. A *proper homogeneous set* is a homogeneous set A such that $|A| \leq |V| - 2$. Observe that two vertices are twins iff they form a homogeneous set of size 2. Evidently, every vertex $v \in V\setminus A$ is equidistant from the vertices of a homogeneous set A.

For the next two lemmas, let $A \subset V$ be a proper homogeneous set of graph G, and x be a vertex of A with $r(x) = min\{r(y) : y \in A\}$.

Lemma 3. *Assume that either $r(x) \geq 2$ or $r(x) = 1$ and there exists a vertex $v \in V\setminus A$ with $dist(v, x) > r(v)$. A subset $D \subseteq (V\setminus A) \cup \{x\}$ is a minimum connected r-dominating set of graph G if D is a minimum connected r-dominating set of graph $\langle (V\setminus A) \cup \{x\}\rangle$.*

Proof. First, we show that every connected r-dominating set D of graph $G' = \langle (V\setminus A) \cup \{x\}\rangle$ is also a r-dominating set of G. Consider a vertex $u \in D$ r-dominating the vertex x in G'. If $u \neq x$ or each vertex $y \in A\setminus\{x\}$ with $r(y) = 1$ is adjacent to x, then u r-dominates all vertices of A. So, assume that $u = x$ and $r(y) = 1$ for some vertex $y \in A\setminus\{x\}$ nonadjacent to x. In this case, we conclude that $r(x) = 1$ and there exists a vertex $v \in V\setminus A$ such that $dist(v, x) > r(v)$. Consider now a path which connects in graph $\langle D\rangle$ the vertex $x \in D$ and some vertex $w \in D$ r-dominating v. Since A is a homogeneous set in G, the vertex from this path which is adjacent to x r-dominates all vertices of A.

Now let D be a minimum connected r-dominating set of graph G. Since A is a homogeneous set of G, every vertex $v \in V\setminus A$ is equidistant from the vertices of A. So, in the case $D \cap A = \{y\}$, the set $D\setminus\{y\} \cup \{x\}$ is a connected r-dominating set of graph G. Analogously, if $|D \cap A| \geq 2$, then either $D\setminus A \cup \{x\}$, when $D\setminus A \neq \emptyset$, or $D\setminus A \cup \{x, z\}$ with $z \in N(x)\setminus A$, otherwise, is a connected r-dominating set of graph G.

Thus, there is a minimum connected r-dominating set D of graph G such that $D \subseteq (V\setminus A) \cup \{x\}$. Since graph G' is a distance-preserving subgraph of G, set D is also a connected r-dominating set of graph G'. ∎

Lemma 4. *Assume that $r(x) = 0$ and $r(u) = 0$ for some vertex $u \in V\setminus A$. A set D is a minimum connected r-dominating set of graph G if $D = D' \cup \{y \in A : r(y) = 0\}$, where D' is a minimum connected r-dominating set of graph $\langle V\setminus A \cup \{x\}\rangle$.*

Proof. Evidently, D is a connected r-dominating set of graph G whenever D' is a connected r-dominating set of graph $G' = \langle V\setminus A \cup \{x\}\rangle$. Also, if D is a minimum connected r-dominating set of G, then a set $D' = D\setminus\{y \in A\setminus\{x\} : r(y) = 0\}$ is a connected r-dominating set of G' because every vertex of $V\setminus A$ is equidistant from the vertices of the homogeneous set A. ∎

For the next two lemmas, let x be a vertex of graph G such that $N(x)$ forms a proper homogeneous set in G, and y be a vertex of $N(x)$ with $r(y) = min\{r(z) : z \in N(x)\}$.

Lemma 5. *Assume that $r(x) > r(y)$. A set $D \subseteq V\setminus\{x\}$* is a minimum connected *r*-dominating set of graph G if D is a minimum connected *r*-dominating set of graph $G - x$.

Proof. Again, every connected r-dominating set D of graph $G - x$ is also r-dominating in G. Assume now that D is a minimum connected r-dominating set of graph G. Let z be any vertex of G with $dist(x, z) = 2$. Then, either D when $x \notin D$ or $D\setminus\{x\} \cup \{z\}$ otherwise is a minimum connected r-dominating set of graph $G - x$. Since $N(x)$ is a homogeneous set, $N(x) \subset N(z)$ holds. ∎

Lemma 6. *Assume that $r(x) = 0$ and $r(u) = 0$ for some vertex $u \in V\setminus N[x]$. A set D is a minimum connected r-dominating set of graph G if $D = D' \cup \{x\}$,* where D′ is a minimum connected r'-dominating set of graph $G - x$ with $r'(v) = r(v)$ *when* $v \neq y$ *and* $r'(y) = 0$.

Proof. Obviously, if a set D' is a connected r'-dominating set of graph $G - x$, then the set $D = D' \cup \{x\}$ is a connected r-dominating set of graph G.

Let D be a minimum connected r-dominating set of graph G. Since x and u must be in D, there exist two vertices $v \in N(x)$ and $z \in V\setminus N[x]$ with $dist(x, z) = 2$ belonging also to set D. So, every vertex w r-dominated in G by vertex x is r-dominated also by an arbitrary vertex of $N(x)$ if $w \in V\setminus N[x]$ or by vertex z if $w \in N(x)$. Hence, the set $D\setminus\{x, v\} \cup \{y\}$ when $r(v) > 0$ or $D\setminus\{x\}$ when $r(v) = r(y) = 0$ is a connected r'-dominating set of graph $G - x$. ∎

These lemmas will be used in the correctness proof of the subsequent algorithm which has a structure similar to the linear-time recognition algorithm presented in [9] and the linear-time algorithm for finding a minimum r-dominating clique of a distance-hereditary graph presented in [7].

4. THE ALGORITHM

Algorithm CRD (Find a minimum connected r-dominating set of a distance-hereditary graph)

Input: A distance-hereditary graph $G = (V, E)$ and an n-tuple $(r(v_1), \ldots, r(v_n))$ of nonnegative integers.
Output: A minimum connected r-dominating set CD of G.

begin
(1) **if** for all $v \in V$ $r(v) > 0$ **then**
(2) for an arbitrary vertex $u \in V$ build its i-th neighborhoods $N^1(u), N^2(u), \ldots, N^k(u)$;
(3) **for** $i = k, k - 1, \ldots, 2$ **do**
(4) find the connected components $A_1, A_2, \ldots, A_p$ of $N^i(u) \cap V$;
(5) in each component A_j pick a vertex x_j such that $r(x_j) = min\{r(y) : y \in A_j\}$;
(6) order the vertices of $X = \{x_1, x_2, \ldots, x_p\}$ by increasing degree $d'(x_j) = |N(x_j) \cap N^{i-1}(u)|$;
(7) **for all** vertices $x_j \in X$ taken by increasing degree $d'(x_j)$ **do**
(8) **if** $(r(x_j) \geq 2$ or $r(x_j) = 1$ and $\exists v \in V \setminus A_j$ $dist(x_j, v) > r(v))$ **then**
(9) delete from V all vertices of $A_j \setminus \{x_j\}$;
(10) in set $B = N(x_j) \cap N^{i-1}(u) \cap V$ pick a vertex y such that $r(y) = min\{r(z) : z \in B\}$;
(11) **do case**
(12) **case** $r(y) = 1$ and $r(x_j) \geq 2$;
(13) delete from V vertex x_j;
(14) **case** $r(y) \geq 2$ or $r(y) = r(x_j) = 1$ and $\exists v \in V \setminus B$ $dist(y, v) > r(v)$;
(15) delete from V all vertices of $(B \cup \{x_j\}) \setminus \{y\}$;
(16) put $r(y) := min\{r(x_j) - 1, r(y)\}$;
(17) **if** $r(y) = 0$ **then goto** *outloop* **endif**
(18) **otherwise** /* $r(y) = r(x_j) = 1$ and vertex y r-dominates all vertices of $V \setminus B$*/
(19) **if** (in set B there is a vertex w adjacent to all vertices $v \in B \setminus \{w\}$ with $r(v) = 1$)
(20) **then** $CD := \{w\}$ **else** $CD := \{x_j, y\}$ **endif**
(21) **stop**
(22) **endcase**
(23) **else** /* $r(x_j) = 1$ and vertex x_j r-dominates all vertices of $V \setminus A_j$ */
(24) **if** (in set A_j there is a vertex w adjacent to all vertices $v \in A_j \setminus \{w\}$ with $r(v) = 1$)
(25) **then** $CD := \{w\}$
(26) **else** choose in set $B = N(x_j) \cap N^{i=1}(u) \cap V$ a vertex w adjacent to all vertices $v \in B \setminus \{w\}$ with $r(v) = 1$;
(27) **if** (there is such a vertex which r-dominates also all vertices from $V \setminus B$) **then**
(28) $CD := \{w\}$
(29) **else** $CD := \{x_j, y\}$, where y is an arbitrary vertex from B;
(30) **endif**
(31) **endif**
(32) **stop**
(33) **endif**
(34) **endfor**
(35) **endfor**
(36) **stop** with output $CD := \{u\}$
(37) **else** /* now $r(u) = 0$ for some $u \in V$ */
outloop:
(38) in the rest of graph G build the i-th neighborhoods $N^1(u), \ldots, N^k(u)$ of vertex u with $r(u) = 0$;
(39) **for** $i = k, k - 1, \ldots, 2$ **do**
(40) repeat the steps (4), (5), (6);
(41) **for all** vertices $x_j \in X$ taken by increasing degree $d'(x_j)$ **do**
(42) delete from V all vertices of A_j;
(43) in set $B = N(x_j) \cap N^{i-1}(u) \cap V$ pick a vertex y such that $r(y) = min\{r(z) : z \in B\}$;
(44) **if** $r(x_j) \geq 1$ **then**
(45) **if** $r(y) \geq 1$ **then** delete from V all vertices $v \in B \setminus \{y\}$;
(46) put $r(y) := min\{r(x_j) - 1, r(y)\}$
(47) **endif**
(48) **else** /* $r(x_j) = 0$ */
(49) $CD := CD \cup \{x \in A_j : r(x) = 0\}$;
(50) put $r(y) := 0$
(51) **endif**
(52) **endfor**
(53) **endfor** /* now $N[u] = V$ */
(54) $CD := CD \cup \{v \in N[u] : r(v) = 0\}$;
(55) CD is a minimum connected r-dominating set of G
(56) **endif**
end

Theorem. *Algorithm CRD is correct and works in linear time $O(|E|)$.*

Proof. The correctness proof of this algorithm is similar to the correctness proof of the correspondent algorithm from [7] for the minimum r-dominating clique problem.

The time bound of the algorithm is obviously linear [it is enough to note that during the work of the algorithm the condition $r(x_j) = 1$ on line (8) will be true only once].

The correctness proof is based on Lemmas 1–6 and on the following claims:

(a) every connected component A_j of $N^i(u)$ [see line (4)] is a proper homogeneous set of the current graph;
(b) if x is a vertex of $N^i(u)$ with minimal degree $d'(x)$, then $B = N(x) \cap N^{i-1}(u)$ is a homogeneous set of the current graph.

Assertion (a) is a trivial consequence of Proposition 1(4). To prove assertion (b), we suppose that a vertex $z \in V \setminus B$ is adjacent to a vertex v of B and nonadjacent to another vertex $y \in B$. From the minimality of degree $d'(x)$, Proposition 2 and Proposition 1, we immediately obtain that vertex z must be in $N^{i-1}(u) \setminus N(x)$ and there exists a vertex $w \in N^{i-2}(u)$ such that $y, v, z \in N(w)$. But in this case, vertices x, y, v, z, w induce a forbidden subgraph (see Fig. 1), a contradiction.

Thus, by the lemmas, lines (8)–(17) and (42)–(51) of the algorithm are correct. After the i-th step, we delete all vertices from $N^i(u)$ and some vertices from $N^{i-1}(u)$ with updating the radii of other vertices. So, we reduce the initial minimum connected r-dominating set problem to the same problem on a smaller graph or in lines (18)–(30) find the solution of our problem.

If the case "$r(y) = r(x_j) = 1$ and vertex y r-dominates all vertices of $V \setminus B$" [see line (18)] occurs, then the vertices x_j, y form a connected r-dominating set of graph G. It remains to check whether graph G has a single vertex r-dominating it. If such a vertex exists, it must be in B. Since every vertex from $V \setminus B$ is equidistant from the vertices of B, it is necessary (and sufficient) to check the existence of vertex $w \in B$ r-dominating all vertices of $B \setminus \{w\}$.

If the case "$r(x_j) = 1$ and vertex x_j r-dominates all vertices of $V \setminus A_j$" [see line (23)] occurs, then the vertices x_j, y, where y is an arbitrary vertex from $N(x_j) \cap N^{i-1}(u)$, form a connected r-dominating set of graph G. Again, it remains to check whether graph G has a single vertex r-dominating it. If such a vertex exists, it must be in $N[x_j] \subset A_j \cup B$.

Thus, the first part of the algorithm [lines (1)–(35)] is correct. If this part ends up on line (36) [without any vertex v with $r(v) = 0$], then, evidently, u is a single vertex which r-dominates G and so line (36) is also correct. The second part of the algorithm repeats the lines of the first part, but with the additional condition $r(u) = 0$.

After line (53), we obtain that all remaining vertices are adjacent to vertex u with $r(u) = 0$. The algorithm simply collects all vertices v with $r(v) = 0$ in CD. Thus, the set CD is a minimum connected r-dominating set of graph G. ∎

Now consider the problem of finding a minimum cardinality Steiner tree on distance-hereditary graphs. As we already mentioned, this problem is a particular instance of the connected r-domination problem. However, below, we present a direct algorithm for the Steiner tree problem which repeats the essence of the Algorithm *CRD* without using a radius function.

Algorithm ST (Find a minimum cardinality Steiner tree of a distance-hereditary graph)

Input: A distance-hereditary graph $G = (V, E)$ and a set $S \subset V$ of terminal vertices.
Output: A minimum cardinality Steiner tree $T(S, G)$.

begin
for an arbitrary vertex $u \in S$ build its i-th neighborhoods $N^1(u), N^2(u), \ldots, N^k(u)$;
for $i = k, k-1, \ldots, 2$ **do**
  if $S \cap N^i(u) \neq \emptyset$ **then**
    find the connected components $A_1, A_2, \ldots, A_p$ of $N^i(u)$;
    in each component A_j pick an arbitrary vertex x_j;
    order these components by increasing degree $d'(A_j) = |N(x_j) \cap N^{i-1}(u)|$;
    for all components A_j taken by increasing degree $d'(A_j)$ **do**
      put $B := N(x_j) \cap N^{i-1}(u)$;
      if $(S \cap A_j \neq \emptyset$ and $S \cap B = \emptyset)$ **then**
        add an arbitrary vertex y from B to set S;
      endif
    endfor
  endif
endfor
$T(S, G) :=$ a spanning tree of graph $\langle S \rangle$;
end

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