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r -Dominating cliques in graphs with hypertree structure

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Abstract

Let $G = (V, E)$ be an undirected graph and r be a vertex weight function with positive integer values. A subset (clique) $D \subseteq V$ is an r -dominating set (clique) in G iff for every vertex $v \in V$ there is a vertex $u \in D$ with $\text{dist}(u, v) \leq r(v)$. This paper contains the following results:

(i) We give a simple necessary and sufficient condition for the existence of r -dominating cliques in the case of Helly graphs and of chordal graphs.

(ii) For Helly graphs with an r -dominating clique the minimum size of an r -dominating clique coincides with the minimum size of any r -dominating set.

(iii) We give a linear-time algorithm for finding a minimum r -dominating clique in dually chordal graphs (a generalization of strongly chordal graphs).

These results improve and extend earlier results on r -dominating cliques in chordal and strongly chordal graphs.

1. Introduction — notions and basic properties

The central notion of this paper is that of an r -dominating clique of a graph. We give necessary and sufficient conditions for the existence of such cliques in Helly and in chordal graphs, and for dually chordal graphs we give a linear-time algorithm for finding a minimum size r -dominating clique if there is one.

Tree structure of graphs and hypergraphs is sometimes helpful in designing efficient graph algorithms. A good and well-known example is the chordal graphs and their maximal cliques. Repeated deletion of simplicial vertices in chordal graphs (which is a generalization of the repeated deletion of leaves in a tree) leads to perfect elimination orderings. Some linear-time algorithms work along such orderings.

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(cf. e.g. [7, 16]). The special structure of a strongly chordal graph G is reflected by its simple elimination ordering. The linear ordering of neighbourhoods $\{N_i[u]: u \in N_i[v_i]\}$, $i \in \{1, \dots, n\}$, however, can be generalized by only demanding that each of these sets of neighbourhoods has a maximum. For some problems this is sufficient for obtaining an efficient algorithm. This leads to graphs with a maximum neighbourhood ordering recently introduced and characterized in several papers [1, 5, 14, 26]. Ref. [24] also introduces maximum neighbourhoods but only in connection with chordal graphs (chordal graphs with maximum neighbourhood ordering are called there *doubly chordal graphs*).

A vertex $u \in N[v]$ is a *maximum neighbour of v* iff for all $w \in N[v]$ $N[w] \subseteq N[u]$ holds (note that $u = v$ is not excluded). A linear ordering (v_1, v_2, \dots, v_n) of the vertex set V of a graph G is a *maximum neighbourhood ordering* of G iff for all $i \in \{1, \dots, n\}$ v_i has a maximum neighbour u_i in G_i : for all $w \in N_i[v_i]$ $N_i[w] \subseteq N_i[u_i]$ holds.

There is a close connection between chordality and the existence of a maximum neighbourhood ordering which can be expressed in terms of hypergraphs (for hypergraph notions we follow [2]).

Let $\mathcal{N}(G) = \{N[v]: v \in V\}$ be the *neighbourhood hypergraph* of G and let $\mathcal{C}(G) = \{C: C \text{ is a maximal clique in } G\}$ be the *clique hypergraph* of G .

For any two vertices u, v denote by $\text{dist}(u, v)$ the length (i.e. number of edges) of a shortest path between u and v in G .

Let v be a vertex of G . The *disk* centered at v with radius k is the set of all vertices having distance at most k to v : $N^k[v] = \{u: u \in V \text{ and } \text{dist}(u, v) \leq k\}$. Let $\mathcal{D}(G) = \{N^k[v]: v \in V, k \text{ a positive integer}\}$ be the *disk hypergraph* of G .

Now let \mathcal{E} be a hypergraph with underlying vertex set V . The *dual hypergraph* \mathcal{E}^* has \mathcal{E} as its vertex set and for all $v \in E$ an edge of the form $\{e \in \mathcal{E}: v \in e\}$. The *line graph* $L(\mathcal{E}) = (\mathcal{E}, E)$ of \mathcal{E} is the intersection graph of \mathcal{E} , i.e. $ee' \in \mathcal{E}$ iff $e \cap e' \neq \emptyset$.

A hypergraph \mathcal{E} has the *Helly property* iff any subfamily $\mathcal{E}' \subseteq \mathcal{E}$ of pairwise intersecting edges has a nonempty intersection.

A hypergraph \mathcal{E} with underlying vertex set V is a *hypertree* iff there is a tree $T = (V, E)$ such that for all $e \in \mathcal{E}$ the induced subgraph $T(e)$ is connected, i.e. a subtree of T . A hypergraph \mathcal{E} is a *dual hypertree* iff \mathcal{E}^* is a hypertree.

It is well-known that a hypergraph \mathcal{E} is a hypertree iff it has the Helly property and its line graph is chordal (cf. [15, 18]) and, as a consequence, G is chordal iff $\mathcal{C}(G)$ is a dual hypertree (cf. [19] for further sources).

Subsequently we use the following equivalence [14], cf. also [4]. Let $G = (V, E)$ be a graph. Then the following conditions are equivalent:

- (i) G has a maximum neighbourhood ordering.
- (ii) $\mathcal{N}(G)$ is a dual hypertree.
- (iii) $\mathcal{N}(G)$ is a hypertree.
- (iv) $\mathcal{C}(G)$ is a hypertree.
- (v) $\mathcal{D}(\mathcal{C})$ is a hypertree.

Condition (iv) now justifies the name ‘dually chordal graphs’ for graphs with maximum neighbourhood ordering: A graph G is *dually chordal* iff $\mathcal{C}(G)$ is a hypertree.

(Note that unlike chordal graphs the dually chordal graphs are in general not perfect because by adding a new vertex adjacent to all vertices we obtain a graph with maximum neighbourhood ordering from an arbitrary graph. Thus the property to be dually chordal is not hereditary.) It should be remarked that it can be recognized in linear time whether a given graph is dually chordal. Section 5 recalls an algorithm for the construction of a maximum neighbourhood ordering of a given dually chordal graph.

A graph G is a *Helly graph* (sometimes called *disk-Helly graph*) iff $\mathcal{D}(G)$ has the Helly property. Thus due to condition (v) and the characterization of hypertrees mentioned above Helly graphs are a generalization of dually chordal graphs. Several of our results even hold for Helly graphs.

There are many papers investigating the problem of finding minimum dominating sets in graphs with (and without) additional requirements to the dominating sets. The problems are in general NP-complete. For more special graphs the situation is sometimes better (for a bibliography on domination cf. [21], for a recent survey on special graph classes cf. [3]).

In [7] efficient solutions of the r -domination problem and variants on strongly chordal graphs are studied: Let $(r(v_1), \dots, r(v_n))$ be a sequence of positive integers $r(v_i) \geq 0$ which is given together with the input graph. For a vertex set $M \subseteq V$ a subset $D \subseteq V$ is *r -dominating for M* in G iff for all $v \in M$ there is a $u \in D$ with $\text{dist}(u, v) \leq r(v)$. If the set D r -dominates V and is a clique (connected) then D is an *r -dominating clique (connected r -dominating set)* for the graph G .

Refs. [22, 23] investigate the dominating clique problem on strongly chordal and chordal graphs. The main results of the present paper are the following.

(1) Even for Helly graphs there exists a simple necessary and sufficient condition for the existence of dominating cliques, and the condition known from [23] on chordal graphs is shown to be valid also in the more general case of r -domination.

(2) For Helly graphs with an r -dominating clique the minimum size of an r -dominating clique coincides with the minimum size of any r -dominating set.

(3) We give a linear-time algorithm for the r -dominating clique problem on graphs with maximum neighbourhood orderings which improves and generalizes the results of [23] for strongly chordal graphs since for these graphs as a part of the input a simple elimination ordering is assumed for which no linear-time construction is known if only the graph is given.

(4) We investigate some related problems on chordal graphs and on dually chordal graphs.

2. The existence of r -dominating cliques in Helly and in chordal graphs

It is known from [6] that the problem whether a given graph has a dominating clique is NP-complete even for weakly chordal graphs (a graph G is *weakly chordal* iff G does not contain induced cycles C_k of length $k \geq 5$ and no complements \bar{C}_k of such cycles — cf. [20]).

In this section we give a simple necessary and sufficient condition for the existence of an r -dominating clique in a given graph, if the graph is Helly or chordal. Note, however, that this condition does not yield an efficient solution for finding a minimum size r -dominating clique if one exists.

Theorem 1. *Let $G = (V, E)$, $|V| = n$, be a Helly graph with n -tuple $(r(v_1), \dots, r(v_n))$ of positive integers and $M \subseteq V$ be any subset of V . Then M has an r -dominating clique C iff for every pair of vertices $v, w \in M$, the inequality*

$$\text{dist}(v w) \leq r(v) + r(w) + 1$$

holds. Moreover such a clique C can be determined within time $O(|M| \cdot |E|)$.

Proof. ‘ \Rightarrow ’: If such a clique C exists for M then obviously the inequality is fulfilled.

‘ \Leftarrow ’: Consider the set

$$X_M = \{x: x \in V \text{ and } \text{dist}(x, v) \leq r(v) + 1 \text{ for all } v \in M\} = \bigcap_{v \in M} N^{r(v)+1}[v].$$

Since the disks of the family $\mathcal{M} = \{N^{r(v)+1}[v]: v \in M\}$ have pairwise nonempty intersection and G is Helly there is a common vertex of all these disks, i.e. $X_M \neq \emptyset$.

Consider an \subseteq -maximal clique $C \subseteq X_M$. We show that C is r -dominating. If there is a vertex $y \in M$ such that $\text{dist}(y, C) = r(y) + 1$ then the disks $\{N[v]: v \in C\} \cup \{N^{r(v)+1}[v]: v \in M\} \cup \{N^{r(y)}[y]\}$ pairwise intersect. By the Helly property there exists a vertex $x \in X_M$ which has an edge to all vertices of C and has distance $r(y)$ to y . Thus C is not maximal.

Time bound: First determine with $O(|M| \cdot |E|)$ steps the distances of every vertex $v \in M$ to all other vertices of the graph. Then X_M can be found within time $O(|M| \cdot |V|)$ which is bounded by $O(|M| \cdot |E|)$. A maximal clique $C \subseteq X_M$ can be found within time $O(|E|)$. \square

Now we give an analogous theorem for the case of chordal graphs for which we need the following notion: A subset $S \subseteq V$ is m -convex iff S contains every vertex on every chordless path between vertices of S . We use the following well-known properties:

Lemma 1 (Farber and Jamison [17]). *Any disk $N^r[v]$ of a chordal graph $G = (V, E)$ is m -convex.*

Lemma 2 (Chang and Nemhauser [8] and Voloshin [28]). *Let $G = (V, E)$ be a chordal graph. If all vertices v_i of a clique $C = \{v_1, \dots, v_k\}$ have the same distance from a vertex $v \in V$ then there is a common neighbour u of all elements of C which has distance $\text{dist}(v, C) - 1$ to v .*

Furthermore we need the following notion: For a subset $S \subseteq V$ and a vertex $v \in V$ let

$$\text{dist}(v, S) = \min \{ \text{dist}(v, w) : w \in S \} \quad \text{and}$$

$$\text{proj}(v, S) = \{ u \in S : \text{dist}(v, u) = \text{dist}(v, S) \}$$

(the *metric projection* of v to S). For a subset $X \subseteq V$ let $\text{proj}(X, S) = \bigcup_{v \in X} \text{proj}(v, S)$.

Lemma 3. *Let $G = (V, E)$ be a (not necessarily chordal) graph. The metric projection $\text{proj}(C, S)$ of any clique C to an m -convex set S of G is a clique.*

Proof. Let C be a clique in G and $u_1, u_2 \in \text{proj}(C, S)$. We consider only the case that there are vertices $v_1, v_2 \in C \setminus S$ with $u_i \in \text{proj}(v_i, S)$, $i \in \{1, 2\}$. Let P_i be a shortest path between u_i and v_i , $i \in \{1, 2\}$. Since $u_i \in \text{proj}(v_i, S)$ no other vertex of P_i belongs to S . Thus the neighbour x_i of u_i in P_i is not in S .

Note that because of the m -convexity of S any path P between vertices $x, y \in S$ containing a vertex from $V \setminus S$ is not chordless.

The path Q_1 between u_1 and u_2 consisting of P_1 , the edge v_1v_2 and P_2 is thus not chordless. Let the edge $e = \{r, s\}$ with r on P_1 and s on P_2 be a chord of Q_1 . If this is the edge u_1u_2 we are done. If not then construct a shorter path Q_2 between u_1 and u_2 from Q_1 using the part of P_1 between u_1 and r , the chord e , and the part of P_2 between s and u_2 . Q_2 is again not chordless and the construction can be repeated until a chord u_1u_2 is found. \square

Lemma 4. *Let $G = (V, E)$ be a (not necessarily chordal) graph. For any m -convex set S and vertices $v \in V$, $u \in S$ of G there is a vertex $w \in \text{proj}(v, S)$ such that $\text{dist}(v, u) = \text{dist}(v, w) + \text{dist}(w, u)$, i.e. w lies on a shortest path between v and u .*

Proof. Assume to the contrary that there are vertices $v \in V$, $u \in S$ such that there is no $w \in \text{proj}(v, S)$ with $\text{dist}(v, u) = \text{dist}(v, w) + \text{dist}(w, u)$. Obviously in this case $v \notin S$ holds since for $v \in S$, $u \in S$ $w = v$ is a suitable choice.

Let P be a shortest path between u and v , let x be the last vertex from u on P inside S , $x \notin \text{proj}(v, S)$ and let y be the neighbour of x on P outside S . Since $x \notin \text{proj}(v, S)$ $\text{dist}(x, v) > \text{dist}(v, S)$ holds. Let w be a vertex of $\text{proj}(v, S)$ and let P' be a shortest path between v and w , i.e. $|P'| = \text{dist}(v, w) = \text{dist}(v, S)$.

Then $Q = PP'$ is a path between $u \in S$ and $w \in S$ with a vertex $v \notin S$ and because of the convexity of S this path is not chordless.

Case 1: Q has a chord e inside S . This means that $e = zw$ for some vertex $z \in S$ on P since w is the only vertex of S on P' . Let P_z be the path between u and z on P . Then the path $P_z e P'$ is a path between u and v which is either shorter than P if $z \neq x$ or is shortest path containing a vertex w from $\text{proj}(v, S)$ — a contradiction.

Case 2: Assume that Q has no chord inside S . Let $e = rs$ be a chord of Q with r on P and s on P' . Then as in the proof of Lemma 3 consider the shorter path Q' between u and w which consists of the initial part of P between u and r , the chord $e = rs$ and

the initial part of P' between s and w . At least one of the vertices r, s is not in S . Thus the same construction can be repeated and finally leads to a chord inside S — a contradiction. \square

Theorem 2. *Let $G = (V, E)$ be a chordal graph with n -tuple $(r(v_1), \dots, r(v_n))$ and $M \subseteq V$ be any subset of V . Then M has an r -dominating clique iff for every pair of vertices $v, w \in M$, the inequality*

$$\text{dist}(v, w) \leq r(v) + r(w) + 1$$

holds. Moreover, such a clique can be determined within time $O(|M| \cdot |E|)$.

Proof. ‘ \Rightarrow ’: Obvious.

‘ \Leftarrow ’: Assume that (v_1, \dots, v_n) is an ordering of V such that M consists of the first $|M|$ vertices of this ordering. Let i be the largest index for which a clique C exists such that $\text{dist}(v_j, C) \leq r(v_j)$ for every $v_j \in M$ with $1 \leq j \leq i$. If $i < |M|$ then for $v_{i+1} \in M$ $\text{dist}(v_{i+1}, C) \geq r(v_{i+1}) + 1$ holds.

Now consider the projection of C to the set $N^{r(v_{i+1})+1}[v_{i+1}]$. Due to the m -convexity of disks in chordal graphs the projection $\text{proj}(C, N^{r(v_{i+1})+1}[v_{i+1}])$ induces a clique according to Lemma 3. Denote this clique by C' . For all $j, j \leq i$, consider a vertex $u_j \in C \cap N^{r(v_j)}[v_j]$ and $w_j \in N^{r(v_j)}[v_j] \cap N^{r(v_{i+1})+1}[v_{i+1}]$. According to Lemma 4 there is a vertex $u'_j \in C'$ such that

$$\text{dist}(u_j, w_j) = \text{dist}(u_j, u'_j) + \text{dist}(u'_j, w_j)$$

holds. Due to the m -convexity of disks for all $j, j \leq i, u'_j \in N^{r(v_j)}[v_j]$ is fulfilled, i.e. for all $j \leq i$ $\text{dist}(v_j, C') \leq r(v_j)$. Furthermore, $\text{dist}(u'_j, v_{i+1}) = r(v_{i+1}) + 1$ holds for all $j \leq i$. But then according to Lemma 2 there is a vertex u'_{i+1} which is a common neighbour of all vertices of C' and which has the distance $r(v_{i+1})$ from v_{i+1} . This is a contradiction to the maximality of i .

Time bound: As in the proof of Theorem 1 first determine within $O(|M| \cdot |E|)$ steps the distances of every vertex $v \in M$ to all other vertices of the graph.

ith step. If the distances of v_{i+1} to all other vertices are known in advance then the projection of clique C to the disk $N^{r(v_{i+1})+1}[v_{i+1}]$ can be determined within $O(|E|)$ steps. The vertices u'_{i+1} can be determined also within $O(|E|)$ steps. There are at most $|M|$ such steps, and each step requires at most $O(|E|)$ time. \square

Let $\text{diam}(G) = \max\{\text{dist}(u, v) : u, v \in V\}$.

Corollary 1. *Let $r(v) = k$ for all $v \in V$ and let G be a Helly or a chordal graph. Then G has a k -dominating clique iff $\text{diam}(G) \leq 2k + 1$.*

Corollary 2. *Let G be a Helly or a chordal graph. Then G has a dominating clique iff $\text{diam}(G) \leq 3$.*

(For the case of chordal graphs Corollary 2 occurs already in [23].)

3. A relation between the r -domination and the r -dominating clique problem

Let

$$\gamma_r(G) = \min\{|D|: D \text{ an } r\text{-dominating set in } G\},$$

$$\gamma_{r\text{-conn}}(G) = \min\{|D|: D \text{ a connected } r\text{-dominating set in } G\},$$

$$\gamma_{r\text{-clique}}(G) = \min\{|D|: D \text{ an } r\text{-dominating clique in } G\}.$$

It is clear that for every graph G which has an r -dominating clique

$$\gamma_r(G) \leq \gamma_{r\text{-conn}}(G) \leq \gamma_{r\text{-clique}}(G)$$

holds but in general the parameters do not coincide. The next theorem shows the coincidence for Helly graphs which have an r -dominating clique. Hereby for an r -dominating set D and a vertex $v \in D$ let u be a *personal r -neighbour* of v if $N^{r(u)}[u] \cap D = \{v\}$. It is obvious that for a minimum r -dominating set D each member $v \in D$ has a personal r -neighbour. These personal r -neighbours play an important role in the subsequent construction.

Theorem 3. *Let D be a minimum r -dominating set of a Helly graph G which has an r -dominating clique. Then there is an r -dominating clique D' with $|D'| = |D|$.*

Proof. Let $D = \{v_1, \dots, v_k\}$. We construct an r -dominating clique $D' = \{u_1, \dots, u_k\}$ stepwise in k steps as follows:

Step 1. Let u_1 be a vertex from the common intersection of all disks

$$\{N^{r(v)}[v]: N^{r(v)}[v] \cap D = \{v_1\}\} \cup \{N^{r(v)+1}[v]: v \in V\}.$$

Step 2: Let u_2 be a vertex from the common intersection of all disks

$$\{N^{r(v)}[v]: N^{r(v)}[v] \cap \{u_1, v_2, \dots, v_k\} = \{v_2\}\} \cup \{N^{r(v)+1}[v]: v \in V\} \cup N[u_1]$$

⋮

Step i : Let u_i be a vertex from the common intersection of all disks

$$\begin{aligned} &\{N^{r(v)}[v]: N^{r(v)}[v] \cap \{u_1, u_2, \dots, u_{i-1}, v_i, v_{i+1}, \dots, v_k\} = \{v_i\}\} \\ &\cup \{N^{r(v)+1}[v]: v \in V\} \cup \{N[u]: u \in \{u_1, u_2, \dots, u_{i-1}\}\}, \quad i \leq k, \end{aligned}$$

and so on until the k th step is carried out.

Now to the correctness of this construction:

In step 1 all disks have a pairwise nonempty intersection. Since G has an r -dominating clique it is obvious that for every pair of vertices $u, v \in V$ $N^{r(u)}[u] \cap N^{r(v)+1}[v] \neq \emptyset$. Furthermore for all personal r -neighbours u, v of $v_1 \in D$

$v_1 \in N^{r(u)}[u] \cap N^{r(v)}[v]$. Thus there is a vertex u_1 in the common intersection of all disks of step 1.

In step 2 and in further steps because of the same reasons as in step 1 the disks have pairwise nonempty intersection. Additionally we have to consider neighbourhoods $N[u_i], i \in \{1, 2, \dots, k\}$. In step 2 the intersection of all disks with $N[u_1]$ is nonempty due to the choice of u_1 . In the subsequent step i the pairwise intersection of $N[u_i]$ and $N[u_j], j \neq i$ and the intersections of these neighbourhoods with the disks are pairwise nonempty due to the choice of u_i . Thus in step i due to the Helly property there is a vertex u_i from the common intersection of all disks of step i . The set $D' = \{u_1, \dots, u_k\}$ defined in this way is obviously an r -dominating clique with $|D'| = |D|$. \square

Note that the procedure above can be carried out in at most $O(n^3)$ steps.

Corollary 3. *Let G be a Helly graph which has an r -dominating clique. Then*

$$\gamma_r(G) = \gamma_{r\text{-conn}}(G) = \gamma_{r\text{-clique}}(G).$$

(For the case of strongly chordal graphs and $r(v) = 1$ for all $v \in V$ this is contained already in [23].)

Theorem 3 has some important consequences: A graph $G = (V, E)$ is a *bridged graph* iff each cycle C of length at least 4 contains two vertices whose distance from each other in G is strictly less than their distance in C .

An induced subgraph G' in a graph G is an *isometric subgraph* in G iff the distance of any two vertices in G' is equal to their distance in G . A k -sun S_k is a graph with $2k$ vertices consisting of two disjoint sets $W = \{w_1, \dots, w_k\}, U = \{u_1, \dots, u_k\}$ such that W is independent, U forms a clique and for each i and j w_i is adjacent to u_i iff $i = j$ or $i \equiv j + 1 \pmod k$.

From Theorem 3 we immediately obtain that the r -dominating clique problem is solvable in polynomial time for any class of Helly graphs for which the r -domination problem is solvable in polynomial time. In particular, the r -dominating clique problem is solvable in polynomial time for graphs with a maximum neighbourhood ordering and for Helly graphs without C_4, \bar{S}_3 (and hence for bridged graphs without isometric sun $S_k, k \geq 3$, bridged graphs without S_3, \bar{S}_3 , doubly chordal graphs and strongly chordal graphs) [10, 11, 13]. Moreover, if \mathcal{G} is a class of Helly graphs G for which the minimum size of an r -dominating clique in G is equal to the maximum number of pairwise disjoint r -neighbourhoods in G for any n -tuple $(r(v_1), \dots, r(v_n))$, then the r -dominating clique problem is solvable in polynomial time on \mathcal{G} [10, 13, 14].

The r -domination problem is NP-complete for any subclass of Helly graphs for which the r -dominating clique problem is NP-complete.

Note that not for all chordal graphs G with a dominating clique $\gamma_r(G) = \gamma_{r\text{-clique}}(G)$ holds which means that this equality for strongly chordal graphs (given also in [23, Corollary 5]) is not due to chordality but to the Helly property.

4. A linear-time algorithm for finding minimum r -dominating cliques in graphs with maximum neighbourhood ordering

In the subsequent linear-time algorithm for the minimum r -dominating clique problem on dually chordal graphs (i.e. graphs with maximum neighbourhood ordering) the special kind of a maximum neighbourhood ordering of G turns out to be of importance.

For graphs G with maximum neighbourhood ordering such an ordering can be determined in linear time $O(|V| + |E|)$ if the input is the graph G . Ref. [12] contains the following algorithm for this purpose. The algorithm is based on the maximum cardinality search (MCS) algorithm of [27] for finding a perfect elimination ordering of chordal graphs and constructs the ordering from right to left. (Perhaps the characterization of dually chordal graphs as the graphs with maximum neighbourhood orderings makes it more transparent why the MCS approach is successful for chordal graphs. For graphs with maximum neighbourhood ordering this approach is almost straightforward.)

The following algorithm describes a numbering of the vertices from n to 1 by decreasing numbers. No number can be used twice.

Algorithm MNO (*Find a maximum neighbourhood ordering of G*)

Input: A dually chordal graph $G = (V, E)$

Output: A maximum neighbourhood ordering of G .

- (0) Initially all $v \in V$ are unnumbered and unmarked;
- (1) Choose an arbitrary $v \in V$, number v with n , i.e. $v_n := v$ and $\text{mn}(v_n) := v$;
repeat
- (2) among all unmarked vertices select a numbered vertex u such that $N[u]$ contains a maximum number of numbered vertices;
- (3) number all unnumbered vertices x from $N[u]$ consecutively by decreasing numbers between $n - 1$ and 1;
 for all of them let $\text{mn}(x) := u$;
- (4) mark u ;
- until** all vertices are numbered.

It should be stressed that in (3) the numbers (i.e. positions) for vertices are the largest possible ones among the numbers between 1 and n which are still free. The meaning of $\text{mn}(x)$ is a maximum neighbour of x .

In [4] it is shown that this algorithm MNO is correct, i.e. it yields a maximum neighbourhood ordering of G iff G has such an ordering.

Note that the algorithm also yields a maximum neighbour for each vertex and all vertices of $N[v_n]$ occur consecutively in the ordering on the left of v_n and have v_n as their maximum neighbour. Furthermore, for all v_i , $i \leq n - 1$, $\text{mn}(v_i) \neq v_i$.

This will be used in the subsequent algorithm which has a structure similar to corresponding algorithms for the case of strongly chordal graphs in [7].

The algorithm works from left to right along a maximum neighbourhood ordering of G . Hereby for every vertex v_i which is not included into an r -dominating clique D an information about $r(v_i)$ is passed over to $\text{mn}(v_i)$ by modifying the function r to r' on the vertex $\text{mn}(v_i)$ and reducing the problem to the rest graph $G_{i+1} = G_i - v_i$.

Algorithm DC (Find a minimum r -dominating clique of a dually chordal graph if there is one, and answer NO otherwise)

Input: A dually chordal graph $G = (V, E)$ and an n -tuple $(r(v_1), \dots, r(v_n))$ of positive integers.

Output: A minimum r -dominating clique of G if there is one, and answer NO otherwise.

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(1)  $D := \emptyset$ ;
(2) if for all  $v \in V$   $r(v) > 0$  then
(3)   begin
(4)     with MNO find a maximum neighbourhood ordering  $(v_1, \dots, v_n)$  of  $V$ ;
(5)     for  $i := 1$  to  $n$  do
(6)       begin
(7)         let  $r(\text{mn}(v_i)) := \min\{r(\text{mn}(v_i)), r(v_i) - 1\}$ ;
(8)          $G := G - v_i$ ;  $\{v_i \neq \text{mn}(v_i)$  for  $i \leq n - 1\}$ 
(9)         if  $r(v_i) = 0$  then goto outloop
(10)      end
(11)   end
(12) if for all  $v \in V$   $r(v) \neq 0$  then STOP with output  $D := \{v_n\}$ 
(13)  $\{v_n$  is an  $r$ -dominating vertex for  $G\}$ ;
(14) else  $\{\text{now } r(v) = 0$  for some  $v \in V$ ; suppose that  $G$  has now  $p$  vertices $\}$ 
outloop:
(15)   begin
(16)     with MNO find a maximum neighbourhood ordering  $(v'_1, \dots, v'_p)$  of the
(17)     rest graph  $G$  with  $r(v'_p) = 0$ ;
(18)     for  $i := 1$  to  $p$  do
(19)       begin
(20)         if  $r(v'_i) = 0$  then  $D := D \cup \{v'_i\}$ 
(21)         else if for all  $x \in N[v'_i]$   $r(x) > 0$  then
(22)           let  $r(\text{mn}(v'_i)) := \min\{r(\text{mn}(v'_i)), r(v'_i) - 1\}$ 
(23)         end;
(24)     if  $D$  is no clique then output ‘there is no  $r$ -dominating clique in  $G$ ’
(25)     else  $D$  is a minimum  $r$ -dominating clique of  $G$ 
(26)   end

```

Lemma 5. *Assume that the graph G has an r -dominating clique and let $N[v] \subseteq N[u]$ for some vertices u and v . Then G has a minimum r -dominating clique D with $v \notin D$ iff $r(v) \neq 0$.*

The next lemma gives the justification of passing over $r(v)$ to a maximum neighbour of v for a vertex v which is not included into the r -dominating clique D .

Let v be a vertex in G with $r(v) > 0$ and let u be a maximum neighbour of v . Define r' on $G - v$ as follows: Let $r'(u) := r(u)$ if $r(w) = 0$ for some $w \in N[v]$ and let $r'(u) := \min\{r(u), r(v) - 1\}$ otherwise. For all other vertices $w \neq u$ in $G - v$ the function r remains unchanged: $r'(w) := r(w)$.

Lemma 6. (Reduction Lemma). *If $D \subseteq V \setminus \{v\}$ is a minimum r' -dominating clique (set) of $G - v$ then D is a minimum r -dominating clique (set) of G . In particular, the graph G has an r -dominating clique iff the graph $G - v$ has an r' -dominating clique.*

Proof. Let D be an r' -dominating clique of $G - v$. Then D is also an r -dominating clique in G since v is $r(v)$ -dominated by a vertex $w \in N[v]$ with $r(w) = 0$ (which means $w \in D$) or by a vertex x which $r'(x)$ -dominates the vertex u in $G - v$.

Let now $D \subseteq V$ be a minimum r -dominating clique of G . According to Lemma 5 D can be chosen such that $v \notin D$. Thus $D \subseteq V - v$.

Since $N[v] \subseteq N[u]$ the distance between any two vertices in $G - v$ equals their distance in G . Thus if there is a $w \in N[v]$ with $r(w) = 0$ then D is also an r' -dominating clique in $G - v$ since r and r' are the same on $G - v$ in this case and the distances are the same on G and $G - v$. Now assume that for all $w \in N[v]$ $r(w) > 0$.

Case 1: There is a $w \in D \cap N[v]$. Then the clique $D' = (D \setminus \{w\}) \cup \{u\}$ is also an r' -dominating clique in $G - v$.

Case 2: $D \subseteq V \setminus N[v]$. Consider a vertex $w \in D$ such that $\text{dist}(v, w) \leq r(v)$ in G . Then $\text{dist}(u, w) = \text{dist}(v, w) - 1 \leq r(v) - 1$. Thus D' is an r' -dominating clique in $G - v$ also if $r'(u) = r(v) - 1$.

The case of r -dominating sets instead of cliques has a completely analogous proof. \square

Note that in the first assertion of Lemma 6 also the converse direction holds — for the correctness proof of the algorithm we need only the given direction.

Lemma 7. *If the label ‘outloop’ is never reached by a goto command then $\{v_n\}$ is r -dominating for G .*

Proof. Assume that the label ‘outloop’ is never reached by a goto command, i.e. in the first part of the algorithm never $r(v_i) = 0$ occurs. This means that in the directed tree defined by the $\text{mn}(\cdot)$ function for all vertices v_i $r(v_i) \geq \text{depth}_T(v_i) + 1$. On the other hand, the directed edges in T correspond to edges in G and thus v_n is r -dominating all vertices v_i , $i \in \{1, \dots, n\}$, i.e. $D = \{v_n\}$ is a minimal r -dominating clique of G . \square

Theorem 4. *Algorithm DC is correct and works in linear time $O(|V| + |E|)$.*

Proof. The time bound of the algorithm is obviously linear. Now to the correctness. For the first part of the algorithm (lines 1–11) the correctness immediately follows from Lemmas 5 and 6. If this part ends up without any vertex v with $r(v) = 0$ then because of Lemma 7 v_n is a single vertex which r -dominates G and thus line 12 is correct. Otherwise, the reduction starts again with maximum neighbourhood ordering (v'_1, \dots, v'_p) of the rest which is up to now correct according to Lemmas 5 and 6. As long as all vertices v'_i have $r(v'_i) > 0$ the correctness of the reduction step in line 22 follows from Lemmas 5 and 6. If a vertex v'_i with $r(v'_i) = 0$ is reached then there are two cases: Recall that the algorithm MNO has the property that all neighbours of the last vertex v'_p form an interval directly to the left of v'_p . Let k be the leftmost index in this interval.

Case 1: $i < k$, i.e. v'_i is not a neighbour of v'_p . In this case necessarily $v'_i \in D$, $v'_p \in D$ but D is no clique. Since up to here the algorithm was correct it follows that there is no r -dominating clique in G .

Case 2: $i \geq k$. Then for all $v'_j, j > i$, v'_p is a neighbour of v'_j which means that there is no further reduction step and the algorithm simply collects all vertices v with $r(v) = 0$ in D . If D is no clique then there is no r -dominating clique in G , otherwise D is a minimum r -dominating clique in G . \square

5. r -Domination by cliques and r -dominant cliques

Since a graph does not necessarily have a dominating clique it is natural to consider the following weaker *r -dominating by cliques* problem: Given an undirected graph $G = (V, E)$ with weight function $(r(v_1), \dots, r(v_n))$ of positive integers find a minimum number of cliques C_1, \dots, C_k such that $\bigcup_{i=1}^k C_i$ r -dominates G .

Note that for the special case $r(v_i) = 0$ for all $i \in \{1, \dots, n\}$ this is the well-known problem *clique partition*.

Another problem closely related to that is to find a clique in G which r -dominates a maximum number of vertices — we call this the *r -dominant clique* problem. For the special case $r(v_i) = 0$ for all $i \in \{1, \dots, n\}$ this is again a well-known problem namely the *maximum clique* problem.

It is well-known that these two problems are NP-complete. The following results show that for chordal graphs the problems are solvable in polynomial time whereas they are NP-complete on dually chordal graphs.

For a graph $G = (V, E)$ with disks $\mathcal{D}(G) = \{N^k[v] : v \in V \text{ and } k \geq 0 \text{ a positive integer}\}$ let $\Gamma(G)$ be the following graph whose vertices are the disks of G and two disks $N^p[v], N^q[w]$ are adjacent iff $N^{p+1}[v] \cap N^q[w] \neq \emptyset$ (or, equivalently, $N^p[v] \cap N^{q+1}[w] \neq \emptyset$, i.e. $0 < \text{dist}(v, w) \leq p + q + 1$).

By $l(G)$ we denote the length (number of edges) in a longest induced cycle of G .

Lemma 8. For each graph G $l(G) = l(\Gamma(G))$ holds.

Proof. Obviously $l(G) \leq l(\Gamma(G))$ since for disks $N^k[v]$ $k=0$ is admitted. Now to the converse: Among all induced cycles of maximum length choose a cycle $Z = (N^{r_1}[v_1], \dots, N^{r_k}[v_k], N^{r_1}[v_1])$ with smallest sum $s = r_1 + \dots + r_k$. We show that $s=0$, i.e. $r_i=0$ for all $i \in \{1, \dots, k\}$. Assume to the contrary that $r_1 \geq 1$. Consider vertices $x, y \in N(v_1)$ in G such that

$$\text{dist}(x, v_2) = \text{dist}(v_1, v_2) - 1 \quad \text{and} \quad \text{dist}(y, v_k) = \text{dist}(v_1, v_k) - 1.$$

If

$$N^{r_1-1}[x] \cap N^{r_k+1}[v_k] = \emptyset \quad \text{and} \quad N^{r_1-1}[y] \cap N^{r_2+1}[v_2] = \emptyset,$$

then

$$N^{r_1-1}[x], N^{r_2}[v_2], \dots, N^{r_k}[v_k], N^{r_1-1}[y], N^r[v_1]$$

with $r=0$ form an induced cycle with $k+1$ edges if $r_1 > 1$ or $r_1 = 1$ and $\text{dist}(x, y) = 1$, and with $k+2$ edges otherwise which contradicts to the maximality of the length of Z .

Assume now w.l.o.g. $N^{r_1-1}[x] \cap N^{r_k+1}[v_k] \neq \emptyset$. Then replacing the disk $N^{r_1}[v_1]$ by $N^{r_1-1}[x]$ in the cycle Z one gets an induced cycle with k edges and smaller sum s which is again a contradiction. This means that in Z all disks have radius 0, i.e. Z is an induced cycle in G itself. \square

A consequence of this lemma is that for chordal graphs G the graph $\Gamma(G)$ is also chordal. This together with Theorem 2 will be used in the following.

Theorem 5. Let $G = (V, E)$ be a chordal graph. Then the problem r -domination by cliques can be solved for chordal graphs within time $O(|V| \cdot |E|)$.

Proof. Let $G = (V, E)$ be chordal with weight function $(r(v_1), \dots, r(v_n))$ of positive integers. Then within time $O(|V| \cdot |E|)$ the matrix of distances in G can be determined. Using this matrix the following graph H can be constructed within $O(|V|^2)$ steps: H has the same vertices as G , and two vertices $u, v \in V$ are adjacent in H iff $0 < \text{dist}(v, w) \leq r(v) + r(w) + 1$. The graph H as an induced subgraph of the chordal graph $\Gamma(G)$ is chordal. It is well-known that for such graphs (cf. [25]) a partition into a minimum number of cliques M_1, \dots, M_k can be found within $O(|V|^2)$ steps. For these cliques M_i in H which fulfill in G the suppositions of Theorem 2 find an r -dominating clique C_i as described there. This can be done within $O(\sum_i |M_i| \cdot |E|) = O(|V| \cdot |E|)$ steps. The cliques C_1, \dots, C_k obtained in this way are also a solution of the problem r -domination by cliques in G : Every clique F in H corresponds in G to a set of vertices which pairwise fulfill $\text{dist}(v, w) \leq r(v) + r(w) + 1$. Consequently according to Theorem 2 there is an r -dominating clique for F in G . Conversely every set of vertices F in H which has an r -dominating clique in G is a clique in H . \square

Theorem 6. *The r -dominant clique problem can be solved for chordal graphs within time $O(|V| \cdot |E|)$.*

The proof of this theorem is completely analogous to the proof of Theorem 5. Now we consider the same problems on dually chordal graphs.

Theorem 7. *The problems r -domination by cliques and r -dominant clique are NP-complete on dually chordal graphs even for $r(v) = 1$ for all vertices v .*

Proof. Let $G = (V, E)$ be a graph. Then let $H = (V', E')$ be the following graph:

- For each vertex $v \in V$ add a new vertex v' which is adjacent only to v .
- Add a single vertex w which is adjacent to all vertices $v \in V$.

Obviously H has a maximum neighbourhood ordering, i.e. H is dually chordal. Now it is easy to see that G has a partition into k cliques iff H has k cliques which dominate H . Likewise, G has a clique of size k iff H has a clique which dominates $|V| + k + 1$ vertices in H . \square

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