Discrete Approximations for Singularly Perturbed Boundary Value Problems with Parabolic Layers, III

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Discrete Approximations for Singularly Perturbed Boundary Value Problems with Parabolic Layers, III *

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[1] [2]

Abstract

In his series of three papers we study singularly perturbed (SP) boundary value problems for equations of elliptic and parabolic type. For small values of the perturbation parameter parabolic boundary and interior layers appear in these problems. If classical discretisation methods are used, the solution of the finite difference scheme and the approximation of the diffusive flux do not converge uniformly with respect to this parameter. Using the method of special, adapted grids, we can construct difference schemes that allow approximation of the solution and the normalised diffusive flux uniformly with respect to the small parameter.

We also consider singularly perturbed boundary value problems for convection-diffusion equations. Also for these problems we construct special finite difference schemes, the solution of which converges ε-uniformly. We study what problems appear, when classical schemes are used for the approximation of the spatial derivatives. We compare the results with those obtained by the adapted approach. Results of numerical experiments are discussed.

In the three papers we first give an introduction on the general problem, and then we consider respectively (i) Problems for SP parabolic equations, for which the solution and the normalised diffusive fluxes are required; (ii) Problems for SP elliptic equations with boundary conditions of Dirichlet, Neumann and Robin type; (iii) Problems for SP parabolic equation with discontinuous boundary conditions.

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Part III
Parabolic equations with a discontinuous boundary condition

1 Introduction

The solution of partial differential equations that are singularly perturbed and / or have discontinuous boundary conditions generally have only limited smoothness. Due to this fact difficulties appear when we solve these problems by numerical methods. For example for regular parabolic equations with discontinuous boundary conditions, classical methods (FDM or FEM) on regular rectangular grids do not converge in the ℓ^∞-norm on a domain that includes a neighbourhood of the discontinuity [8, 9, 4].

If the parameter multiplying the highest-order derivative vanishes, boundary- and interior layers generally appear. When a discontinuity is present in the initial function (given at $t = 0$), an interior layer is generated. Outside a neighbourhood of the discontinuity classical difference schemes converge in the ℓ^∞-norm for each fixed value of the small parameter, but they do not converge in the ℓ^∞-norm in the neighbourhood of the discontinuity. Neither do they converge uniformly in ε in any neighbourhood of the interior layer [8, 9]. Therefore, it is interesting to construct special methods which are ℓ^∞-convergent for parabolic PDEs with discontinuous initial functions, both in the regular and in the singularly perturbed case. In the latter case it is important to see if and when such convergence can be uniform in the small parameter on the whole domain of definition.

In [8, 9] singularly perturbed parabolic equations with discontinuous boundary conditions were studied. There, special difference schemes were constructed for these problems. In order to be able to construct a method that was uniformly convergent (in the small parameter ε), special variables were used in the neighbourhood of the discontinuity. By introducing the variables $\theta = x/(2\varepsilon\sqrt{t})$ and t, the singularity was removed from the boundary value problem and the solution became a smooth function in the new variables. This behaviour of the transformed solution allows the use of a classical scheme in the transformed variables in the neighbourhood of the singularity. Away from the singularity the classical scheme can be used with the original variables.

This transformation in the neighbourhood of the singularity implied the use of a specially condensed grid in the neighbourhood of the boundary and interior layers. So we can say that the technique is based on: (i) fitted methods in which the coefficients of the difference equations are adapted to the singularities; (ii)

methods that use special, refined meshes in the neighbourhood of singularities. For these schemes ℓ^∞-convergence on the whole domain is proved, uniformly in the small parameter, but a disadvantage of these schemes is that they are very hard to realise in practice.

Because fitting of the coefficients, combined with fitting of the mesh is generally too complex for practical application, in the present paper we propose a new method in which only the coefficients are adapted. We use a uniform rectangular grid and a special difference equation with fitted coefficients. This method is much easier to realise.

For the construction of the new scheme the coefficients are selected such that the solution of a model problem with a piecewise constant, discontinuous initial function is the exact solution of the difference equations. This difference scheme with adapted coefficients is studied in this paper and it is compared with the classical scheme.

As was shown in [8, 9], no scheme exists that converges uniformly on a uniform grid for the general problem with a parabolic layer. However, for problems with an interior layer, the present method has this favourable property, and, in addition, numerical examples show that the method has practical value for far more general equations with discontinuous boundary conditions.

2 Problem formulation

We consider the Dirichlet boundary value problem for the following singularly perturbed equation of parabolic type

$$\begin{aligned} L_{(2.1)}u(x,t) &= f(x,t), \qquad (x,t)\in G, \\ u(x,t) &= \phi(x,t), \qquad (x,t)\in S, \end{aligned} \tag{2.1a}$$

where

$$G = \{(x,t)|\ -1 < x < 1,\ 0 < t \le T\},\quad S = \overline{G}\backslash G, \tag{2.1b}$$

$$L_{(2.1)} \equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} - p(t)\frac{\partial}{\partial t} - c(t)\,. \tag{2.1c}$$

The parameter ε may take any value $\varepsilon\in(0,1]$. The coefficients $c(t)$, $p(t)$ and the source $f(x,t)$ are sufficiently smooth functions on $\overline{G}$ and the coefficients are positive:

$$c(t) \ge 0,\quad p(t) \ge p_0 > 0,\quad (x,t)\in\overline{G}. \tag{2.2}$$

The boundary function $\phi(x,t)$ has a discontinuity[3] of the first kind on the set $S^\star$:

$$S^\star = \{(x,t)|\, x = 0, t = 0\}.$$

[3] A piecewise continuous function $v(x,t)$, $(x,t)\in S\backslash S^\star$, is redefined at the discontinuity by

$$v(x,t) = \frac{1}{2}\left\{ \lim_{s\searrow 0} v(x+s,t) + \lim_{s\nearrow 0} v(x+s,t) \right\},\ (x,t)\in S^\star, \tag{2.3}$$

For simplicity $S^\star$ consists of a single point only. Outside $S^\star$ the function $\phi(x,t)$ is sufficiently smooth on S.

Such boundary value problems with discontinuous boundary condition describe for example the temperature in a heat transfer problem, when two parts of a material with different temperatures are instantaneously connected [5]. Then, the small parameter ε corresponds to a small heat conduction coefficient.

The solution of the boundary value problem (2.1) is a function $u \in C(\overline{G} \backslash S^\star) \cap C^{2,1}(G)$, that is on G it is C^2 in x and C^1 in t.

We say that the discrete approximation converges ϵ-uniformly (or uniformly in ε) on $\overline{G}$ if the ℓ^∞-norm of the error converges to zero on $\overline{G}$, uniformly in ε.

For the construction of a special difference scheme we shall use the standard function $w_0(x,t)$, which is discontinuous on $S^\star$,

$$w_0(x,t) = w_0(x,t;p_1) = \frac{1}{2} v\Big(\frac{x}{2\varepsilon} \sqrt{\frac{p_1}{t}} \Big), \quad (x,t) \in \overline{G} \backslash S^\star \,, \tag{2.5}$$

where $p_1 = p(0)$ and $v(\xi) = \operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi \exp(-\alpha^2)\, d\alpha$ is the error function. For $t = 0$, at the point $x = 0$ the function (2.5) is defined by continuous extension. The function $w_0(x,t)$ is the solution of the constant coefficient equation

$$L_{(2.6)} u(x,t) \equiv \left(\varepsilon^2 \frac{\partial^2}{\partial x^2} - p_1 \frac{\partial}{\partial t} \right) \; u(x,t) = 0, \quad (x,t) \in G. \tag{2.6}$$

This function is piecewise constant on S at $t = 0$ and has a discontinuity of the first kind in $S^\star$:

$$[w_0(0,0)] = 1.$$

Suppose

$$W_0(x,t) = \exp\left(- \int_0^t \frac{c(\xi)}{p(\xi)}\, d\xi \right) w_0(x, \eta(t); p_1), \quad \text{with } \eta(t) = \int_0^t \frac{p_1}{p(\xi)}\, d\xi \,. \tag{2.7}$$

Then the function $W_0(x,t)$ is continuous on $\overline{G} \backslash S^\star$, it is a solution of the homogeneous equation

$$L_{(2.1)} u(x,t) = 0, \quad (x,t) \in G \,. \tag{2.8}$$

On S the function $u(x,t)$ is continuous and piecewise smooth. For simplicity we suppose that $u(x,t)$ is sufficiently smooth on the boundary of G, and that a compatibility condition is satisfied at the corner points. We are interested in the solution of problem (2.1) in the neighbourhood of the point of discontinuity and in the neighbourhood of the generated interior layer. Therefore, we suppose that the boundary conditions at $x = \pm 1$ are such that no boundary layers appear.

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and the jump in the discontinuity is defined by

$$[v(x,t)] = \left\{ \lim_{s \searrow 0} v(x+s,t) - \lim_{s \nearrow 0} v(x+s,t) \right\}, (x,t) \in S^\star. \tag{2.4}$$

3 An ε-uniformly convergent scheme

On the set $\overline{G}$ we introduce the rectangular grid

$$\overline{G}_h = \omega \times \omega_0 \,. \tag{3.9}$$

Here ω and ω_0 are uniform grids on the segments $[-1,1]$ and $[0,T]$ respectively. For some $N, N_0 > 0$ we take $x_i = ih$, $i \in \mathbb{Z}$; $-1 \le x_i \le 1$; $h = 2/N$; $t^j = j\tau$; $j = 0, 1, 2, \ldots, N_0$, $\tau = T/N_0$; and

$$G_h = G \cap \overline{G}_h\,; \quad S_h = S \cap \overline{G}_h\,; \quad S_h^\star = S^\star \cap \overline{G}_h\,.$$

On the set $S_h^\star$ the boundary function $\phi(x,t)$ is defined by

$$\phi(x,t) = \frac{1}{2}\left\{ \lim_{s \nearrow x} \phi(s,t) + \lim_{s \searrow x} \phi(s,t) \right\}, \quad (x,t) \in S_h^\star. \tag{3.10}$$

For the numerical approximation of (2.1) we may use classical difference approximations (see [6, 7]). For example, in the case of the implicit central difference scheme we have

$$\begin{aligned} \Lambda_{(3.11)} z(x,t) &= f(x,t), \qquad (x,t) \in G_h\,, \\ z(x,t) &= \phi(x,t), \qquad (x,t) \in S_h\,, \end{aligned} \tag{3.11a}$$

where

$$\Lambda_{(3.11)} \equiv \varepsilon^2 \delta_{x\overline{x}} - p(t)\delta_{\overline{t}} - c(t), \tag{3.11b}$$

with $\delta_{\overline{t}} z(x,t)$ and $\delta_{x\overline{x}} z(x,t)$ the usual first and second difference of $z(x,t)$ on the uniform grids ω_0 and ω respectively; the bar denotes the backward difference. It is well known that the operator $\Lambda_{(3.11)}$ is monotone [7], which implies that the maximum principle holds for (3.11).

Nevertheless, the classical difference scheme (i) does not converge on the whole domain $\overline{G}_h^\star = \overline{G}_h \backslash S_h^\star$ for a fixed value of ε, and (ii) outside a neighbourhood of the discontinuity it does not converge uniformly with respect to ε in the interior layer (see Section 4). To obtain uniform convergence, in the present paper we introduce a specially fitted scheme for the approximation of equation (2.1a),

$$\begin{aligned} \Lambda_{(3.12)} z(x,t) &= f(x,t), \qquad (x,t) \in G_h\,, \\ z(x,t) &= \phi(x,t), \qquad (x,t) \in S_h\,, \end{aligned} \tag{3.12a}$$

where

$$\Lambda_{(3.12)} \equiv \varepsilon^2 \gamma(x,t) \delta_{x\overline{x}} - p(t)\delta_{\overline{t}} - c(t)\,. \tag{3.12b}$$

According to the principle mentioned in the introduction, here $\gamma(x,t)$ is a fitting coefficient or fitting factor, which is chosen such that the singular solution, $W(_0x,t)$, is the exact solution of the homogeneous difference equation (3.13):

$$\Lambda_{(3.12)} W_0(x,t) \equiv \left\{ \varepsilon^2 \gamma(x,t) \delta_{x\overline{x}} - p(t)\delta_{\overline{t}} - c(t) \right\} W_0(x,t) = 0, \quad (x,t) \in G_h\,. \tag{3.13}$$

More generally we can select $\gamma(x,t)$ such that (3.13) is satisfied by $v(x,t) = W_0(x,t) + u_0(x,t)$, where W_0 is the singular solution and u_0 is some smooth solution of the homogeneous equation

$$L_{(2.1)}u(x,t) = 0, \quad (x,y)\in G\,. \tag{3.14}$$

This leads to the following expression for γ:

$$\gamma(x,t) = \frac{p(t)\ \delta_{\overline{t}}v(x,t) + c(t)v(x,t)}{\varepsilon^2\ \delta_{x\overline{x}}v(x,t)}, \quad (x,t)\in G_h\,, \tag{3.15}$$

for any point (x,t) where $\delta_{x\overline{x}}v(x,t) \neq 0$.

We notice that for $u_0(x,t) \equiv 0$ the differences $\delta_{x\overline{x}}v(x,t)$ and $\delta_{\overline{t}}v(x,t)$ can be very small because of the exponentially small derivatives of $W_0(x,t)$ for large $x/(\varepsilon\sqrt{t})$. To improve the numerical behaviour in the computation of $\gamma(x,t)$, we choose the function u_0 such that the differences $\delta_{x\overline{x}}W_0$ and $\delta_{x\overline{x}}u_0$ have the same sign, for $(x,t)\in G_h$. In particular, in the remaining part of this paper we take, for example,

$$u_0(x,t) = -\left\{x^3 + 6\varepsilon^2 x\int_0^t \frac{1}{p(\xi)}\,d\xi\right\}\ \exp\left(-\int_0^t \frac{c(\nu)}{p(\nu)}d\nu\right)\,, \quad (x,t)\in\overline{G}, \tag{3.16}$$

so that, for example for $c(t) \equiv 0$ and $p(t) \equiv 1$, we obtain

$$u_0(x,t) = u_{(3.17)}(x,t) = -x^3 - 6\varepsilon^2 xt\,, \quad (x,t)\in\overline{G}\,. \tag{3.17}$$

Then, for $\gamma(x,t)$ we have the general representation

$$\gamma(x,t) = \frac{p(t)(\delta_{\overline{t}}W_0(x,t) + \delta_{\overline{t}}u_0(x,t)) + c(t)(W_0(x,t) + u_0(x,t))}{\varepsilon^2\delta_{x\overline{x}}W_0(x,t) + \varepsilon^2\delta_{x\overline{x}}u_0(x,t)}, x \neq 0\,, \tag{3.18}$$

where the functions W_0 and u_0 are defined by (2.7) and (3.16) respectively. Note that $\delta_{x\overline{x}}v = \delta_{\overline{t}}v = v = 0$, at $x = 0$, $t > 0$. Therefore, for definiteness, we set $\gamma(x,t) = 1$ at $x = 0$. [4] Now we define the resulting difference scheme as (3.12), where $\gamma(x,t)$ is determined by (3.18).

Under the condition that

$$\frac{\tau^{3/2}}{h} \leq \psi(h,\tau) \tag{3.19}$$

where $\psi(h,\tau) > 0$ and $\psi(h,\tau) \to 0$ for $h,\tau \to 0$, then the scheme (3.12,3.18) converges uniformly in ε:

$$\max_{\overline{G}_h}|u(x,t) - z(x,t)| \leq M\left\{(h+\tau)^\nu + \psi(h,\tau)\right\}\,, \tag{3.20}$$

[4] According to (2.7) to compute $\delta_{x\overline{x}}W_0(x,t)$ on time layer $t = j\tau$ we use the difference derivative $\delta_{x\overline{x}}w_0(x,\eta(t))$. The difference derivatives $\delta_{\overline{t}}W_0(x,t)$, $\delta_{x\overline{x}}W_0(x,t)$, $\delta_{\overline{t}}u_0(x,t)$, $\delta_{x\overline{x}}u_0(x,t)$ can easily be found, for example when the functions $p(t)$ and $c(t)$ are analytical.

for any $\nu \in (0, 1/3)$.

If, for instance,

$$h \geq \mathcal{O}(\tau^{\frac{3}{2(1+\nu)}}) \tag{3.21}$$

then

$$\max_{\overline{G}_h} |u(x,t) - z(x,t)| \leq M(h^{\nu_1} + \tau^{\nu_1}), \tag{3.22}$$

for any $\nu_1 \in (0, 1/3)$. Thus, we have the following theorem [3]

Theorem 3.1 Under condition (3.19), the solution of the difference scheme (3.12,3.18) converges on $\overline{G}_h$ in the discrete ℓ^∞-norm to the solution of problem (2.1) uniformly in ε. Under the conditions (3.19) or (3.21) respectively, the estimates (3.20) or (3.22) hold for the solution of the difference problem.

4 Numerical results

By theory [8, 9] and by numerical experiments [2] it is shown that, for discontinuous initial conditions, classical difference schemes do not converge in the ℓ^∞-norm everywhere on the set $\overline{G}_h \backslash S^\star$, even for a fixed value of ε. Neither do they converge uniformly in ε in the neighbourhood of the interior layer, outside a neighbourhood of $S^\star$. However, both the true solution $u(x,t)$ and the numerical approximation $z(x,t)$ are bounded, uniformly in ε and it may be the case that the error $\max_{\overline{G}_h} |z(x,t) - u(x,t)|$ is relatively small for the classical difference scheme. That would reduce the need for a special scheme.

On the other hand, the above theorem shows that the fitted scheme converges uniformly in ε on $\overline{G}_h$, but no indication is given about the value of the order constant M in (3.22). Moreover, the order of convergence is rather small. It might be possible that the error is rather large for any reasonable value of h or τ. This might also reduce the practical value of our fitted scheme. To decide on the practical value of the new scheme numerical experiments are necessary to provide a more detailed comparison.

4.1 The model problem

To see the effect of the fitted scheme in practice, for the approximation of the model problem for a singularly perturbed heat equation with a discontinuous initial condition

$$\begin{aligned} L_{(4.23)} u(x,t) \equiv \varepsilon^2 \tfrac{\partial^2}{\partial x^2} - \tfrac{\partial}{\partial t} u(x,t) = 0, \qquad (x,t) \in G, \\ u(x,t) = \phi(x,t), \qquad (x,t) \in S, \end{aligned} \tag{4.23}$$

we compare the numerical results for the classical scheme (3.11) and the fitted scheme (3.12, 3.18). For problem (4.23) we have

$$v(x,t) = w_0(x,t;1) + u_{(3.17)}(x,t),$$

so that the coefficient $\gamma(x,t)$ in (3.12) takes the form

$$\gamma(x,t) = \begin{cases} \frac{\delta_{\overline{t}} w_0(x,t) - 6\varepsilon^2 x}{\varepsilon^2 \delta_{x\overline{x}} w_0(x,t) - 6\varepsilon^2 x} & \text{for } (x,t) \in G_h, x \neq 0, \\ 1 & \text{for } (x,t) \in G_h, x = 0. \end{cases} \tag{4.24}$$

For $\epsilon = 1/8$, $N = 32$, $N_0 = 40$ the solution of the model problem (4.23), with

$$\phi(x,t) = \frac{5}{2} w_0(x,t) + u_2(x,t),\ (x,t) \in S\,, \tag{4.25}$$

$$u_2(x,t) = -(x+0.5)^2 - 2\varepsilon^2 t\,, \tag{4.26}$$

for which we have the representation

$$u(x,t) = u_2(x,t) + \frac{5}{2} w_0(x,t), \quad (x,t) \in \overline{G} \backslash S^\star\,, \tag{4.27}$$

is shown in Figure 1. The fitting coefficient (4.24) is shown in Figure 2. We can

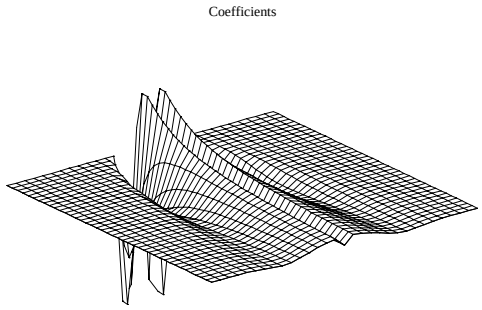

Figure 1: Computed solution with the fitted scheme.

The solution of problem (4.23,4.25) with $u_{(4.23)}(x,t) = \frac{5}{2} w_0(x,t) + u_2(x,t)$;
$\varepsilon = 1/8$; $N = 32$; $N_0 = 40$.

see that the solution has a jump at $S^\star$ for $t = 0$, and for $t > 0$ it is smooth. The space derivatives of the solution are large in the neighbourhood of the interior layer. The fitted coefficient varies strongly in the neighbourhood of the set $S^\star$ and becomes almost constant (equal to 1) away from $S^\star$.

4.2 Results with the classical difference approximation for the model problem

We show the behaviour of the classical difference scheme (3.11), central in x and backward in t, for the model problem (4.23,4.25). We know that this

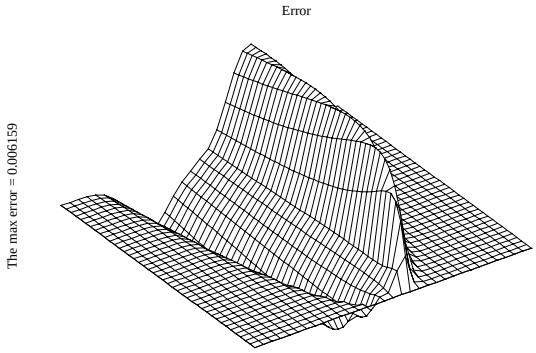

Figure 2: Coefficients $\gamma(x,t)$ in the fitted scheme.

Scheme (3.18), for the same problem as used in Figure 1.

scheme converges for a fixed parameter ε on each smooth part of the solution of (4.23,4.25). Therefore we are primarily interested in the singular part of the solution for problem (4.23,4.25). Hence, we select the boundary conditions such that $u(x,t) = w_0(x,t)$,

$$\phi(x,t) = w_0(x,t), \quad (x,t) \in S\,. \tag{4.28}$$

For the approximation of problem (4.23,4.28) we use the classical scheme (3.11). We solve the problem for different values of the mesh, $h = 2/N$, and the time step, $\tau = 1/N_0$, and for different values of the small parameter ε. The results for a set numerical experiments are summarised in Table 1 and Table 2.

We notice that asymptotically for larger N or N_0 and smaller ε, the ℓ^∞-norm of the error does not depend on ε, N and N_0 independently , but behaves as depending on a single parameter $N_0\varepsilon^{-2}$ or $N\varepsilon^{-1}$ for Table 1, and $N_0\varepsilon^{-2}$ or N for Table 2. Note that $|w_0(x,t)| \le 0.5$. From Table 1 we see that for no value of the parameter ε we can guarantee the error on $\overline{G}$ to be less than 12% for any sufficiently large $N,\ N_0$:

$$\eta_1(K,\varepsilon) = \max_{N,\ N_0 \ge K} \{[\max_{(x,t)\in\overline{G}} |w_0(x,t)|]^{-1} E(N,N_0,\varepsilon)\} \ge 12\%$$

when K is sufficiently large. From the results in Table 2 we see that for no values of $N_0,\ N$ we can guarantee the error on $\overline{G},\ t \ge 0.2$ to be less than 6% for $\varepsilon \in (0,1]$:

$$\eta_2(N,N_0) = \max_{\varepsilon}\{[\max_{(x,t)\in\overline{G}, t\ge 0.2} |w_0(x,t)|]^{-1} E(N,N_0,\varepsilon)\} \ge 6\%\,.$$

Table 1: Table of errors $E(N, N_0, \varepsilon)$ for the classical scheme.

N_0		N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	5.76(-2)	6.08(-2)	6.16(-2)	6.25(-2)	6.26(-2)	6.26(-2)
40		2.48(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)	6.20(-2)
160		2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)
640		3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
10	$\varepsilon = 1/8$	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
40		3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)
160		3.29(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)
640		3.29(-2)	3.29(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)
	ε						
40	1	2.48(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)	6.20(-2)
	0.5	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)
	2^{-2}	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
	2^{-3}	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)
	2^{-4}	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)
	2^{-5}	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)
	2^{-6}	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)
	2^{-7}	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)
	2^{-8}	1.22(-4)	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)
	2^{-9}	3.05(-5)	1.22(-4)	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)

In this table $E(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$.
The solution w_0 is as defined in (2.5) with $p_1 = 1$.

Table 2: Table of errors $E_{0.2}(N, N_0, \varepsilon)$ for the classical scheme.

N_0		N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	3.08(-2)	3.39(-2)	3.40(-2)	3.40(-2)	3.40(-2)	3.40(-2)
40		1.01(-2)	9.37(-3)	9.28(-3)	9.22(-3)	9.21(-3)	9.21(-3)
160		3.77(-3)	2.73(-3)	2.45(-3)	2.38(-3)	2.37(-3)	2.36(-3)
640		2.12(-3)	9.97(-4)	6.98(-4)	6.22(-4)	6.02(-4)	5.98(-4)
10	$\varepsilon = 1/8$	3.18(-2)	2.05(-2)	2.47(-2)	3.01(-2)	3.32(-2)	3.33(-2)
40		3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)
160		3.29(-2)	2.56(-2)	7.40(-3)	2.59(-3)	2.29(-3)	2.20(-3)
640		3.29(-2)	2.50(-2)	7.57(-3)	2.17(-3)	7.35(-4)	5.89(-4)
	ε						
40	1	1.01(-2)	9.37(-3)	9.28(-3)	9.22(-3)	9.21(-3)	9.21(-3)
	0.5	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)	8.62(-3)	8.61(-3)
	2^{-2}	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)	8.62(-3)
	2^{-3}	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)
	2^{-4}	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)
	2^{-5}	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)
	2^{-6}	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)
	2^{-7}	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)
	2^{-8}	1.19(-4)	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)
	2^{-9}	2.98(-5)	1.19(-4)	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)

In this table $E_{0.2}(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h, x\geq 0.2} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$.
The solution w_0 is as defined in (2.5) with $p_1 = 1$.

Thus, the computations also confirm that: (i) the classical scheme converges on the set $\overline{G}_h$ with $t \geq t_0 > 0$ for a fixed value of ϵ; (ii) on $\overline{G} \backslash S^\star$ the classical scheme does not converge for any fixed ε; (iii) on the set $\overline{G}_h$ with $t \geq t_0 > 0$ the scheme does not converge uniformly in ε.

4.3 A fitted difference approximation

Let us study the behaviour of the fitted scheme applied to model problem (4.23,4.25), where the function $u(x,t)$ is the sum of a smooth and a singular part

$$u(x,t) = u_2(x,t) + \frac{5}{2} w_0(x,t), \quad (x,t) \in \overline{G} \backslash S^\star \, . \tag{4.29}$$

Because the problem is linear, we can study both parts of the error independently. First we consider the behaviour of the fitted scheme for the singular part, that is for the model problem with

$$\phi(x,t) = w_0(x,t) \, , \ (x,t) \in \overline{G} \backslash S^\star \, , \tag{4.30}$$

as we did for the classical scheme. This initial function $w_0(x,t)$ is a representative example from the class of initial functions with a discontinuity. For problem (4.23,4.30) we have the solution

$$u(x,t) = w_0(x,t), \ (x,t) \in \overline{G} \backslash S^\star \, . \tag{4.31}$$

Then, considering the smooth part of the solution in the expression (4.29) we study problem (4.23) with

$$\phi(x,t) = u_2(x,t) = -(x+0.5)^2 - 2\varepsilon^2 t \, , \qquad (x,t) \in \overline{G} \, . \tag{4.32}$$

For problem (4.23,4.32), we have the solution

$$u(x,t) = u_2(x,t), \quad (x,t) \in \overline{G} \, . \tag{4.33}$$

The results of these numerical experiments are given in Tables 3 and 4.

From the results in Tables 3 and 4 we see that the errors for singular and regular parts, $w_0(x,t)$ and $u_2(x,t)$ respectively, decrease for N, N_0 large enough, and a fixed value of the parameter $\varepsilon = 2^{-K}$, $K = 0, 1, \ldots$. Also the errors decrease with increasing N uniformly in ε. The relative error is less than 1% for $N \geq 8$, $N_0 \geq 160$, $\varepsilon = 2^{-K}$, $K \geq 0$ when $u(x,t) = w_0(x,t)$. The relative error is also less than 1% for the same parameters when $u(x,t) = u_2(x,t)$.

The functions $\frac{5}{2} w_0(x,t)$ and $u_2(x,t)$ are components of the solution of the problem (4.23), (4.25). Thus we have: (i) for the model problem (4.23), (4.25) the numerical scheme converges for a fixed ε in the discrete ℓ^∞-norm on $\overline{G}_h$; (ii) we observe ε-uniform convergence for the model problem (4.23, 4.25); (iii) the relative error for the model problem is less than 2% for N, N_0 sufficiently large.

Table 3: Table of errors $E(N, N_0, \varepsilon)$ for the new scheme.

N_0		N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	2.26(-2)	1.96(-2)	1.89(-2)	1.87(-2)	1.87(-2)	1.87(-2)
40		1.27(-2)	1.06(-2)	1.01(-2)	1.01(-2)	1.00(-2)	1.00(-2)
160		7.74(-3)	5.30(-3)	4.30(-3)	4.16(-3)	4.08(-3)	4.07(-3)
640		6.13(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
10	$\varepsilon = 1/8$	5.46(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
40		5.56(-3)	2.30(-3)	9.47(-4)	5.28(-4)	4.17(-4)	3.88(-4)
160		5.57(-3)	2.12(-3)	7.00(-4)	2.64(-4)	1.44(-4)	1.12(-4)
640		5.58(-3)	2.07(-3)	6.36(-4)	1.92(-4)	6.90(-5)	3.66(-5)
	ε						
40	1	1.27(-2)	1.07(-2)	1.01(-2)	1.01(-2)	1.00(-2)	1.00(-2)
	0.5	7.74(-3)	5.30(-3)	4.30(-3)	4.16(-3)	4.08(-3)	4.07(-3)
	2^{-2}	6.13(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
	2^{-3}	5.56(-3)	2.30(-3)	9.47(-4)	5.28(-4)	4.17(-4)	3.88(-3)
	2^{-4}	4.48(-3)	1.70(-3)	6.56(-4)	2.60(-4)	1.44(-4)	1.12(-4)
	2^{-5}	1.23(-3)	6.55(-4)	3.69(-4)	1.46(-4)	5.90(-5)	3.27(-5)
	2^{-6}	3.08(-4)	1.79(-4)	8.37(-5)	5.67(-5)	2.63(-5)	1.12(-5)
	2^{-7}	7.71(-5)	4.47(-5)	2.28(-5)	1.05(-5)	7.77(-6)	4.25(-6)
	2^{-8}	1.93(-5)	1.12(-5)	5.71(-6)	2.86(-6)	1.34(-6)	1.15(-6)
	2^{-9}	4.82(-6)	2.80(-6)	1.43(-6)	7.15(-7)	3.58(-7)	2.09(-7)

In this table the scheme (3.12) is used to solve a problem (4.23,4.30) with an interior layer. $E(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$; the solution w_0 is as defined in (2.5) with $p_0 = 1$.

Table 4: Table of errors $E(N, N_0, \varepsilon)$.

N_0	ε	N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	5.10(-2)	8.72(-2)	1.16(-1)	1.36(-1)	1.47(-1)	1.53(-1)
40		1.46(-2)	2.27(-2)	3.15(-2)	3.89(-2)	4.50(-2)	4.87(-2)
160		7.19(-3)	5.87(-3)	7.00(-3)	8.44(-3)	9.83(-3)	1.10(-2)
640		7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
10	$\varepsilon = 1/8$	7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
40		7.44(-3)	3.17(-3)	1.64(-3)	1.03(-3)	8.52(-4)	8.00(-4)
160		7.46(-3)	2.94(-3)	1.25(-3)	6.06(-4)	3.74(-4)	3.04(-4)
640		7.46(-3)	2.89(-3)	1.15(-3)	4.63(-4)	2.09(-4)	2.55(-4)
	ε						
40	1	1.46(-2)	2.27(-2)	3.14(-2)	3.89(-2)	4.50(-2)	4.87(-2)
	0.5	7.19(-3)	5.87(-3)	7.00(-3)	8.44(-3)	9.83(-3)	1.10(-2)
	2^{-2}	7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
	2^{-3}	7.44(-3)	3.17(-3)	1.64(-3)	1.03(-3)	8.52(-4)	8.00(-4)
	2^{-4}	5.98(-3)	2.39(-3)	1.25(-3)	6.06(-4)	3.74(-4)	3.04(-4)
	2^{-5}	1.64(-3)	1.75(-3)	1.00(-3)	4.59(-4)	2.09(-4)	1.26(-4)
	2^{-6}	4.11(-4)	4.77(-4)	4.47(-4)	3.04(-4)	1.47(-4)	6.90(-5)
	2^{-7}	1.03(-4)	1.19(-4)	1.22(-4)	1.12(-4)	8.30(-5)	4.45(-5)
	2^{-8}	2.57(-5)	2.98(-5)	3.04(-5)	3.05(-5)	2.80(-5)	2.17(-5)
	2^{-9}	6.42(-6)	7.45(-6)	7.61(-6)	7.63(-6)	7.63(-6)	7.01(-6)

In this table the scheme (3.12) is used to solve a problem (4.23,4.32) with a smooth solution. In this table $E(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - u_2(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$; the solution u_1 is as defined in (4.31).

4.4 The error analysis for the fitted difference scheme

To determine the quality of the convergence, using the data from the Tables 3 and 4 we can examine the experimental order of convergence of the fitted scheme.

When we use the classical scheme (3.11) for problem (4.23,4.33) then, according to the classical theory, we typically find an estimate of the form

$$\max_{\overline{G}_h \setminus S^\star} |u_2(x,t) - z_{(3.11)}(x,t)| \leq Q(\varepsilon)(h^2+\tau),\ (x,t) \in \overline{G}_h\,. \tag{4.34}$$

This estimate means that the function $z_{(3.11)}(x,t)$ converges to the function $u_2(x,t)$ for each fixed value of ε. The constant $Q(\varepsilon)$ tends to infinity for $\varepsilon \to 0$.

From theory we know that the solution of the fitted difference scheme (3.12, 3.18) $z(x,t)$ converges ε-uniformly to the solution of problem (4.23,4.33). To investigate the ε-uniform convergence of a function $z(x,t) = z(x,t;\varepsilon,h,\tau)$, it is natural to express an error estimate in the form

$$\max_{\varepsilon} \max_{\overline{G}_h \setminus S^\star} |u(x,t,\varepsilon) - z(x,t;\varepsilon,h,\tau)| \leq M(h^2+\tau)^{\nu}\,, \tag{4.35}$$

where ν does not depend on the parameters ϵ, h or τ. To compute ν we shall use an inequality of the form

$$\max_{\overline{G}_h \setminus S^\star} |u(x,t,\varepsilon) - z(x,t;\varepsilon,h,\tau)| \leq M(h^2+\tau)^{\nu(\varepsilon)}\,. \tag{4.36}$$

We call $\nu(\varepsilon)$ in expression (4.36) the *generalised order of convergence for a fixed value of the parameter* ε, and ν in expression (4.35) the *generalised order of ε-uniform convergence.*

We determine the experimental generalised order *at the point* (N, N_0) by

$$\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4\,, \tag{4.37}$$

where $E(N, N_0, \varepsilon) = \max_{\overline{G}_h \setminus S^\star} |u(x,t,\varepsilon) - z(x,t;\varepsilon,h,\tau)|$, $hN = 2$ and $\tau N_0 = 1$. We introduce the *experimental generalised order of convergence* for fixed ε as

$$\overline{\nu}(\varepsilon) = \min_{N, N_0} \overline{\nu}(N, N_0, \varepsilon)\,, \tag{4.38}$$

and the *experimental generalised order of ε-uniform convergence* as

$$\overline{\nu} = \min_{\varepsilon} \overline{\nu}(\varepsilon)\,. \tag{4.39}$$

Similarly the the *experimental ε-uniform generalised order at the point* (N, N_0) is

$$\overline{\nu}(N, N_0) = \min_{\varepsilon} \overline{\nu}(N, N_0, \varepsilon)\,. \tag{4.40}$$

The results are given in the Tables 6 and 5.

Table 5: Experimental generalised order of convergence $\overline{\nu}(N, N_0, \varepsilon)$.

N_0		N				
		8	16	32	64	128
10	$\varepsilon = 1$	0.544	0.479	0.454	0.450	0.450
40		0.631	0.653	0.640	0.650	0.651
160		0.681	0.782	0.792	0.818	0.818
10	$\varepsilon = 1/8$	0.625	0.834	0.882	0.891	0.892
40		0.696	0.858	0.922	0.938	0.945
160		0.714	0.868	0.932	0.967	0.987
	ε					
40	1	0.631	0.653	0.640	0.650	0.651
	0.5	0.681	0.782	0.792	0.818	0.818
	2^{-2}	0.708	0.834	0.882	0.891	0.892
	2^{-3}	0.854	0.904	0.934	0.939	0.946
	2^{-4}	1.387	1.104	1.084	1.070	1.067
	2^{-5}	1.391	1.484	1.351	1.237	1.195
	2^{-6}	1.392	1.485	1.498	1.433	1.314
	2^{-7}	1.393	1.485	1.498	1.486	1.378
	2^{-8}	1.393	1.485	1.498	1.500	1.340

The fitted scheme (3.12,3.18) for the problem (4.23,4.30), applied to the solution $u(x,t) = w_0(x,t)$ with the interior layer. $\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4$,
$E(N, N_0, \varepsilon)$ from Table 3.

Table 6: Experimental generalised order of convergence $\overline{\nu}(N, N_0, \varepsilon)$.

N_0	ε	N				
		8	16	32	64	128
10	$\varepsilon = 1$	0.583	0.736	0.789	0.795	0.798
40		0.656	0.850	0.949	0.992	1.016
160		0.413	0.551	0.828	1.012	1.041
10	$\varepsilon = 1/8$	0.604	0.652	0.702	0.691	0.687
40		0.668	0.669	0.719	0.733	0.744
160		0.685	0.676	0.718	0.769	0.276
	ε	N				
40	1	0.656	0.850	0.949	0.992	1.016
	0.5	0.413	0.551	0.828	1.012	1.041
	0.25	0.603	0.652	0.702	0.691	0.687
	0.125	0.820	0.669	0.719	0.733	0.744
	2^{-4}	0.887	0.627	0.724	0.769	0.783
	2^{-5}	0.891	0.984	0.859	0.820	0.799
	2^{-6}	0.892	0.985	0.998	0.938	0.863
	2^{-7}	0.893	0.985	0.998	0.999	0.966
	2^{-8}	0.893	0.985	0.998	1.000	1.000

$\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4$,
$E(N, N_0, \varepsilon)$ from Table 4.
Computation with the new scheme (3.12,3.18) for the smooth solution $u(x,t) = u_2(x,t)$.

From the results in the Tables 5 and 6 we see: (i) for $w_0(x,t)$ and $u_2(x,t)$ the experimental generalised order of ε-uniform convergence for the fitted scheme is approximately 0.413 and 0.450 respectively; (ii) for $N \geq 16$ and $N_0 \geq 40$ the generalised orders of ε-uniform convergence for $w_0(x,t)$ and $u_2(x,t)$ are apparently not less than 0.50. This means that in practice

$$\max_{\overline{G}_h} |u(x,t) - z(x,t)| \leq M(h + \tau^{1/2})$$

for $N \geq 16$ and $N_0 \geq 40$, $0 < \varepsilon \leq 1$, for each value of the parameter ε. In accordance with the theory, for each value of ε, the experimental generalised order of convergence tends to 1 for decreasing h and τ. Thus, the experimental generalised order of convergence for the fitted scheme (3.12, 3.18) for the full model problem (4.23, 4.25) is not less than predicted by the theory. The behaviour of the errors $e(x,t;N,N_0,\epsilon) = z(x,t) - u(x,t)$ for the fitted scheme (3.12, 3.18) and for the classical scheme (3.11) are shown in the Figures 3 and 4. We can see that the largest errors are in the neighbourhood of the set $S^\star$ and that the errors for the classical scheme are significantly larger than for the fitted scheme.

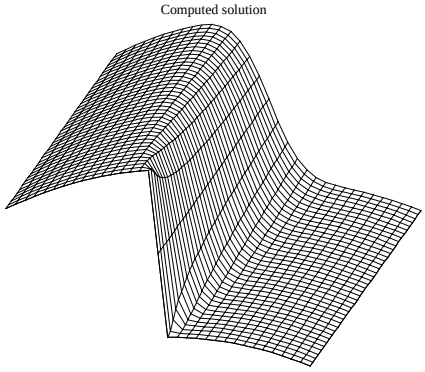

Figure 3: Discretisation error the fitted scheme.

Scheme (3.12,3.18) is used for the same problem as used in Figure 1.

5 Conclusion

For a singularly perturbed boundary value problem of parabolic type with discontinuous initial condition (2.1), we have constructed a specially fitted difference scheme that converges in $\overline{G}\backslash S^\star$ ε-uniformly in the ℓ^∞-norm.

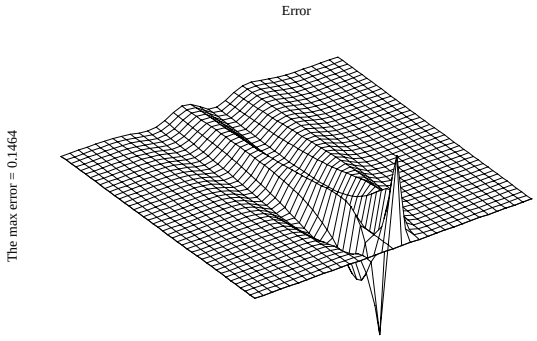

Figure 4: Discretisation error the classical scheme.

Scheme (3.11) is used for the same problem as used for Figure 1.

Numerical experiments for a model boundary value problem with discontinuous boundary function show that a classical difference scheme does not converge ε-uniformly. Moreover, for a fixed value of ε this scheme doesn't converge in the ℓ^∞-norm in the neighbourhood of the discontinuity, and away from the discontinuity it does not converge ε-uniformly in the neighbourhood of the interior layer. In the case of the constant coefficient problem and a simple discontinuity, for which the error-function is the solution, an error of less than 6% on $\overline{G}$, $t \geq t_0 = 0.2$, and less than 12% on $\overline{G} \backslash S^\star$ can not be guaranteed for arbitrarily small h or τ.

Theoretically and numerically it is also shown, that the fitted difference scheme converges ε-uniformly in the ℓ^∞-norm on $\overline{G}_h$. Moreover in the case of the fitted scheme, for a model problem, an experimental generalised order of convergence of not less than 0.5 is observed if $h \leq 1/8$ and $\tau \leq 0.025$ e.g. $\nu(\varepsilon, N, N_0) \geq 0.5$ at $N \geq 16,\ N_0 \geq 40$. The experimental generalised order of convergence is substantially larger than the bound guaranteed by the theory. Both for the singular and for the regular part of the solution an error less than 1% is guaranteed for $N \geq 8$, $N_0 > 40$ and for any $\epsilon \in (0, 1]$.

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