Schwartz Methods for Singularly Perturbed Convection-Diffusion Problems *

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Abstract

In this paper we consider singularly perturbed ordinary differential equations with convection terms. An ε-uniformly convergent finite difference scheme is constructed for such boundary value problems by using an upwind finite difference operator and a piecewise uniform mesh. Our primary interest is the design of a numerical method, for which a parallel technique will later be applicable. In this paper we consider the solution of the discrete problem using a domain decomposition method, based on the Schwartz alternating process with overlapping subdomains with essential and minimal overlapping width. We analyse and discuss the influence of various parameters, such as the perturbation parameter ε, the number of mesh points and the number of iterations, on the convergence of the iterative process. We give conditions under which the iterative difference schemes converge ε-uniformly. Illustrative numerical examples are presented.

1 Introduction

In this paper, we consider iterative domain decomposition algorithms for the solution of non-selfadjoint linear singularly perturbed two-point boundary value problems, that is linear singularly perturbed problems having a convective term. The methods considered are based on the Schwartz alternating process. The problem is discretized using an upwind finite difference operator and a piecewise uniform mesh. In this paper we consider only domain decomposition schemes with overlapping subdomains having essential and minimal overlapping width. The ultimate aim is to provide the basis for effective parallel methods for the solution of similar problems. We remark that, in practice, one would never use

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an iterative scheme to solve a one-dimensional linear problem such as this. Such iterative methods only become effective for non-linear problems, such as in [1], and for problems in two or more dimensions. However, we confine ourselves to linear problems so that we can decouple the analysis of the effects of the Schwartz process from those that arise from nonlinearities in the problem. The objective is to analyse and discuss the influence of various parameters, such as the perturbation parameter ε, the number of mesh points and the number of iterations, on the convergence of the iterative process. To this end we give conditions under which the iterative difference schemes converge ε-uniformly, and present some numerical results to clarify the analysis.

On the segment

$$\overline{D}, \quad \text{where} \quad D = (0, d), \tag{1.1}$$

we consider the Dirichlet problem for the singularly perturbed equation [1]

$$\begin{aligned} L_{(1.2)} u(x) &= f(x), \qquad x \in D, && (1.2a) \\ u(x) &= \varphi(x), \qquad x \in \Gamma. && (1.2b) \end{aligned}$$

Here $\Gamma = \overline{D} \setminus D$,

$$L_{(1.2)} u(x) \equiv \left\{ \varepsilon\, a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} - c(x) \right\} u(x),$$

the functions $a(x),\ b(x),\ c(x),\ f(x)$ are sufficiently smooth on $\overline{D}$, and $a(x) \geq a_0,\ \ b(x) \geq b_0,\ \ c(x) \geq 0, \quad x \in \overline{D},\ \ a_0,\, b_0 > 0$. The singular perturbation parameter ε takes arbitrary values from the half-interval $(0, 1]$. As the parameter $\varepsilon \to 0$, a boundary layer appears in a small neighbourhood of the boundary point $x = 0$.

To construct grid approximations for problem (1.2), (1.1), we use the well-known difference scheme combining a classical finite difference operator and a special piecewise uniform mesh [4, 3]. This special scheme is given in Section 3. Its solution converges to the solution of the boundary value problem ε-uniformly

In this paper we construct ε-uniformly convergent difference schemes, for the problem (1.2), (1.1), based on domain decomposition with overlapping subdomains, where coherent meshes are used in the subdomains, which form the covering of the set D. The meshes on the subdomains are generated by a single mesh that is introduced on the whole domain $\overline{D}$.

2 Difference schemes on essentially overlapping subdomains

Iteration-Free Schemes

For the boundary value problem (1.2), (1.1) we will consider iterative difference schemes based on domain decomposition in the case of overlapping subdomains.

[1]The notation $L_{(j.k)}$ ($M_{(j.k)}$) denotes that this operator (constant) is introduced in the formula $(j.k)$.

First we introduce the standard iteration-free classical and special difference schemes. On the set $\overline{D}$ we introduce the mesh

$$\overline{D}_h = \left\{ x^i,\ i = 0, 1, ..., N,\ 0 = x^0 < ... < x^N = d \right\}, \tag{2.1}$$

which is an arbitrary, generally speaking, non-uniform mesh; $N+1$ is the number of nodes in the grid $\overline{D}_h$. Define $h^i = x^{i+1} - x^i$, $x^i,\, x^{i+1} \in \overline{D}_h$, $h = \max_i h^i$. We assume [2] that $h \le M N^{-1}$.

On the mesh $\overline{D}_h$, define the difference scheme for the problem (1.2), (1.1) by:

$$\begin{aligned} \Lambda_{(2.2)} z(x) &= f(x), \qquad x \in D_h, \\ z(x) &= \varphi(x), \qquad x \in \Gamma_h. \end{aligned} \tag{2.2}$$

Here $D_h = D \cap \overline{D}_h,\ \Gamma_h = \Gamma \cap \overline{D}_h$,

$$\Lambda_{(2.2)} z(x) \equiv \{\varepsilon\, a(x)\, \delta_{\overline{x}\widehat{x}} + b(x)\, \delta_x - c(x)\}\, z(x),$$

$\delta_{\overline{x}\widehat{x}}\, z(x)$, $\delta_x\, z(x)$ are the second central and the first (forward) differences on the non-uniform grid, $\delta_{\overline{x}\widehat{x}}\, z(x) = 2\left(h^{i-1} + h^i\right)^{-1} [\, \delta_x\, z(x) - \delta_{\overline{x}}\, z(x)\,]$, $x = x^i$ (the bar denotes the backward difference).

As is well known from theory, the monotone difference scheme (2.2), (2.1) converges, as $N \to \infty$, for a fixed value of the parameter ε, however, it does not converge ε-uniformly.

Now we define a special finite difference scheme (see, for example, [4, 3, 2]) the convergence of which is ε-uniform. We introduce on $\overline{D}$ the special grid

$$\overline{D}_h = \overline{D}^{\star}_h, \tag{2.3}$$

where $\overline{D}^{\star}_h = \overline{D}^{\star}_h(\sigma)$ is a *piecewise uniform* mesh depending on the parameter σ, and σ is a function of ε and N. To construct the mesh $\overline{D}^{\star}_h(\sigma)$, we divide $\overline{D}$ into two parts $[0, \sigma]$ and $[\sigma, d]$, and in each part we use a unifppporm grid. The step-size of the mesh $\overline{D}^{\star}_h$ is equal to $h^{(1)} = 2\sigma N^{-1}$ in $[0, \sigma]$, and to $h^{(2)} = 2(d - \sigma) N^{-1}$ in $[\sigma, d]$. We take $\sigma = \min\,[\, 2^{-1} d,\, \varepsilon\, m^{-1} \ln N\,]$, where m is an arbitrary number satisfying $0 < m < m_0$, $m_0 = \min\limits_{\overline{D}}\, [\, a^{-1}(x)\, b(x)\,]$.

Under the condition

$$a,\ b,\ c,\ f \in C^{\alpha}(\overline{D}), \quad \alpha > 0, \tag{2.4}$$

the difference scheme (2.2), (2.3) converges ε-uniformly as $N \to \infty$. We are thus led to the following lemma.

Lemma 2.1. *Let the data of the boundary value problem* (1.2), (1.1) *satisfy condition* (2.4). *Then the difference scheme* (2.2), (2.3) *is ε-uniformly convergent. If* (2.4) *holds, with $\alpha \ge 2$, the solution of the difference scheme satisfies the estimate*

$$|\, u(x) - z(x)\,| \le M\, N^{-1} \ln^2 N, \quad x \in \overline{D}^{\star}_h. \tag{2.5}$$

[2]Throughout this paper, the symbol M (or m) denotes any sufficiently large (small) positive constant which does not depend on either the value of the parameter ε or on the discretization parameters.

Continuous Schwartz Schemes on Overlapping Meshes

We begin with consideration of the classical Schwartz overlapping method for problem (1.2), (1.1). Let the connected sets

$$D^k, \;\; k = 1, \dots, K, \tag{2.6a}$$

each of which overlaps with the adjacent sets, form the covering of the domain D:

$$D = \bigcup_{k=1}^{k} D^k. \tag{2.6b}$$

Assume $\Gamma^k = \overline{D}^k \setminus D^k$. By $D^{[k]}$ we denote the union of the subdomains $D^1, \dots, D^K$ excluding the set D^k

$$D^{[k]} = \bigcup_{i=1,\, i \neq k}^{K} D^i.$$

We denote the minimal width of the overlap of the sets D^k and $D^{[k]}$ by δ^k. Let δ denote the least value of δ^k, that is

$$\begin{gathered} \delta = \min_{k,\, x^1,\, x^2} \rho(x^1, x^2), \;\; x^1 \in \overline{D}^k, \;\; x^2 \in \overline{D}^{[k]}, \;\; x^1,\, x^2 \notin \left\{ \, D^k \cap D^{[k]} \, \right\}, \\ k = 1, \dots, K, \end{gathered} \tag{2.6c}$$

where $\rho(x^1, x^2)$ is the distance between the points $x^1, \; x^2 \in \overline{D}$.

The value δ, generally speaking, may depend on the parameter ε: $\delta = \delta_{(2.6)}(\varepsilon)$. It may also be the case that $D \setminus \bigcup\limits_{k=1}^{K} D^k \neq \emptyset$, that is to say, not all the subdomains D^k have nonempty intersection with the adjacent subdomains. In this case $\delta = 0$. In this case, we would say the method is a non-overlapping Schwartz method. In this paper we confine our attention to overlapping Schwartz methods.

Let

$$u^0(x), \quad x \in \overline{D} \tag{2.7a}$$

be an arbitrary bounded function satisfying condition (1.2b). Assume that some function $v^r(x)$, $x \in \overline{D}$ is defined for the iteration r. We take $v^r(x)$ as a first approximation for the function $v^{r+1}(x)$ on the next iteration $r+1$, and denote this first approximation by $\widehat{v}^{\,r}(x)$, $x \in \overline{D}$. In other words, $\widehat{v}^{\,r}(x)$ is the extension of $v^r(x)$ to the set $\overline{D}$ on the next iteration $r+1$.

Suppose that the functions $u^i(x)$, $x \in \overline{D}$, $i = 1, 2, ..., r$ have been already constructed; $u^0(x) = u^0_{(2.7)}(x)$. The function $u^{r+1}(x)$, $x \in \overline{D}$ is determined by solving the problems

$$L_{(1.1)}\, u^{r+\frac{k}{K}}(x) = f(x), \quad x \in D^k, \tag{2.7b}$$

$$u^{r+\frac{k}{K}}(x) = \left\{ \begin{array}{ll} \widehat{u}^{\,r}(x), & k = 1, \\ u^{r+\frac{k-1}{K}}(x), & k \geq 2 \end{array} \right\}, \qquad x \in \Gamma^k$$

$$\text{for } x \in \overline{D}^{\,k}, \;\; k = 1, \ldots, K,$$

where

$$u^{r+\frac{k}{K}}(x) = \left\{ \begin{array}{ll} u^{r+\frac{k}{K}}(x), & x \in \overline{D}^{\,k}, \\ \left. \begin{array}{ll} \widehat{u}^{\,r}(x), & k = 1, \\ u^{r+\frac{k-1}{K}}(x), & k \geq 2 \end{array} \right\}, & x \in \overline{D} \setminus \overline{D}^{\,k} \end{array} \right\}$$

$$\text{for } x \in \overline{D}; \quad k = 1, \ldots, K, \;\; r = 0, 1, 2, \ldots.$$

The function $u^{r+1}(x)$ is defined by the relation

$$u^{r+1}(x) = u^{r+\frac{K}{K}}(x), \quad x \in \overline{D}.$$

In the case of overlapping subdomains the function $\widehat{u}^{\,r}(x)$ is constructed in such a way:

$$\widehat{u}^{\,r}(x) = u^{\,r}(x), \quad x \in \overline{D}, \;\; r = 0, 1, 2, \ldots. \tag{2.7c}$$

Solving the problem (2.7), (2.6) using the Schwartz classical alternating method with overlapping subdomains involves finding the sequence of functions $u^r(x)$, $x \in \overline{D}$, $r = 1, 2, \ldots$. Note that Dirichlet interface conditions are given on the boundaries of the subdomains in the intermediate problems.

The differential problem (2.7), (2.6) is monotone. The following analog of the maximum principle is valid.

Lemma 2.2. *Let the functions* $u^{r+\frac{k}{K}}(x)$, $x \in \overline{D}$, $k = 1, \ldots, K$, $r = 0, 1, 2, \ldots$ *satisfy the conditions*

$$L_{(1.1)}\, u^{r+\frac{k}{K}}(x) \leq 0, \quad x \in D^k,$$

$$u^{r+\frac{k}{K}}(x) \geq \left\{ \begin{array}{ll} \widehat{u}^{\,r}(x), & k = 1, \\ u^{r+\frac{k-1}{K}}(x), & k \geq 2 \end{array} \right\}, \qquad x \in \Gamma^k;$$

$$u^0(x) \geq 0, \quad x \in \overline{D},$$

$$u^{r+\frac{k}{K}}(x) \geq 0, \;\; x \in \Gamma; \quad k = 1, \ldots, K, \;\; r = 0, 1, 2, \ldots .$$

Then $u^{r+\frac{k}{K}}(x) \geq 0$, $x \in \overline{D}$, $k = 1, \ldots, K$, $r = 0, 1, 2, \ldots$.

We say that the sets $\overline{D}^{\,k}$ overlap essentially if the following condition holds:

$$\delta = \delta_{(5.2)}(\varepsilon) > 0, \;\; \varepsilon \in (0, 1], \qquad \inf_{\varepsilon \in (0,1]} \left[\varepsilon^{-1} \delta_{(5.2)}(\varepsilon) \right] > 0. \tag{2.8}$$

Using the technique of majorant functions, we establish the following assertion.

Lemma 2.3. *The condition* (2.8) *is necessary and sufficient for* ε*-uniform convergence (as* $r \to \infty$*) of the solutions of problem* (2.7), (2.6) *to the solution of the original problem* (1.2), (1.1).

Discrete Schwartz Schemes on Overlapping Subdomains

In this subsection we construct grid approximations of problem (2.7), (2.6). On the subdomains $\overline{D}^k$ we first construct the coherent meshes

$$\overline{D}_h^k = \overline{D}^k \cap \overline{D}_h, \tag{2.9}$$

where $\overline{D}_h$ is one of the above-introduced meshes, either $\overline{D}_{h(2.1)}$ or $\overline{D}^\star_{h(2.3)}$. Assume that the boundaries of the sets $\overline{D}^k$ pass through the nodes of the grid $\overline{D}_h$; the number of the nodes in each mesh $\overline{D}_h^k$ is commensurable with N.

Let the grid function $v^r(x)$, $x \in \overline{D}_h$ be defined on the r-th iteration. Its extension onto the set $\overline{D}_h$ on the next iteration $r+1$ is denoted by $\widehat{v}^{\,r}(x)$.

Suppose that the functions $z^i(x)$, $x \in \overline{D}_h$, $i = 1, 2, ..., r$ have been already constructed; $z^0(x) = u^0_{(2.7)}(x)$, $x \in \overline{D}_h$. The function $z^{r+1}(x)$, $x \in \overline{D}_h$ is determined by solving the problems

$$\Lambda_{(2.2)}\, z^{r+\frac{k}{K}}(x) = f(x), \quad x \in D_h^k, \tag{2.10a}$$

$$z^{r+\frac{k}{K}}(x) = \left\{ \begin{array}{ll} \widehat{z}^{\,r}(x), & k = 1, \\ z^{r+\frac{k-1}{K}}(x), & k \geq 2 \end{array} \right\}, \qquad x \in \Gamma_h^k$$

$$\text{for } \ x \in \overline{D}_h^k, \ \ k = 1, \ldots, K,$$

where

$$z^{r+\frac{k}{K}}(x) = \left\{ \begin{array}{ll} z^{r+\frac{k}{K}}(x), & x \in \overline{D}_h^k, \\ \left. \begin{array}{ll} \widehat{z}^{\,r}(x), & k = 1, \\ z^{r+\frac{k-1}{K}}(x), & k \geq 2 \end{array} \right\}, & x \in \overline{D}_h \setminus \overline{D}_h^k \end{array} \right\} \tag{2.10b}$$

$$\text{for } \ x \in \overline{D}_h; \quad k = 1, \ldots, K, \ \ r = 0, 1, 2, \ldots.$$

Now define $z^{r+1}(x)$ by

$$z^{r+1}(x) = z^{r+\frac{K}{K}}(x), \quad x \in \overline{D}_h.$$

In the case of overlapping subdomains the extension $\widehat{z}^{\,r}(x)$ of the function $z^r(x)$, $x \in \overline{D}_h$ is defined by the relations

$$\widehat{z}^{\,r}(x) = \left\{ \begin{array}{ll} z^r(x), & x \in D_h, \\ \varphi(x), & x \in \Gamma_h; \end{array} \right. \qquad x \in \overline{D}_h, \ \ r = 0, 1, 2, \ldots . \tag{2.10c}$$

We must find the sequence of the functions $z^r(x)$, $x \in \overline{D}_h$, $r = 1, 2, \ldots$, that is, the solution of the grid analog of the Schwartz method.

The difference scheme (2.10), (2.9) is monotone. The following analog of the discrete maximum principle is valid.

Lemma 2.4. *Let the functions* $z^{r+\frac{k}{K}}_{(2.10b)}(x)$, $x \in \overline{D}_h$, $k = 1, \ldots, K$, $r = 0, 1, 2, \ldots$ *satisfy the conditions*

$$\Lambda_{(2.2)}\, z^{r+\frac{k}{K}}(x) \leq 0, \quad x \in D_h^k,$$

$$z^{r+\frac{k}{K}}(x) \geq \left\{ \begin{array}{ll} \widehat{z}^{\,r}(x), & k = 1, \\ z^{r+\frac{k-1}{K}}(x), & k \geq 2 \end{array} \right\}, \qquad x \in \Gamma_h^k;$$

$$z^0(x) \geq 0, \quad x \in \overline{D}_h,$$

$$z^{r+\frac{k}{K}}(x) \geq 0, \;\; x \in \Gamma_h; \quad k = 1, \ldots, K, \;\; r = 0, 1, 2, \ldots \,.$$

Then $z^{r+\frac{k}{K}}(x) \geq 0, \;\; x \in \overline{D}_h, k = 1, \ldots, K, \;\; r = 0, 1, 2, \ldots.$

If condition (2.8) holds, for the solutions of the iterative difference scheme (2.10), (2.9), (2.3) we have the estimate

$$|\, z(x) - z^r(x) \,| \leq M\, q^r, \quad x \in \overline{D}_h^{\star}, \tag{2.11}$$

where $q \leq 1 - m$ and $z(x)$ and $z^r(x)$ are the solutions of the difference schemes (2.2) and (2.10), (2.9), respectively.

The estimates (2.5) and (2.11) imply the ε-uniform convergence (for $N, \; r \to \infty$) of the difference scheme (2.10), (2.9), (2.3):

$$|\, u(x) - z^r(x) \,| \leq M \left[N^{-1} \ln^2 N + q^r \right], \quad x \in \overline{D}_h^{\star}, \tag{2.12}$$

where $q = q_{(2.11)}$.

Theorem 2.1. *The condition* (2.8) *is necessary and, when condition* (2.4) *is satisfied, it is also sufficient for* ε*-uniform convergence (as* $N, \; r \to \infty$*) of the iterative difference scheme* (2.10), (2.9), (2.3). *For the solutions* $z^r(x)$ *of the iterative scheme the estimate* (2.11) *holds, and also under condition* (2.4) *with* $\alpha \geq 2$ *the estimate* (2.12) *is valid.*

3 Difference schemes on minimally overlapping subdomains

For the boundary value problem (1.2), (1.1) we construct iterative difference schemes and study their convergence on overlapping subdomains with minimal overlapping width.

We begin with consideration of the iterative scheme in the case when the mesh

$$\overline{D}_h \tag{3.1}$$

is uniform with step-size $h = d\, N^{-1}$, and the overlapping of the subdomains has minimal width, which equals the step-size h. Then the condition $\delta = h$ holds. For $\varepsilon = 1$, the solutions of the iterative scheme satisfy the estimate

$$\left|\, z_{(2.2)}(x) - z^r_{(2.10)}(x) \,\right| \leq M\, q^r, \quad x \in \overline{D}_{h(3.1)}, \tag{3.2a}$$

where

$$1 - q \;\leq\; m\, N^{-1}. \tag{3.2b}$$

Here $z_{(2.2)}(x)$ and $z^r_{(2.10)}(x)$ are the solutions of schemes (2.2) and (2.10), (2.9), respectively.

If estimates (3.2a) and (3.2b) are valid for $1 - q \le m\,\mu(\varepsilon, N)$, but estimate (3.2a) is generally false in the case of the condition $\mu(\varepsilon, N) = o(1 - q)$ as $1 - q \to 0$, then we shall say that the estimate for q is best possible (or unimprovable) with respect to the quantity $\mu(\varepsilon, N)$.

Note that estimate (3.2b) is unimprovable with respect to N^{-1}. Thus, the solutions of the iterative difference scheme for fixed ε do not converge, as $r \to \infty$, to the solution of scheme (2.2) uniformly in N.

In the case of minimally overlapping subdomains, if we use the piecewise uniform meshes (2.3), the two following cases can be realized:

$$\delta = h^{(1)}_{(2.3)} \ge m \, \min \left[\varepsilon^{-1}, \, \ln N\right] \varepsilon \, N^{-1}, \tag{3.3a}$$

$$\delta = h^{(2)}_{(2.3)} \ge m \, N^{-1}. \tag{3.3b}$$

The case (3.3b) takes place if, for all the boundaries Γ^k, the condition

$$\left\{\Gamma^k \setminus \Gamma\right\} \subset [\,\sigma, d\,], \quad \text{where } \sigma = \sigma_{(2.3)}, \;\; k = 1, \dots, K,$$

is fulfilled, that is all the overlapping subdomains are outside the region $[0, \sigma]$, which contains the boundary layer; and/or if the parameter ε satisfies the condition $\varepsilon \ge m \, \ln^{-1} N$.

Under conditions (3.3) the following estimate is valid:

$$\left| \, z_{(2.2)}(x) - z^r_{(2.10)}(x) \, \right| \le M \, q^r, \quad x \in \overline{D}^{\star}_{h(2.3)}, \tag{3.4a}$$

where

$$1 - q \le m \left\{ \begin{array}{ll} \min \left[\varepsilon^{-1}, \ln N\right] N^{-1} \equiv \mu_1(\varepsilon, N) & \text{under condition (3.3a),} \\ \min \left[\varepsilon^{-1}, N\right] N^{-1} \equiv \mu_2(\varepsilon, N) & \text{under condition (3.3b)} \end{array} \right\} \equiv \tag{3.4b}$$
$$\equiv m \, \mu(\varepsilon, N).$$

Case (3.3a) holds when at least one of the overlapping subdomains is entirely within the fine mesh region $[0, \sigma]$. The estimate (3.4b) is unimprovable with respect to the quantity $\mu(\varepsilon, N)$; note that $\inf\limits_{\varepsilon \in (0,1]} \mu(\varepsilon, N) = N^{-1}$, and also $\mu(\varepsilon, N) = O(N^{-1})$ for fixed values of the parameter ε.

Thus, the solutions of the iterative difference scheme (2.10), (2.9), (2.3) under condition (3.3) (that is, with minimal overlapping width) converge, as $r \to \infty$, to the solution of scheme (2.2), (2.3) ε-uniformly for fixed N; however, as $r, \; N \to \infty$, they do not converge for fixed ε if the values of r and N are independent.

The solutions of this iterative scheme do converge ε-uniformly for $r, \; N \to \infty$ only in that case, when the values of r and N obey some restriction. In other words, the solutions converge conditionally under the condition

$$\rho_1(r, N, \varepsilon) \equiv r \mu_{(3.4)}(\varepsilon, N) \to \infty \quad \text{for} \quad r, \; N \to \infty; \tag{3.5a}$$

in particular, provided that the values of $r,\ N$ satisfy the condition

$$\rho_2(r,N) \equiv r\,N^{-1} \to \infty \quad \text{for} \quad r,\ N \to \infty. \tag{3.5b}$$

Under condition (3.5a) the following estimates are valid

$$|\,z(x) - z^r(x)\,| \le M\ \exp\left(-m\rho_1(r,N,\varepsilon)\right), \quad x \in \overline{D}^\star_{h(2.3)}, \tag{3.6}$$

$$|\,u(x) - z^r(x)\,| \le M\left[N^{-1}\ln^2 N + \exp\left(-m\rho_1(r,N,\varepsilon)\right)\right], \quad x \in \overline{D}^\star_{h(2.3)}, \tag{3.7}$$

that is, in the case of conditions (3.3) and (3.5a) the iterative scheme converges ε-uniformly as $r,\ N \to \infty$. If condition (3.5b) holds, then we have the estimate

$$|\,u(x) - z^r(x)\,| \le M\left[N^{-1}\ln^2 N + \exp(-m\,r\,N^{-1})\right], \quad x \in \overline{D}^\star_{h(2.3)}. \tag{3.8}$$

Theorem 3.1. *The iterative difference scheme* (2.10), (2.9) *on the meshes* (2.1) *and* (2.3), *that is, the scheme based on the overlapping Schwartz method with coherent meshes on the subdomains, does not converge, as* $r,\ N \to \infty$, *for fixed values of the parameter* ε *in the case of minimally overlapping subdomains. On the meshes* (2.3) *the solutions of this scheme converge* ε*-uniformly for* $r,\ N \to \infty$, *but conditionally (with respect to* r, N *and* ε*): the condition* $(3.5a)$ *is necessary and, if condition* (2.4) *holds, it is also sufficient (as well as condition* $(3.3b)$*) for* ε*-uniform convergence. For the solutions of the iterative difference scheme the estimates* (3.4) *and* (3.6) *hold, and also under condition* (2.4) *with* $\alpha \ge 2$ *the estimates* (3.7) *and* (3.8) *are valid.*

4 Numerical Results

In the following tables we present results for the sample problem

$$\varepsilon u''(x) + 2u'(x) = 0, \;\; 0 < x < 1, \qquad u(0) = 1.0, \quad u(1) = 0.0$$

solved on a uniform mesh with mesh spacing $h = 1/N$ and on a piecewise uniform mesh $\overline{D}^\star_h$ of the form given in (2.3). To simplify the presentation, we restrict ourselves to the case of a Schwartz domain decomposition method with two intervals. In all cases considered, the initial guess was $u(x) \equiv 0,\ \ x \in (0,1], \quad u(0) = 1.0$, and the iteration proceeded until the difference between successive iterates was 10^{-5}. We note that in theory this may not be sufficient to exhibit uniform convergence on the piecewise uniform mesh, and in practice it was seen that the errors for N **and** ε large were still significant relative to the discretization error. In this case, this would not effect the rate of uniform convergence determined in the usual double mesh manner, since this is dominated by the errors for small ε.

The intervals are chosen to provide minimal overlap of one mesh interval or slightly more than minimal overlap consisting of two mesh intervals. In the case of a uniform mesh, the overlap interval is chosen to have the $N/2^{nd}$ mesh point as one endpoint. Thus it is either $[x_{N/2-1}, x_{N/2}]$ or $[x_{N/2}, x_{N/2+1}]$. Table 1 gives the error reduction per iteration q

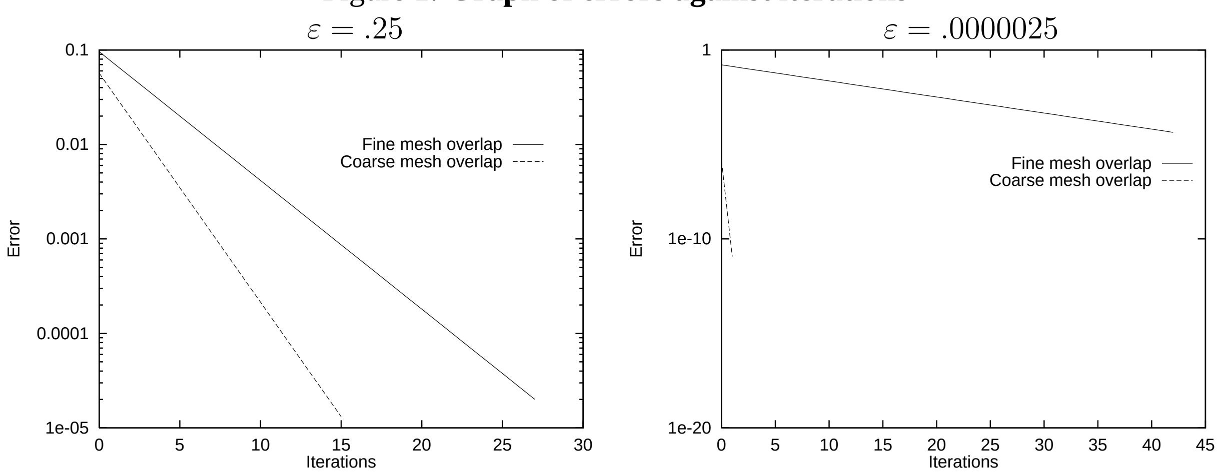
for these cases. It should be noted that although the factors differ somewhat they are close and the number of iterations to achieve an accuracy of 10^{-5} in each case differs by at most one iteration. Note that, when $\varepsilon < 2^{-5}$, the error is reduced below the rounding error level on the first iteration and thus no accurate error reduction q can be determined.

Table 1: **Error reduction per iteration** q

	Overlap [N/2-1,N/2]							Overlap [N/2,N/2+1]						
	Number of Mesh Points N							Number of Mesh Points N						
ε	8	16	32	64	128	256	512	8	16	32	64	128	256	512
2^{-1}	.540	.730	.852	.922	.960	.980	.990	.540	.730	.852	.922	.960	.980	.990
2^{-2}	.452	.644	.790	.884	.939	.969	.984	.452	.644	.790	.884	.939	.969	.984
2^{-3}	.322	.497	.666	.800	.889	.941	.970	.322	.497	.666	.800	.889	.941	.970
2^{-4}	.198	.333	.500	.667	.800	.889	.941	.198	.333	.500	.667	.800	.889	.941
2^{-5}	.111	.200	.333	.500	.667	.800	.889	.111	.200	.333	.500	.667	.800	.889
2^{-6}	.588-1	.111	.200	***	***	***	.100+1	.588-1	.111	.200	***	***	***	.100+1
2^{-7}	.303-1	.588-1	***	***	***	.500	.100+1	.303-1	.588-1	***	***	***	.500	.100+1

In the case of the piecewise uniform mesh given by (2.3) the overlap is also chosen to be one or two intervals at or around the $N/2^{nd}$ point. This is also the transition point σ between the fine mesh and the coarse mesh. Thus in the case of a one interval overlap, it comprises either an interval in the fine mesh $[x_{N/2-1}, x_{N/2}]$ or one in the coarse mesh $[x_{N/2}, x_{N/2+1}]$. The errors for a mesh of 16 intervals and for values of $\varepsilon = .25$ and $.0000025$ are given in Figure 1. Note that the error is plotted on a logarithmic scale. It is clear that the plot is log-

Figure 1: **Graph of errors against iterations**



linear indicating the error reduces exponentially as shown by (3.7) and (3.8). Note that this evidence indicates that the error reduction per iteration q can be determined as the slope of the line. In practice, the q was determined as the slope for the second iteration, provided this did not differ by more than 10% from that of the third iteration. This was necessary, since as can be seen from Tables 2 and 4 the error reduces very rapidly for small ε. Thus after 3 or 4 iterations we commonly reach errors of the order of the rounding error even using double precision floating point numbers. The values after one iteration could be influenced by starting errors and hence were not used. In the tables below, cases where the 10% criterion was not met are indicated by ***. Table 2 gives the error reduction for the one interval overlap case. Note that it is of critical importance that the overlap be in the coarse mesh. In this case, the overlap size δ satisfies (3.3b) that is $\delta = h^{(2)}_{(2.3)} \geq m\,N^{-1}$. When the overlap is in the fine mesh δ satisfies (3.3a) that is $\delta = h^{(1)}_{(2.3)} \geq m \, \min\,[\varepsilon^{-1}, \ln N]\,\varepsilon\,N^{-1}$. This is somewhat easier to see by considering Table 3, which gives the corresponding values of $1-q$ for these two cases. The significance of the rates q may be seen from Table 4 which gives the number of iterations required to achieve an error of 10^{-5}.

Table 2: **Error reduction per iteration q for one interval overlap**

	Overlap in Fine Mesh							Overlap in Coarse Mesh						
	Number of Mesh Points N							Number of Mesh Points N						
ε	8	16	32	64	128	256	512	8	16	32	64	128	256	512
2^{-1}	.540	.730	.852	.922	.960	.980	.990	.540	.730	.852	.922	.960	.980	.990
2^{-2}	.615	.731	.815	.884	.939	.969	.984	.331	.572	.767	.884	.939	.969	.984
2^{-3}	.652	.765	.838	.892	.932	.959	.976	.175	.347	.548	.726	.850	.924	.963
2^{-4}	.674	.788	.857	.903	.937	.961	.977	.875-1	.191	.345	.528	.700	.828	.908
2^{-5}	.685	.804	.872	.915	.944	.964	.978	.434-1	.100	.198	.342	.517	.685	.816
2^{-6}	.692	.812	.881	.923	.950	.968	.980	.216-1	.510-1	.106	.200	.339	.510	.677
2^{-7}	.695	.817	.887	.929	.955	.971	.982	.107-1	.257-1	.552-1	.109	.201	.337	.505
2^{-8}	.697	.820	.890	.933	.959	.974	.984	.536-2	.129-1	.281-1	.572-1	.110	.201	.335
2^{-9}	.698	.821	.892	.935	.961	.976	.985	.268-2	.648-2	.142-1	.293-1	.581-1	.111	.200
2^{-10}	.698	.822	.893	.935	.962	.977	.987	.134-2	.324-2	.713-2	.148-1	.298-1	.585-1	.111
2^{-11}	.698	.822	.893	.936	.962	.978	.987	.669-3	.162-2	.357-2	.746-2	.151-1	.301-1	.587-1
2^{-12}	.698	.822	.893	.936	.962	.978	.988	.334-3	.811-3	.179-2	.374-2	.762-2	.153-1	.302-1
2^{-13}	.698	.822	.893	.936	.963	.978	.988	.167-3	.406-3	.895-3	.187-2	.382-2	.769-2	.153-1
2^{-14}	.699	.822	.893	.936	.963	.979	.988	.835-4	.203-3	.448-3	.937-3	.191-2	.386-2	.772-2
2^{-15}	.699	.822	.893	.936	.963	.979	.988	.417-4	.101-3	.224-3	.469-3	.958-3	.193-2	.388-2
2^{-16}	.699	.822	.893	.936	.963	.979	.988	.210-4	.507-4	.112-3	.235-3	.479-3	.967-3	.194-2
2^{-17}	.699	.822	.893	.936	.963	.979	.988	***	.253-4	.560-4	.117-3	.240-3	.484-3	.972-3
2^{-18}	.699	.822	.893	.936	.963	.979	.988	***	***	.281-4	.587-4	.120-3	.242-3	.486-3

Table 3: **Values of** $1-q$

	Overlap in Fine Mesh							Overlap in Coarse Mesh						
	Number of Mesh Points N							Number of Mesh Points N						
ε	8	16	32	64	128	256	512	8	16	32	64	128	256	512
2^{-1}	.460	.270	.148	.778-1	.399-1	.202-1	.102-1	.460	.270	.148	.778-1	.399-1	.202-1	.102-1
2^{-2}	.385	.269	.185	.116	.612-1	.315-1	.160-1	.669	.428	.233	.116	.612-1	.315-1	.160-1
2^{-3}	.348	.235	.162	.108	.684-1	.410-1	.237-1	.825	.653	.452	.274	.150	.758-1	.368-1
2^{-4}	.326	.212	.143	.968-1	.631-1	.390-1	.230-1	.912	.809	.655	.472	.300	.172	.916-1
2^{-5}	.315	.196	.128	.855-1	.565-1	.360-1	.219-1	.957	.900	.802	.658	.483	.315	.184
2^{-6}	.308	.188	.119	.766-1	.500-1	.323-1	.203-1	.978	.949	.894	.800	.661	.490	.323
2^{-7}	.305	.183	.113	.707-1	.448-1	.287-1	.182-1	.989	.974	.945	.891	.799	.663	.495
2^{-8}	.303	.180	.110	.673-1	.414-1	.257-1	.162-1	.995	.987	.972	.943	.890	.799	.665
2^{-9}	.302	.179	.108	.655-1	.394-1	.238-1	.145-1	.997	.994	.986	.971	.942	.889	.800
2^{-10}	.302	.178	.107	.645-1	.384-1	.227-1	.134-1	.999	.997	.993	.985	.970	.941	.889
2^{-11}	.302	.178	.107	.640-1	.378-1	.220-1	.128-1	.999	.998	.996	.993	.985	.970	.941
2^{-12}	.302	.178	.107	.638-1	.375-1	.217-1	.125-1	1.000	.999	.998	.996	.992	.985	.970
2^{-13}	.302	.178	.107	.636-1	.374-1	.216-1	.123-1	1.000	1.000	.999	.998	.996	.992	.985
2^{-14}	.301	.178	.107	.636-1	.373-1	.215-1	.122-1	1.000	1.000	1.000	.999	.998	.996	.992
2^{-15}	.301	.178	.107	.635-1	.372-1	.214-1	.121-1	1.000	1.000	1.000	1.000	.999	.998	.996
2^{-16}	.301	.178	.107	.635-1	.372-1	.214-1	.121-1	1.000	1.000	1.000	1.000	1.000	.999	.998
2^{-17}	.301	.178	.107	.635-1	.372-1	.214-1	.121-1	***	1.000	1.000	1.000	1.000	1.000	.999
2^{-18}	.301	.178	.107	.635-1	.372-1	.214-1	.121-1	***	***	1.000	1.000	1.000	1.000	1.000

Table 4: **Number of iterations for one interval overlap.**

	Overlap in Fine Mesh							Overlap in Coarse Mesh						
	Number of Mesh Points N							Number of Mesh Points N						
ε	8	16	32	64	128	256	512	8	16	32	64	128	256	512
2^{-1}	16	28	49	86	153	270	471	15	27	48	86	152	269	470
2^{-2}	20	27	34	47	78	130	212	9	15	27	46	77	129	211
2^{-3}	22	31	39	49	61	71	69	6	9	13	20	31	45	55
2^{-4}	24	35	45	55	66	75	71	5	6	8	12	17	24	32
2^{-5}	25	38	51	63	74	81	74	4	5	6	8	10	14	19
2^{-6}	26	40	55	71	84	90	80	3	4	4	5	7	9	12
2^{-7}	26	41	58	77	94	102	90	3	3	4	4	5	6	8
2^{-8}	27	42	60	81	101	114	101	2	3	3	3	4	4	5
2^{-9}	27	42	61	83	107	123	112	2	2	3	3	3	3	4
2^{-10}	27	42	61	84	110	129	121	2	2	2	2	3	3	3
2^{-11}	27	42	61	85	111	133	128	2	2	2	2	2	2	3
2^{-12}	27	42	62	85	112	135	131	2	2	2	2	2	2	2
2^{-13}	27	42	62	85	113	136	133	2	2	2	2	2	2	2
2^{-14}	27	42	62	85	113	136	134	2	2	2	2	2	2	2
2^{-15}	27	42	62	86	113	137	135	2	2	2	2	2	2	2
2^{-16}	27	42	62	86	113	137	135	1	1	1	1	1	1	1
2^{-17}	27	42	62	86	113	137	135	1	1	1	1	1	1	1
2^{-18}	27	42	62	86	113	137	135	1	1	1	1	1	1	1

References

[1] P.A. Farrell, I.P. Boglaev, V.V. Sirotkin, Parallel Domain Decomposition Methods for Semi-linear Singularly Perturbed Differential Equations, *Computational Fluid Dynamics Journal*, 2(4), 423–434, 1994.

[2] P.A. Farrell, J.J.H. Miller, E. O'Riordan and G.I. Shishkin. A Uniformly Convergent Finite Difference Scheme for a Singularly Perturbed Semilinear Equation. *SIAM Journal on Numerical Analysis*, 33: 1135–1149, 1996.

[3] J.J.H. Miller, E. O'Riordan and G.I. Shishkin. Fitted Numerical Methods for Singular Perturbation Problems. World Scientific, Singapore, 1996.

[4] G.I. Shishkin. Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations. Ural Branch of Russian Acad. Sci., Ekaterinburg, 1992 (in Russian).

[5] G.I. Shishkin. Acceleration of the process of the numerical solution to singularly perturbed boundary value problems for parabolic equations on the basis of parallel computations. *Russ. J. Numer. Anal. Math. Modelling.*, 12: 271–291, 1997.