Minimum Spanning Tree

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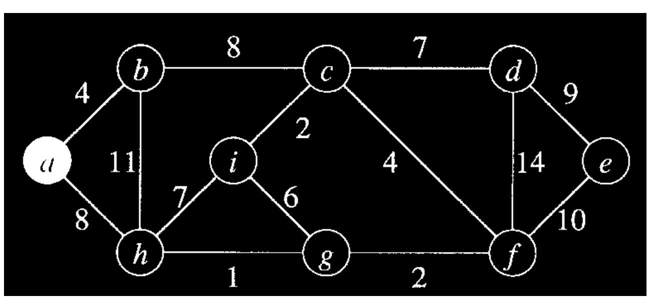
Minimum Spanning Tree

- G=(V,E) is an undirected graph, where V is a set of nodes and E is a set of possible interconnections between pairs of nodes.
- For each edge (u,v) in E, we have a weight W(u,v).
- Find an acyclic subset T of E, that connects all the vertices and whose total weight is minimum.



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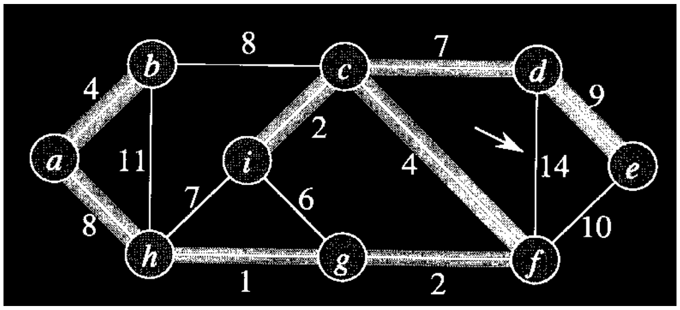
A Spanning Trees



Quiz: Are minimum spanning trees unique?

Kruskal’s Algorithm

- Consider v isolated trees in the forest. Each initially with only one node.
- Pick the shortest path that connects two trees in the forest.
- In other words, select a least-cost edge that does not result in a cycle when added to a set of already selected edges.

Algorithm

Spanning Tree edge set.

Initialize V forests.

Sort edges.

```
MST-KRUSKAL(G, w)
A ← ∅
for each vertex v ∈ V[G]
     do MAKE-SET(v)
sort the edges of E by nondecreasing weight w
for each edge (u, v) ∈ E, in order by nondecreasing weight
     do if FIND-SET(u) ≠ FIND-SET(v)
           then A ← A ∪ {(u, v)}
                UNION(u, v)
return A
```

Make sure two ends are in two trees. No cycle.

Merge the trees in the forest

If no cycle, add the edge in the spanning tree set.

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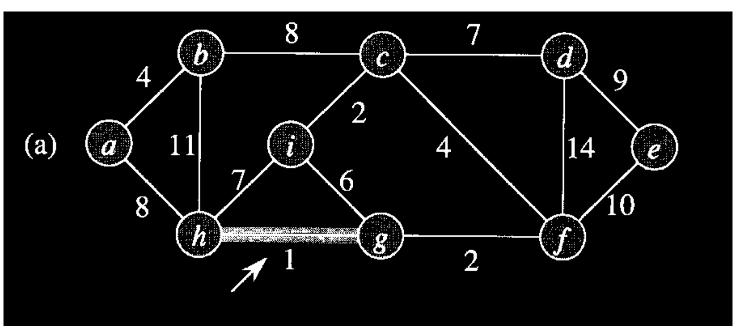
Example: Kruskal's Algorithm



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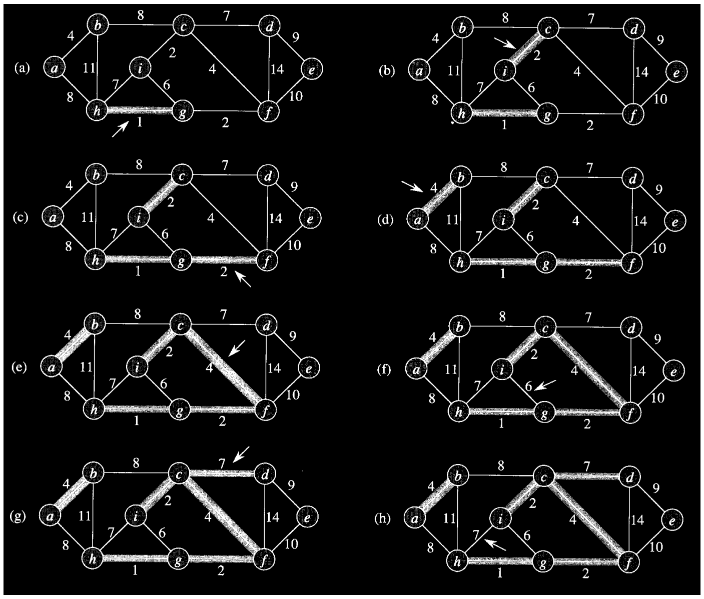
Example: Kruskal’s Algorithm



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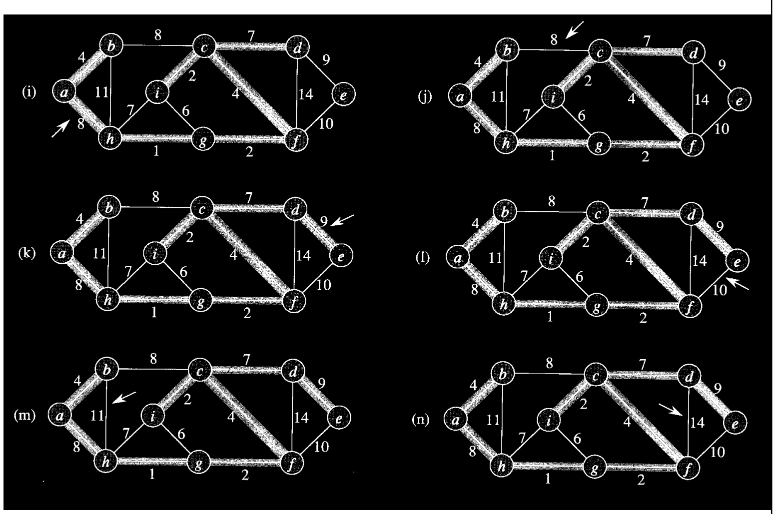
Example: Continued..



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Proof of Correctness of an Algorithm

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Correctness of Krsukal's Algorithm

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- Let T be the tree found by Kruskal's algorithm.
- Let U be the actual minimum spanning tree.
- We will prove cost of T= cost of U.
- Do you agree?
  - T and U and all spanning trees must have exactly V-1 edges.
  - If , k (k>o) number of edges in U are not in T, then exactly k number of edges in T must not be in U.
- We will one by one substitute a unique edge of U by unique edge of T to prove that the cost does not change.

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Correctness of Kruskal's Algorithm (contd..)

- Let e be the least-cost edge in T that is not in U.
- Add e to U.
  - It must create a cycle.
  - There must be an edge f in this cycle which was not in T.
- Take it out. The new spanning tree has cost V=U+{e}-{f}
- Can {e} < {f}?
  - No because , then U cannot be minimum spanning tree.
- Can {e}>{f}?
  - No because, then f will be included by Kruskal's greedy scheme before e. That did not happen!
- Therefore {e}={f}
- Therefore T=U

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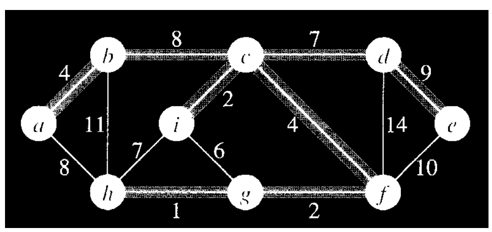
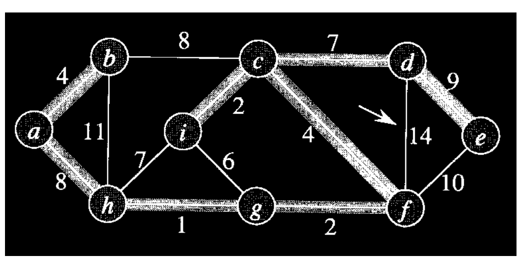
Correctness of Kruskal's algorithm

T



U



Two solution's with same cost.

V=U+{b,c}-{a,h}

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Complexity of Kruskal's Algorithm

```
MST-KRUSKAL(G, w)
1  A ← ∅
2  for each vertex v ∈ V[G]
3       do MAKE-SET(v)
4  sort the edges of E by nondecreasing weight w
5  for each edge (u, v) ∈ E, in order by nondecreasing weight
6       do if FIND-SET(u) ≠ FIND-SET(v)
7             then A ← A ∪ {(u, v)}
8                  UNION(u, v)
9  return A
```

- 1-3: Initialization O(v)
- 4: sorting O(E log E)
- 5: E iterations.
- 6: Each FIND-SET is O(log E) total cost= 2E.O(log E)
- 7: 2E
- 8: UNION is at most V-1
- Overall complexity is O(V+E log E)

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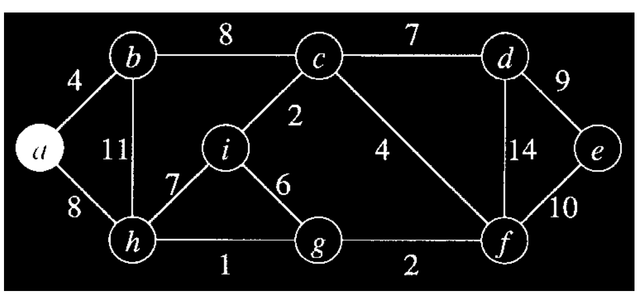
Prim's Algorithm

- Like Kruskal's, but, start with any node.
- Extend the tree to the closest node!



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Prim's Algorithm

```
MST-PRIM(G, w, r)
    Q ← V[G]
    for each u ∈ Q
        do key[u] ← ∞
    key[r] ← 0
    π[r] ← NIL
    while Q ≠ ∅
        do u ← EXTRACT-MIN(Q)
            for each v ∈ Adj[u]
                do if v ∈ Q and w(u, v) < key[v]
                    then π[v] ← u
                        key[v] ← w(u, v)
```

Add the vertices in a Queue

Key[u] is the cost of reaching vertex u from current tree set.

Start from any node r. Its cost is zero. PI[u] is the root of u.

Take the vertex closest to the tree.

For each node adj to u, but not in spanning tree, update the reaching cost.

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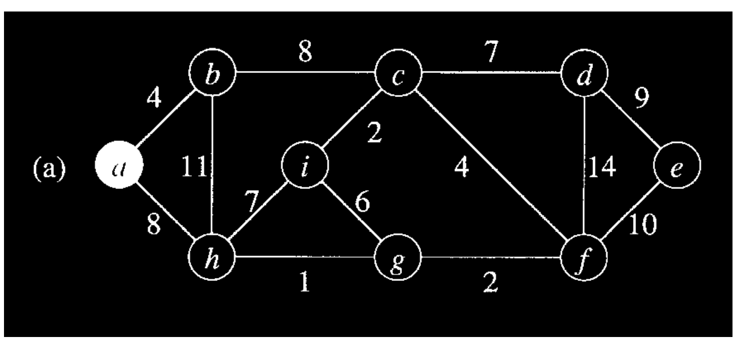
Example: Prim's Algorithm



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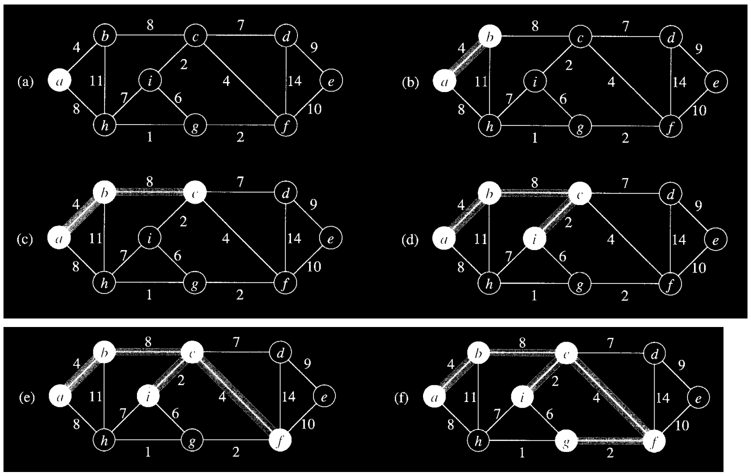
Example: Prim's Algorithm



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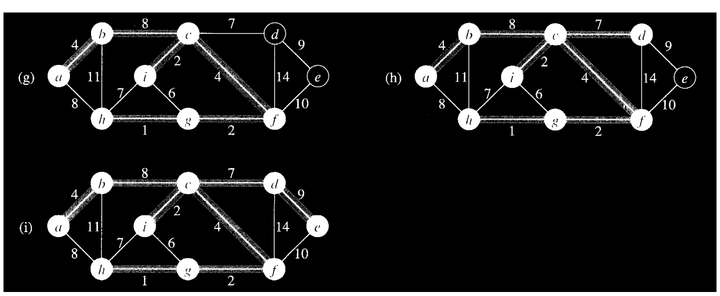
Example: Prim's Algorithm



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Complexity of Prim's Algorithm

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```
MST-PRIM(G, w, r)
1  Q ← V[G]
2  for each u ∈ Q
3       do key[u] ← ∞
4  key[r] ← 0
5  π[r] ← NIL
6  while Q ≠ ∅
7       do u ← EXTRACT-MIN(Q)
8          for each v ∈ Adj[u]
9              do if v ∈ Q and w(u, v) < key[v]
10                   then π[v] ← u
11                        key[v] ← w(u, v)
```

- 1-5: Initialization O(v)
- 6: Loop executes V times.
- 7: Each EXTRACT-MIN is O(log V). Total O(V log V).
- 8: Loop 8-11 executes E times.
- 9: membership can be tested in constant time.
- 11: v have to be deleted from Q (not shown): O(log V)
- Total: O(V log V+E log V)

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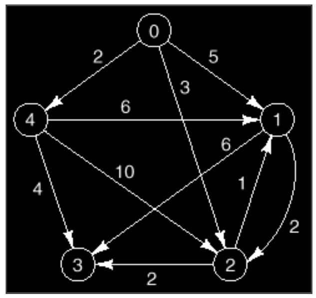
Shortest Path

Shortest Path

Given a directed graph in which each edge has a nonnegative ***weight*** or cost, find a path of least total weight from a given vertex, called the ***source***, to every other vertex in the graph.



- Other Variants:
  - Single Destination shortest-path problem.
  - Single-pair shortest path problem.
  - All pairs shortest-paths problem.

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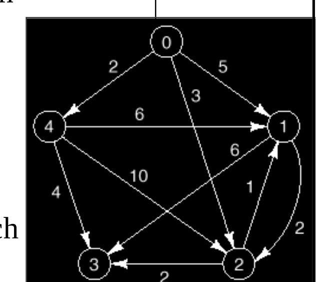
Greedy Method
(Dijkstra's Algorithm)

- We keep a set *S* of vertices whose closest distances to the source, Vertex 0, are known and add one vertex to *S* at each stage.
- We maintain a table *D* that gives, for each vertex *v*, the distance from 0 to *v* along a path all of whose vertices are in *S*, except possibly the last one.
- To determine what vertex to add to *S* at each step, we apply the *greedy* criterion of choosing the vertex *v* with the smallest distance recorded in the table *D*, such that *v* is not already in *S*.



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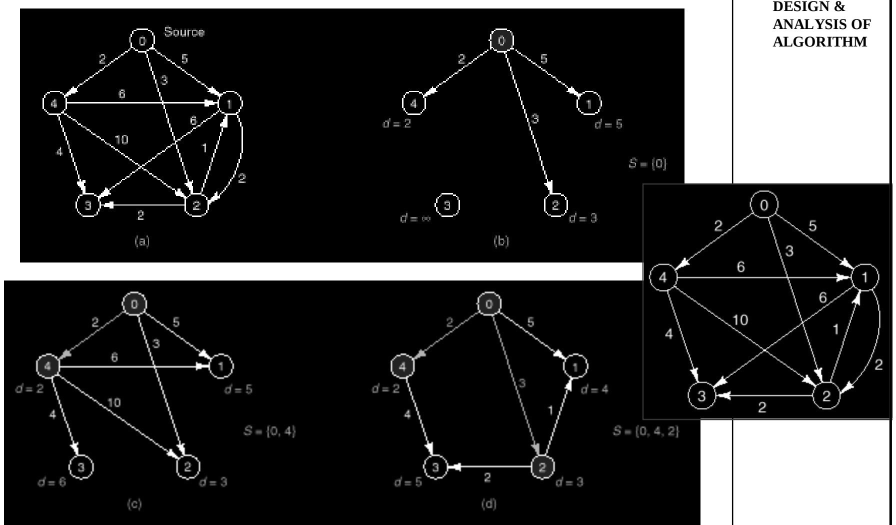
Example



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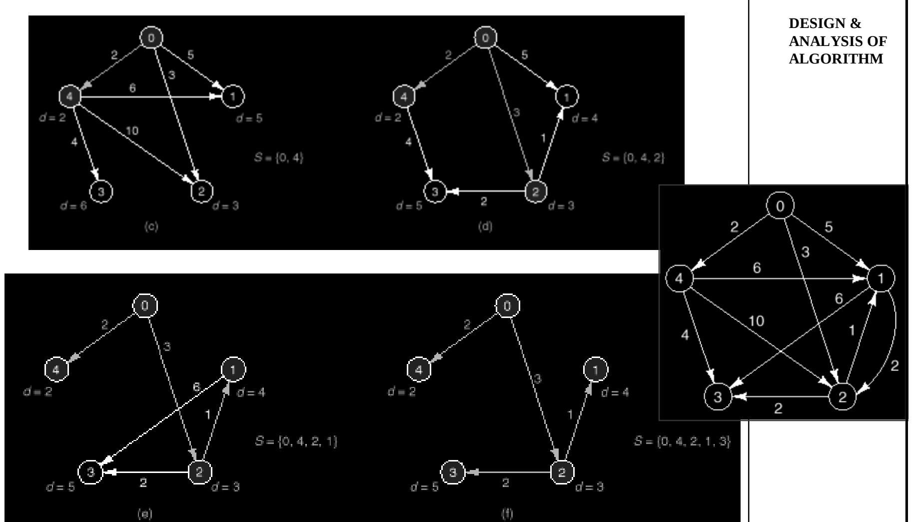
Example (contd..)



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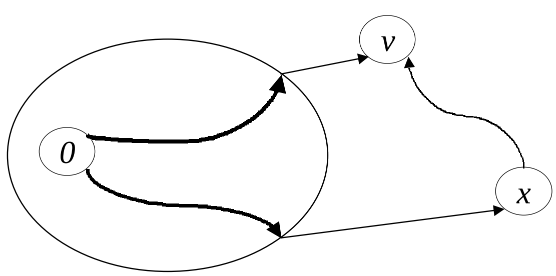
Algorithm

- 1. INITIALIZE_SINGLE-SOURCE(G,s)
- 2. S = EMPTY.
- 3. Q= V[G]
- 4. While Q not EMPTY
- 5.   u= EXTRACT-MIN(Q)
- 6.   Add u in S
- 7.   For each vertex v adjacent to u
- 8.   Do Update cost
    - if D[v] > d[u]+w[u,v]
    - then D[v]=d[u]+w[u,v]
    - GoFrom[v]=u

Proof of Correctness



- Let us assume the path through another node x, which is not yet included in S to v is closer.
- Then D[x] must be smaller than D[v], but in that case x should already be included in S!

Complexity

- Each EXTRACT-MIN takes O(V).
- Each time at least one vertex will be added.
- Therefore it can take at most V iterations.
- Step 5 is $O(v^2)$
- On the other hand, in steps 4-8 each path will be processed only once.
- Thus the complexity is $O(V^2+E)$.

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Bellman-Ford Algorithm

- It can solve the shortest-path problem, even if there are negative weighted links.
- What if there is a negative weighted cycle?
- Its complexity is O(V.E)

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