Dynamic Programming

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Dynamic Programming

DESIGN & ANALYSIS OF ALGORITHM

- Dynamic Programming, like the divide-and-conquer method, solves problems by combining the solutions to sub-problems.

- Pure divide-and-conquer:
  - divides problems into independent sub-problems,
  - solves the sub-problem recursively, and then,
  - combines their solutions to solve the original sub-problem.

- Dynamic programming in contrast is used when the sub-problems are not independent, that is sub-problems share sub-problems.
- It is typically applied to optimization problems.

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Example: Matrix Chain Multiplication

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Matrix Multiplication


```
MATRIX-MULTIPLY(A, B)
if columns[A] ≠ rows[B]
   then error "incompatible dimensions"
   else for i ← 1 to rows[A]
              do for j ← 1 to columns[B]
                       do C[i, j] ← 0
                          for k ← 1 to columns[A]
                               do C[i, j] ← C[i, j] + A[i, k] · B[k, j]
         return C
```

- Cost of multiplying A[p][q] x B[q][r] is p.q.r
- What is the cost of multiplying three matrices A, B, and C of sizes 10x100, 100x5, and 5x50?
- How to find the best way of multiplying?

Matrix Chain Multiplication

- Given a chain $A_1, A_2, A_3, .. A_n$ of n matrices, such than A_i has dimension $p_{i-1} x\, p_i$, find the sequence of multiplication that will result in minimum number of scalar multiplication.
  - Recursive Cost Function Catalan numbers:

$$(A_1.(A_2.(A_3.A_4)))$$
$$((A_1.A_2).(A_3.A_4))$$
$$((A_1.(A_2.A_3)).A_4)$$

$$p(n) = \begin{cases} 1\, if & n=1 \\ \sum_{k=1}^{n-1} p(k).P(n-k),\, if & n>1 \end{cases}$$

$$P(n+1) = \Omega\left(\frac{4^n}{n^{\frac{3}{2}}}\right)$$

Observations

- Existence of Optimal Substructure: In an optimum sequence of decisions, each subsequence must also be optimum.
- $(A_1\ A_2\ A_3\ ).\ (A_4.\ A_5.\ A_6)$
  - Total cost is C(1..3) + C(4..6) + cost of multiplying the two final matrices.
- Recursive Solution Possible: If m[i,j] is the optimum cost of multiplying all matrices between i^{th} and j^{th} matrices ….($A_i\ A_{i+1}$ …. A_j)..
  - if i==j then m[i,j]= 0
  - otherwise,

$$m[i, j] = \min_{i \le k \le j}\{m[i,k] + m[k+1, j] + p_{i-1}.p_k.p_j\}$$

Recursive Solution

```
RECURSIVE-MATRIX-CHAIN(p, i, j)
  if i = j
     then return 0
  m[i, j] ← ∞
  for k ← i to j − 1
      do q ← RECURSIVE-MATRIX-CHAIN(p, i, k)
               + RECURSIVE-MATRIX-CHAIN(p, k + 1, j) + p_{i−1} p_k p_j
         if q < m[i, j]
            then m[i, j] ← q
  return m[i, j]
```

- Running time is exponential $O(2^n)$!

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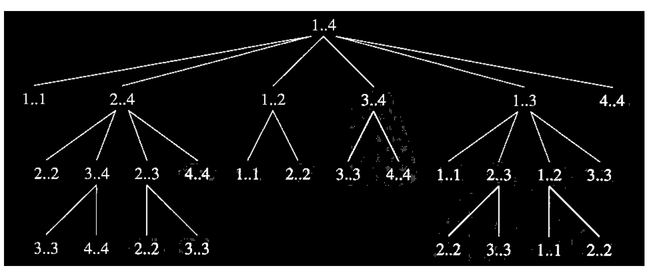
Observation-2



- Existence of Overlapping Sub-problem: the same sub-sequence is part of many super sequences.
- For a string of limited size, the actual number of subproblems are quite small. $O(n^2)$ only!

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Dynamic Programming Solution: A Bottom up approach

- Compute the optimum cost for multiplying all matrix chains of size 2.
- Store them in a matrix m[i,j], when i-j spans two matrices.
- Use the above values to compute optimum cost for multiplying all matrix chains of size 3.
- Then size 4 .. Up to size n.

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Algorithm

```
MATRIX-CHAIN-ORDER(p)
n ← length[p] − 1
for i ← 1 to n
     do m[i, i] ← 0
for l ← 2 to n
     do for i ← 1 to n − l + 1
            do j ← i + l − 1
               m[i, j] ← ∞
               for k ← i to j − 1
                   do q ← m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
                      if q < m[i, j]
                        then m[i, j] ← q
                             s[i, j] ← k
return m and s
```

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Example

matrix	dimension
A_1	30×35
A_2	35×15
A_3	15×5
A_4	5×10
A_5	10×20
A_6	20×25

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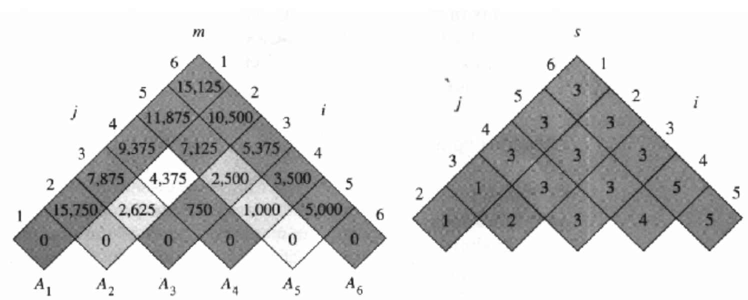

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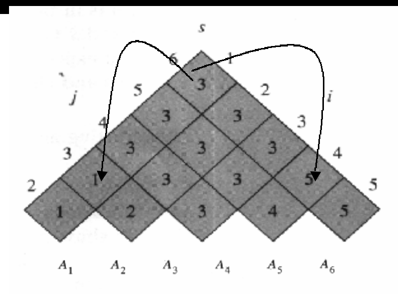
Constructing the Optimal Solution

```
MATRIX-CHAIN-MULTIPLY(A, s, i, j)
if j > i
   then X ← MATRIX-CHAIN-MULTIPLY(A, s, i, s[i, j])
        Y ← MATRIX-CHAIN-MULTIPLY(A, s, s[i, j] + 1, j)
        return MATRIX-MULTIPLY(X, Y)
   else return A_i
```

In the example of Figure 16.1, the call MATRIX-CHAIN-MULTIPLY($A, s,$ $1, 6$) computes the matrix-chain product according to the parenthesization

$$((A_1(A_2A_3))((A_4A_5)A_6)) . \tag{16.3}$$



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Complexity of Algorithm

```
MATRIX-CHAIN-ORDER(p)
n ← length[p] − 1
for i ← 1 to n
     do m[i, i] ← 0
for l ← 2 to n
     do for i ← 1 to n − l + 1
            do j ← i + l − 1
               m[i, j] ← ∞
               for k ← i to j − 1
                   do q ← m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
                      if q < m[i, j]
                         then m[i, j] ← q
                              s[i, j] ← k
return m and s
```

- Running time?
- Space?

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Example: Optimal Polygon Triangulation

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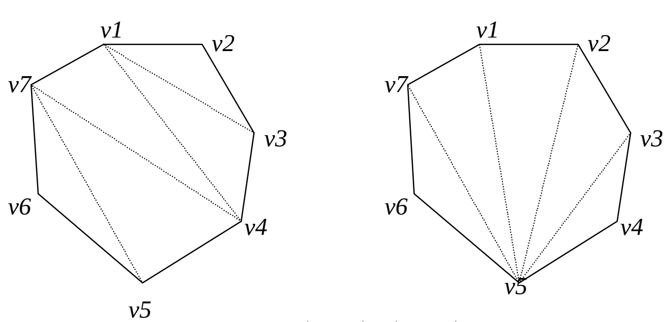
Polygon Triangulation

- We are given a convex polygon $P=\langle v_0, v_1, \ldots v_{n-1} \rangle$ and a weight function w defined on triangles formed by sides and chords of P. The problem is to find a triangulation that minimizes the sum of the weights of the triangles in the triangulation.



$$w(\Delta v_i v_j v_k) = |v_i v_j| + |v_j v_k| + |v_k v_i|$$

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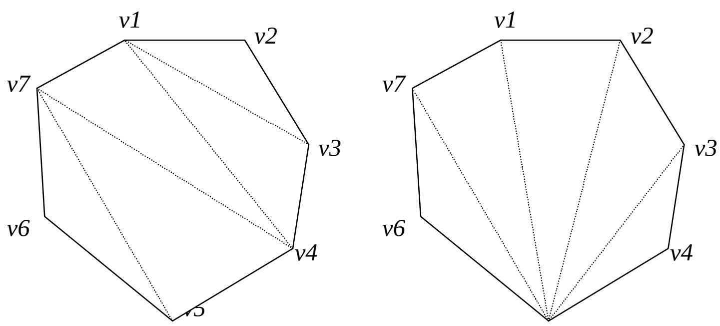
Observations

- Optimum substructure:
- Overlapping subproblems:



Dynamic programming Solution

- For all degenerated polygon of size 2, <v $_{i-1}$, v$_i$> cost = zero.
- For all polygons of size 3 the cost is

$$w(\Delta v_i v_j v_k) = |v_i v_j| + |v_j v_k| + |v_k v_i|$$

- For all polygons of size 4 or more try all division point k and pick the best:

$$t[i, j] = \min_{i \le k \le j-1} \{t[i,k] + t[k+1, j] + w(\Delta v_{i-1} v_k v_j)\}$$

Final Exam

- 4 Questions Total:
  - 1 True-False
  - 1 Overall Concept
  - 1 Tree & Graph
  - 1 String Matching & Dynamic Programming

- Open Book 60 min.

- Project Due/Demo on “Exam Day”
  - Quiz Today or extra 5% goes to project?
  - End Term (20%)
  - Last Assignment (~5%)

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