Eccentricity Approximating Trees ⋆

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Abstract. Using the characteristic property of chordal graphs that they are the intersection graphs of subtrees of a tree, Erich Prisner showed that every chordal graph admits an eccentricity 2-approximating spanning tree. That is, every chordal graph G has a spanning tree T such that $ecc_T(v) - ecc_G(v) \leq 2$ for every vertex v, where $ecc_G(v)$ ($ecc_T(v)$) is the eccentricity of a vertex v in G (in T, respectively). Using only metric properties of graphs, we extend that result to a much larger family of graphs containing among others chordal graphs, the underlying graphs of 7-systolic complexes and plane triangulations with inner vertices of degree at least 7. Furthermore, based on our approach, we propose two heuristics for constructing eccentricity k-approximating trees with small values of k for general unweighted graphs. We validate those heuristics on a set of real-world networks and demonstrate that all those networks have very good eccentricity approximating trees.

1 Introduction

All graphs $G = (V, E)$ occurring in this paper are connected, finite, unweighted, undirected, loopless and without multiple edges. The *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest path connecting u and v in G. If no confusion arises, we will omit subindex G. The *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths, that is, it consists of all vertices (metrically) between u and v: $I(u, v) = \{x \in V : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. The *eccentricity* $ecc_G(v)$ of a vertex v in G is defined by $\max_{u \in V} d_G(u, v)$, i.e., it is the distance to a most distant vertex. The *diameter* of a graph is the maximum over the eccentricities of all vertices: $diam(G) = \max_{u \in V} ecc_G(u) = \max_{u,v \in V} d_G(u, v)$. The *radius* of a graph is the minimum over the eccentricities of all vertices: $rad(G) = \min_{u \in V} ecc_G(u)$. The set of vertices with minimum eccentricity forms the *center* $C(G)$ of a graph G, i.e., $C(G) = \{u \in V : ecc_G(u) = rad(G)\}$.

A spanning tree T of a graph G with $d_T(u, v) - d_G(u, v) \leq k$, for all $u, v \in V$, is known as an *additive tree spanner* of G [18] and, if it exists for a small integer k, then it gives a good approximation of all distances in G by the distances in T. Many optimization problems involving distances in graphs are known to be NP-hard in general but have efficient solutions in simpler metric spaces, with well-understood metric structures, including trees. A solution to such an optimization problem obtained for a tree spanner T of G usually serves as a good approximate solution to the problem in G.

E. Prisner in [29] introduced the new notion of eccentricity approximating spanning trees. A spanning tree T of a graph G is called an *eccentricity k-approximating spanning tree* if $ecc_T(v) - ecc_G(v) \leq k$ holds for all $v \in V$. Such a tree tries to approximately preserve only distances from each vertex v to its most distant vertices and can tolerate larger increases to nearby

⋆ Results of this paper were partially presented at the WG'16 conference vertices. They are important in applications where vertices measure their degree of centrality by means of their eccentricity and would tolerate a small surplus to the actual eccentricities [29]. Note also that Nandakumar and Parthasarasthy considered in [24] eccentricity-preserving spanning trees (i.e., eccentricity 0-approximating spanning trees) and showed that a graph G has an eccentricity 0-approximating spanning tree if and only if: (a) either $diam(G) = 2rad(G)$ and $|C(G)| = 1$, or $diam(G) = 2rad(G) - 1$, $|C(G)| = 2$, and those two center vertices are adjacent; (b) every vertex $u \in V \setminus C(G)$ has a neighbor v such that $ecc_G(v) < ecc_G(u)$.

Every additive tree k-spanner is clearly eccentricity k-approximating. Therefore, eccentricity k-approximating spanning trees can be found in every interval graph for $k = 2$ [18, 20, 28], and in every asteroidal-triple–free graph [18], strongly chordal graph [7] and dually chordal graph [7] for $k = 3$. On the other hand, although for every k there is a chordal graph without a tree k-spanner [18, 28], yet as Prisner demonstrated in [29], every chordal graph has an eccentricity 2-approximating spanning tree, i.e., with the slightly weaker concept of eccentricity-approximation, one can be successful even for chordal graphs.

Unfortunately, the method used by Prisner in [29] heavily relies on a characteristic property of chordal graphs (*chordal graphs are exactly the intersection graphs of subtrees of a tree*) and is hardly extendable to larger families of graphs.

In this paper we present a new proof of the result of [29] using only metric properties of chordal graphs. This allows us to extend the result to a much larger family of graphs which includes not only chordal graphs but also other families of graphs known from the literature.

It is known [9, 33] that every chordal graph satisfies the following two metric properties:

α_1-metric: if $v \in I(u, w)$ and $w \in I(v, x)$ are adjacent, then $d_G(u, x) \geq d_G(u, v) + d_G(v, x) - 1 = d_G(u, v) + d_G(w, x)$.

triangle condition: for any three vertices u, v, w with $1 = d_G(v, w) < d_G(u, v) = d_G(u, w)$ there exists a common neighbor x of v and w such that $d_G(u, x) = d_G(u, v) - 1$.

A graph G satisfying the α_1-metric property is called an α_1-*metric graph*.[1] If an α_1-metric graph G satisfies also the triangle condition then G is called an (α_1, Δ)-*metric graph*. We prove that every (α_1, Δ)-metric graph $G = (V, E)$ has an eccentricity 2-approximating spanning tree and that such a tree can be constructed in $\mathcal{O}(|V||E|)$ total time. As a consequence, we get that the underlying graph of every 7-systolic complex (and, hence, every plane triangulation with inner vertices of degree at least 7 and every chordal graph) has an eccentricity 2-approximating spanning tree.

The paper is organized as follows. In Section 2, we present additional notions and notations and some auxiliary results. In Section 3, some useful properties of the eccentricity function on (α_1, Δ)-metric graphs are described. Our eccentricity approximating spanning tree is constructed and analyzed in Section 4. In Section 5, the algorithm for the construction of an eccentricity approximating spanning tree developed in Section 4 for (α_1, Δ)-metric graphs is generalized and validated on some real-world networks. Our experiments show that all those real-world networks have very good eccentricity approximating trees. Section 6 concludes the paper with a few open questions.

[1] A more general concept of α_i-metric was introduced in [33], however, in this paper, we are interested only in the case when $i = 1$.

2 Preliminaries

For a graph $G = (V, E)$, we use $n = |V|$ and $m = |E|$ to denote the cardinality of the vertex set and the edge set of G. We denote an *induced cycle* of length k by C_k (i.e., it has k vertices) and by W_k an *induced wheel* of size k which is a C_k with one extra vertex universal to C_k. For a vertex v of G, $N_G(v) = \{u \in V : uv \in E\}$ is called the *open neighborhood*, and $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v. The distance between a vertex v and a set $S \subseteq V$ is defined as $d_G(v, S) = \min_{u \in S} d_G(u, v)$ and the set of furthest (most distant) vertices from v is denoted by $F(v) = \{u \in V : d_G(u, v) = ecc_G(v)\}$.

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if for each pair of vertices $u, v \in A$ it includes the interval $I(v, u)$ of G between u, v. An induced subgraph H of G is called *isometric* if the distance between any pair of vertices in H is the same as their distance in G. In particular, convex subgraphs are isometric. The *disk* $D(x, r)$ with center x and radius $r \geq 0$ consists of all vertices of G at distance at most r from x. In particular, the unit disk $D(x, 1) = N[x]$ comprises x and the neighborhood $N(x)$. For an edge $e = xy$ of a graph G, let $D(e, r) := D(x, r) \cup D(y, r)$.

By the definition of α_1-metric graphs clearly, such a graph cannot contain any isometric cycles of length $k > 5$ and any induced cycle of length 4. The following results characterize α_1-metric graphs and the class of chordal graphs within the class of α_1-metric graphs. Recall that a graph is *chordal* if all its induced cycles are of length 3.

Theorem 1 ([33]). *G is a chordal graph if and only if it is an α_1-metric graph not containing any induced subgraphs isomorphic to cycle C_5 and wheel $W_k, k \geq 5$.*

Theorem 2 ([33]). *G is an α_1-metric graph if and only if all disks $D(v, k)$ ($v \in V$, $k \geq 1$) of G are convex and G does not contain the graph W_6^{++} (see Fig. 1) as an isometric subgraph.*

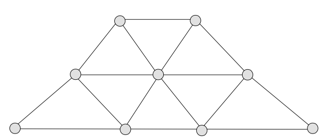

Fig. 1. Forbidden isometric subgraph W_6^{++}.

Theorem 3 ([15, 30]). *All disks $D(v, k)$ ($v \in V$, $k \geq 1$) of a graph G are convex if and only if G does not contain isometric cycles of length $k > 5$, and for any two vertices x, y the neighbors of x in the interval $I(x, y)$ are pairwise adjacent.*

A graph G is called a *bridged graph* if all isometric cycles of G have length three [15]. The class of bridged graphs is a natural generalization of the class of chordal graphs. They can be characterized in the following way.

Theorem 4 ([15, 30]). *$G = (V, E)$ is a bridged graph if and only if the disks $D(v, k)$ and $D(e, k)$ are convex for all $v \in V$, $e \in E$, and $k \geq 1$.*

As a consequence of Theorem 2, Theorem 3 and Theorem 4 we obtain the following equivalences.

Lemma 1. *For a graph $G = (V, E)$ the following statements are equivalent:*

(a) G is an α_1-metric graph not containing an induced C_5;
(b) G is a bridged graph not containing W_6^{++} as an isometric subgraph;
(c) The disks $D(v, k)$ and $D(e, k)$ of G are convex for all $v \in V$, $e \in E$, and $k \geq 1$, and G does not contain W_6^{++} as an isometric subgraph.

Proof. By Theorem 2, if G is an α_1-metric graph then all disks $D(v, k)$ ($v \in V$, $k \geq 1$) of G are convex and G does not contain the graph W_6^{++} as an isometric subgraph. If, in addition, G does not contain induced subgraphs isomorphic to C_5 then, by Theorem 3, G is a bridged graph. Hence, (a) implies (b). Also, (b) implies (c), by Theorem 4, and (c) implies (a), by Theorem 2 and since a graph where $D(e, 1)$ is convex for each $e \in E$ cannot contain an induced C_5. $\square$

As we will show now the class of (α_1, Δ)-metric graphs contains all graphs described in Lemma 1. An induced C_5 is called *suspended* in G if there is a vertex in G which is adjacent to all vertices of the C_5.

Theorem 5. *A graph G is (α_1, Δ)-metric if and only if it is an α_1-metric graph where for each induced C_5 there is a vertex $v \in V$ such that $C_5 \subseteq N(v)$, i.e., every induced C_5 is suspended.*

Proof. Consider an induced $C_5 = (a, b, c, d, e)$ in an (α_1, Δ)-metric graph G. By the triangle condition, for vertex a and edge cd of the C_5, there must exist a vertex v such that $a, c, d \in N(v)$. As G is α_1-metric it cannot have an induced C_4. Thus v must be adjacent to e and b as well.

Consider now an edge xy and a vertex v in an α_1-metric graph where each induced C_5 is suspended. Let $d(x, v) = d(y, v) = k$. Consider an arbitrary neighbor x' of x in $I(v, x)$ and an arbitrary neighbor y' of y in $I(v, y)$. Assume $x'y \notin E$ and $y'x \notin E$ (otherwise, there is nothing to prove). Since $x \in I(x', y)$ and $y \in I(y', x)$, by the α_1-metric property, $d(x', y') \geq 2$ must hold. On the other hand, since the path (x', x, y, y') connecting vertices x' and y' of disk $D(v, k-1)$ has vertices outside the disk, by the convexity of disks of G (see Theorem 2), $d(x', y') \leq 2$ must hold. Thus, $d(x', y') = 2$ and $I(x', y') \subseteq D(v, k-1)$. Consider a common neighbor v' of x' and y' in G. Necessarily, $v' \in D(v, k-1)$. To avoid an induced cycle of length 4 in G, either v' must be adjacent to both x and y (and we are done) or x, y, y', v', x' form an induced C_5. In the latter case, a vertex w, which suspends this C_5, is adjacent to both x and y and is in $I(x', y')$ and, therefore, in $D(v, k-1)$. $\square$

We will also need the following fact.

Lemma 2. *Let $G = (V, E)$ be an (α_1, Δ)-metric graph, let K be a complete subgraph of G, and let v be a vertex of G. If for every vertex $z \in K$, $d(z, v) = k$ holds, then there is a vertex v' at distance $k - 1$ from v which is adjacent to every vertex of K.*

Proof. Consider a vertex x at distance $k-1$ from v which is adjacent to the maximum number of vertices of K. Assume that a vertex $u \in K$ exists which is not adjacent to x. Consider a neighbor y of u in $I(u, v)$. By the choice of x, since $yu \in E$ and $xu \notin E$, there must exist a vertex $w \in K$ such that $xw \in E$ and $yw \notin E$. By the α_1-metric property, applied to $w \in I(x, u)$ and $u \in I(w, y)$, we obtain $d(x, y) \geq 2$. Then, by the convexity of $D(v, k-1)$, vertices y and x must be at distance 2 and any vertex v' from $N(x) \cap N(y)$ must be in $D(v, k-1)$. We claim that at least one vertex $v^* \in N(x) \cap N(y)$ exists which is adjacent to both w and u. Since G cannot have a C_4 as an induced subgraph, if v' from $N(x) \cap N(y)$ is adjacent to one of $\{w, u\}$ then it must be adjacent to the other one as well. Assume $v'w, v'u \notin E$. Then, by the triangle condition, there must exist a vertex v^* which is adjacent to v', w, u and, since any induced C_4 is forbidden, v^* has to be adjacent to x, y. Thus, v^* is at distance $k-1$ from v and adjacent to both w and u. Furthermore, every neighbor z of x in K must be adjacent to v^* as well, since 4-cycle (z, u, v^*, x, z) cannot be induced. Thus, v^* is adjacent to more vertices of K than x is, which is a contradiction. □

We note here, without going into the rich theory of systolic complexes, that the underlying graph of any 7-systolic complex is nothing else than a bridged graph not containing a 6-wheel W_6 as an induced (equivalently, isometric) subgraph (see [11] for this fact and a relation of 7-systolic complexes with CAT(0) complexes). Hence, the class of (α_1, Δ)-metric graphs contains the underlying graphs of 7-systolic complexes and hence all plane triangulations with inner vertices of degree at least 7 [11] (vertices that are not on the outerface are called inner vertices).

3 Eccentricity function on (α_1, Δ)-metric graphs

In what follows, by $C(G)$ we denote not only the set of all central vertices of G but also the subgraph of G induced by this set. We say that the eccentricity function $ecc_G(v)$ on G is *unimodal* if every vertex $u \in V \setminus C(G)$ has a neighbor v such that $ecc_G(v) < ecc_G(u)$. In other words, every local minimum of the eccentricity function $ecc_G(v)$ is a global minimum on G. It this section we will often omit subindex G since we deal only with a graph G here. A spanning tree T of G will be built only in the next section.

In this section, we will show that the eccentricity function $ecc_G(v)$ on an (α_1, Δ)-metric graph G is almost unimodal and that the radius of the center $C(G)$ of G is at most 2.

Lemma 3. *Let G be an α_1-metric graph and x be its arbitrary vertex with $ecc(x) \geq rad(G) + 1$. Then, for every vertex $z \in F(x)$ and every neighbor v of x in $I(x, z)$, $ecc(v) \leq ecc(x)$ holds.*

Proof. Assume, by way of contradiction, that $ecc(v) > ecc(x)$ and consider an arbitrary vertex $u \in F(v)$. Since x and v are adjacent, necessarily, $d(v, u) = ecc(v) = ecc(x) + 1 = d(u, x) + 1$, i.e., $x \in I(v, u)$. By the α_1-metric property,

$$d(u, z) \geq d(u, x) + d(v, z) = ecc(v) - 1 + ecc(x) - 1 = 2ecc(x) - 1 \geq 2rad(G) + 1.$$

The latter gives a contradiction to $d(u, z) \leq diam(G) \leq 2rad(G)$. □

Recall that for every graph G, $diam(G) \leq 2rad(G)$ holds.

Theorem 6. *Let G be an (α_1, Δ)-metric graph and x be an arbitrary vertex of G. If*

(i) $ecc(x) > rad(G) + 1$ *or*
(ii) $ecc(x) = rad(G) + 1$ *and* $diam(G) < 2rad(G)$,

then there must exist a neighbor v of x with $ecc(v) < ecc(x)$.

Proof. For every neighbor v of x, we define the set S_v as the most distant vertices from x which have v on their shortest path from x. Formally,

$$S_v := \{z \in F(x) : v \in I(x, z)\}.$$

Choose a neighbor v of x which maximizes $|S_v|$. We claim that $ecc(v) < ecc(x)$. We know, by Lemma 3, that $ecc(v) \leq ecc(x)$. Assume $ecc(v) = ecc(x)$ and consider an arbitrary vertex $u \in F(v)$.

Suppose first that $x \in I(v,u)$. Then, $d(u,z) \geq d(u,x) + d(v,z) = 2ecc(x) - 2$ holds for every $z \in S_v$ by the α_1-metric property. Hence, if $ecc(x) > rad(G)+1$ then $d(u,z) > 2rad(G)$ and thus a contradiction to $d(u,z) \leq diam(G) \leq 2rad(G)$ arises. If, on the other hand, case (ii) applies, i.e., $ecc(x) = rad(G)+1$ and $diam(G) < 2rad(G)$, then it follows that $d(u,z) \geq 2rad(G) > diam(G)$ and again a contradiction arises.

Now consider the case that $x \notin I(v,u)$. Then $ecc(v) = ecc(x)$ implies that $d(u,x) = d(u,v)$ and $u \in F(x)$. By the triangle condition, there must exist a common neighbor w of x and v such that $w \in I(x,u) \cap I(v,u)$. Since u belongs to S_w but not to S_v, then, by the maximality of $|S_v|$, there must exist a vertex $z \in F(x)$ which is in S_v but not in S_w. Thus, $d(w,z) > d(v,z)$ and $v \in I(w,z)$ must hold. Now, the α_1-metric property applied to $v \in I(w,z)$ and $w \in I(v,u)$ gives $d(u,z) \geq d(u,w) + d(v,z) = 2ecc(x) - 2$. As before we get $d(u,z) > 2rad(G) \geq diam(G)$, if $ecc(x) > rad(G)+1$ (case (i)), and $d(u,z) \geq 2rad(G) > diam(G)$, if $ecc(x) = rad(G)+1$ and $diam(G) < 2rad(G)$ (case (ii)). These contradictions complete the proof. □

Note that the requirement in Theorem 6 that G satisfies the triangle condition cannot be removed. The statement is not true for arbitrary α_1-metric graphs (see Fig. 2).

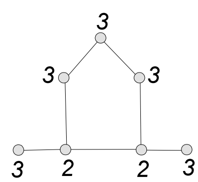

Fig. 2. An α_1-metric graph G with $rad(G) = 2$, $diam(G) = 3 < 2rad(G)$, and with a vertex of eccentricity $3 = rad(G)+1$ that has no neighbor with smaller eccentricity. The numbers next to vertices show their eccentricities.

For each vertex $v \in V \setminus C(G)$ of a graph G we can define a parameter

$$loc(v) = \min\{d(v,x) : x \in V, ecc(x) < ecc(v)\}$$

and call it the *locality* of v. We define the locality of any vertex from $C(G)$ to be 1. Theorem 6 says that if a vertex v with $loc(v) > 1$ exists in an (α_1, Δ)-metric graph G then $diam(G) = 2rad(G)$ and $ecc(v) = rad(G)+1$. That is, only in the case that $diam(G) = 2rad(G)$ the eccentricity function can be not unimodal on G.

Observe that the center $C(G)$ of a graph $G = (V,E)$ can be represented as the intersection of all the disks of G of radius $rad(G)$, i.e., $C(G) = \bigcap\{D(v, rad(G)) : v \in V\}$. Consequently, the center $C(G)$ of an α_1-metric graph G is convex (in particular, it is connected), as the intersection of convex sets is always a convex set. In general, any set $\mathcal{C}_{\leq i}(G) := \{z \in V : ecc(z) \leq rad(G) + i\}$ is a convex set of G as $\mathcal{C}_{\leq i}(G) = \bigcap\{D(v, rad(G)+i) : v \in V\}$.

Corollary 1. *In an α_1-metric graph G, all sets $\mathcal{C}_{\leq i}(G)$, $i \in \{0, \ldots, diam(G) - rad(G)\}$, are convex. In particular, the center $C(G)$ of an α_1-metric graph G is convex.*

The following result gives bounds on the diameter and the radius of the center of an $(\alpha_1, \varDelta)$-metric graph. Previously it was known that the diameter (the radius) of the center of a chordal graph is at most 3 (at most 2, respectively) [10].

Theorem 7. *Let G be an $(\alpha_1, \varDelta)$-metric graph. Then, $diam(C(G)) \leq 3$ and $rad(C(G)) \leq 2$.*

Proof. Assume, by way of contradiction, that there are vertices $s, t \in C(G)$ such that $d(s,t) = 4$. Consider an arbitrary shortest path $P = (s = x_1, x_2, x_3, x_4, x_5 = t)$. Since $C(G)$ is convex any shortest path connecting s and t is in $C(G)$.

First we claim that for any vertex $u \in F(x_3)$ all vertices of P are at distance $r := d(u, x_3) = rad(G)$ from u. As $x_i \in C(G)$, we know that $d(u, x_i) \leq r$ $(1 \leq i \leq 5)$. Assume $d(u, x_i) = r-1$, $d(u, x_{i+1}) = r$, and $i \leq 2$. Then, the α_1-metric property applied to $x_i \in I(u, x_{i+1})$ and $x_{i+1} \in I(x_i, x_{i+3})$ gives $d(x_{i+3}, u) \geq r-1+2 = r+1$ which is a contradiction to $d(u, x_{i+3}) \leq r$. So, $d(u, x_1) = d(u, x_2) = r$. By symmetry, also $d(u, x_4) = d(u, x_5) = r$.

By the triangle condition, there must exist vertices v and w at distance $r-1$ from u such that $vx_1, vx_2, wx_4, wx_5 \in E$. We claim that x_3 is adjacent to neither v nor w. Assume, without loss of generality, that $vx_3 \in E$. Then, $d(x_5, x_1) = 4$ implies $d(x_5, v) = 3$ and therefore $x_3 \in I(x_5, v)$. Now, the α_1-metric property applied to $x_3 \in I(x_5, v)$ and $v \in I(u, x_3)$ gives $d(x_5, u) \geq r+1$ which is impossible. So, $vx_3, wx_3 \notin E$.

Obviously, $vx_4, wx_2 \notin E$. If $d(x_4, v) = 3$ then $x_2 \in I(x_4, v)$. Thus, by $v \in I(x_2, u)$ and the α_1-metric property, we would get $d(x_4, u) \geq r-1+2 = r+1$ which, again, is impossible. Thus, $d(x_4, v) = 2$ must hold. Since, by Theorem 5, every induced C_5 is suspended in G and, further, G cannot contain an induced C_4, we can choose a vertex $y \in N(v) \cap N(x_4)$ which is adjacent both to x_2 and x_3 as well. If $d(y, u) = r$ then again $y \in I(v, x_5)$ and $v \in I(u, y)$ will imply $d(x_5, u) \geq r-1+2 = r+1$, which is impossible. So, $d(y, u) = r-1$ must hold and, by the convexity of disks, y must be adjacent to w.

All the above holds for every shortest path $P = (s = x_1, x_2, x_3, x_4, x_5 = t)$ connecting vertices s and t. Now, assume that P is chosen in such a way that among all vertices in $I(s,t)$ that are at distance 2 from s (we will call this set of vertices $S_2(s,t)$) the vertex x_3 has the minimum number of furthest vertices, i.e., $|F(x_3)|$ is as small as possible. Observe that, by convexity of the center, $S_2(s,t) \subseteq C(G)$. As y also belongs to $S_2(s,t)$ and has u at distance $r-1$, by the choice of x_3, there must exist a vertex $u' \in F(y)$ which is at distance $r-1$ from x_3. Applying the previous arguments to the path $P' := (s = x_1, x_2, y, x_4, x_5 = t)$, we will have $d(x_i, u') = d(y, u') = r$ for $i = 1, 2, 4, 5$, and get two more vertices v' and w' at distance $r-1$ from u' such that $v'x_1, v'x_2, w'x_4, w'x_5 \in E$ and $v'y, w'y \notin E$. By the convexity of disk $D(u', r-1)$, also $v'x_3, w'x_3 \in E$. Now consider the disk $D(x_2, 2)$. Since w, w' are in the disk and x_5 is not, vertices w and w' must be adjacent. But then vertices y, x_3, w', w form a forbidden induced cycle C_4.

The obtained contradictions show that a shortest path P of length 4 cannot exist in $C(G)$, i.e., $diam(C(G)) \leq 3$. As $C(G)$ is a convex set of G, the subgraph of G induced by $C(G)$ is also an α_1-metric graph. According to [33], $diam(G) \geq 2rad(G) - 2$ holds for every α_1-metric graph G. Hence, for a graph induced by $C(G)$ we have $3 \geq diam(C(G)) \geq 2rad(C(G)) - 2$, i.e., $rad(C(G)) \leq 2$. □

As chordal graphs are $(\alpha_1, \varDelta)$-metric graphs, we get the following corollary.

Corollary 2 ([10]). *Let G be a chordal graph. Then, $diam(C(G)) \leq 3$ and $rad(C(G)) \leq 2$.*

For our next arguments we need a generalization of the set $S_2(s,t)$, as used in the proof of Theorem 7. We define a *slice* of the interval $I(u,v)$ from u to v for $0 \le k \le d(u,v)$ to be the set $S_k(u,v) = \{w \in I(u,v) : d(w,u) = k\}$.

Theorem 8. *Let G be an (α_1, Δ)-metric graph. Then, in every slice $S_k(u,v)$ there is a vertex x that is universal to that slice, i.e., $S_k(u,v) \subseteq N[x]$. In particular, if $diam(G) = 2rad(G)$, then $diam(C(G)) \le 2$ and $rad(C(G)) \le 1$.*

Proof. First we prove that for every pair u,v of vertices of G, any slice $S_k(u,v)$ is convex and has diameter at most 2. Since the slice $S_k(u,v)$ can be represented as the intersection of two disks, $D(u,k) \bigcap D(v,\ell)$, where $\ell = d(u,v) - k$, clearly $S_k(u,v)$ is convex (see Theorem 2).

Assume, by way of contradiction, that $S_k(u,v)$ has a shortest path $P := (x_1, x_2, x_3, x_4)$ of length 3. By the triangle condition, there must exist vertices y and z at distance $\ell - 1$ from v such that y is adjacent to x_1,x_2 and z is adjacent to x_3,x_4. Since $D(u,k)$ is convex, y cannot be adjacent to x_3 and z cannot be adjacent to x_2 and y. Furthermore, $d(y,x_4) > 2$ and $d(z,x_1) > 2$ must hold. Additionally, by the convexity of $D(v,\ell-1)$, vertices y and z must be at distance 2, implying $d(y,x_4) = 3 = d(z,x_1)$. Now, applying the α_1-metric property to $y \in I(z,x_1)$ and $x_1 \in I(u,y)$, we obtain $d(z,u) \ge k+2$, which is impossible since $d(u,z) = k+1$.

Now, consider a vertex $x \in S_k(u,v)$ which is adjacent to the maximum number of vertices of $S_k(u,v)$. Assume that there is a vertex $y \in S_k(u,v)$ which is not adjacent to x. We know that $d(x,y) = 2$. Consider a common neighbor w of x and y. Since $wy \in E$, $xy \notin E$ and $w \in S_k(u,v)$, by the choice of x, there must exist a vertex $t \in S_k(u,v)$ such that $wt \notin E$ and $xt \in E$. To avoid an induced C_4, $ty \notin E$ must hold. From $d(t,w) = d(t,y) = 2$, by the triangle condition, there must exist a common neighbor s of t, w and y. Additionally, s must be adjacent to x to avoid an induced C_4. By the convexity of $S_k(u,v)$, s belongs to $S_k(u,v)$. Applying now Lemma 2 to u and triangles $\{x,t,s\}$ and $\{w,y,s\}$, we get two vertices a and b at distance $k-1 = d(x,u)-1$ from u with $\{x,t,s\} \subseteq N(a)$ and $\{w,y,s\} \subseteq N(b)$. Since $D(u,k-1)$ is convex, necessarily a and b are adjacent. On the other hand, since $S_k(u,v)$ is convex, $d(y,x) = d(t,w) = 2$ and $a,b \notin S_k(u,v)$, a cannot be adjacent to w and b cannot be adjacent to x. But then vertices x,w,b,a form an induced C_4. This contradiction shows that such a vertex $y \in S_k(u,v)$, which is not adjacent to x, cannot exist.

It remains to note that if $diam(G) = 2rad(G)$ holds for a graph G, then $C(G) \subseteq S_{rad(G)}(u,v)$ for every pair of vertices u,v with $d(u,v) = diam(G)$. Hence, $diam(C(G)) \le 2$ and $rad(C(G)) \le 1$ in this case. $\square$

4 Eccentricity approximating spanning tree construction

It this section, we construct an eccentricity approximating spanning tree and analyze its quality for (α_1, Δ)-metric graphs. Here, we will use sub-indices G and T to indicate whether the distances or the eccentricities are considered in G or in T. However, $I(u,v)$ will always mean the interval between vertices u and v in G.

4.1 Tree construction for unimodal eccentricity functions

First consider the case when the eccentricity function on G is unimodal, that is, every non-central vertex of G has a neighbor with smaller eccentricity. We will need the following lemmas.

Lemma 4 ([13]). *Let G be an arbitrary graph. The eccentricity function on G is unimodal if and only if, for every vertex v of G, $ecc_G(v) = d_G(v, C(G)) + rad(G)$.*

Proof. Let v be an arbitrary vertex of G and let v' be a vertex of $C(G)$ closest to v, i.e., $d_G(v, C(G)) = d_G(v, v')$. Consider a vertex $u \in F(v)$. We have

$$ecc_G(v) = d_G(v, u) \leq d_G(v, v') + d_G(v', u) \leq d_G(v, C(G)) + rad(G).$$

First assume that the eccentricity function on G is unimodal. We will show that $ecc_G(v) \geq d_G(v, C(G)) + rad(G)$ by induction on $k = d_G(v, C(G))$. If $k = 0$ then $v \in C(G)$ and hence $ecc_G(v) = rad(G)$. If $k = d_G(v, C(G)) > 0$, then, by unimodality, there must exist a neighbor x of v such that $ecc_G(v) = ecc_G(x) + 1$. By the inductive hypothesis,

$$ecc_G(v) = ecc_G(x) + 1 = d_G(x, C(G)) + rad(G) + 1 \geq d_G(v, C(G)) + rad(G)$$

as $d_G(v, C(G)) \leq d_G(v, x') \leq d_G(x, x') + 1 = d_G(x, C(G)) + 1$ (here, x' is a vertex of $C(G)$ closest to x).

Assume now that $ecc_G(v) = d_G(v, C(G)) + rad(G)$ holds for every vertex v of G. Consider a neighbor x of v on a shortest path from v to a vertex of $C(G)$ closest to v. Since $d_G(v, C(G)) = d_G(x, C(G)) + 1$, we get $ecc_G(v) = ecc_G(x) + 1$. □

Lemma 5 ([5]). *Let G be an arbitrary α_1-metric graph. Let x, y, v, u be vertices of G such that $v \in I(x, y)$, $x \in I(v, u)$, and x and v are adjacent. Then $d(u, y) = d(u, x) + d(v, y)$ holds if and only if there exist a neighbor x' of x in $I(x, u)$ and a neighbor v' of v in $I(v, y)$ with $d_G(x', v') = 2$; in particular, x' and v' lie on a common shortest path of G between u and y.*

We construct a spanning tree T of G as follows. First find the center $C(G)$ of G and pick an arbitrary central vertex c of the graph $C(G)$, i.e., $c \in C(C(G))$. Compute a *breadth-first-search tree* T' of $C(G)$ started at c. Expand this tree T' to a spanning tree T of G by identifying for every vertex $v \in V \setminus C(G)$ its parent vertex in the following way: among all neighbors x of v with $ecc_G(x) = ecc_G(v) - 1$ pick that vertex which is closest to c in G (break ties arbitrarily).

Lemma 6. *Let G be an (α_1, Δ)-metric graph whose eccentricity function is unimodal. Then, for a tree T constructed as described above and every vertex v of G, $d_G(v, c) = d_T(v, c)$ holds, i.e., T is a shortest-path-tree of G started at c.*

Proof. Let v be an arbitrary vertex of G and let v' be a vertex of $C(G)$ closest to v in T. By Lemma 4 and by the construction of T, $d_G(v, v') = d_T(v, v')$ and v' is a vertex of $C(G)$ closest to v in G. By the construction of T', also $d_G(c, v') = d_T(c, v')$ (note that, as $C(G)$ is a convex subgraph of G, clearly, $d_{C(G)}(x, y) = d_G(x, y)$ for every pair x, y of $C(G)$). So, in the tree T, we have $d_T(c, v') + d_T(v', v) = d_T(v, c)$. If $d_G(c, v') + d_G(v', v) = d_G(v, c)$, then $d_G(v, c) = d_T(v, c)$, and we are done. Assume, therefore, that $d_G(c, v') + d_G(v', v) > d_G(v, c)$ and among all vertices that fulfill this inequality, let v be the one that is closest to $C(G)$. Consider the neighbor x of v' on the path in T from v' to v. We have $x \in I(v', v)$ and, by Lemma 4, $ecc_G(x) = rad(G) + 1$. Note that $x = v$ is possible.

If $v' \notin I(x, c)$ then $d_G(x, c) \leq d_G(v', c)$. By the convexity of $C(G)$, x with $ecc_G(x) = rad(G)+1$ cannot be on a shortest path between two central vertices c and v'. Hence, $d_G(x, c) = d_G(v', c)$ holds. By the triangle condition, there must exist a common neighbor y of v' and x which is at distance $d_G(v',c)-1$ from c. Since $y \in I(v',c)$, by the convexity of $C(G)$, $ecc_G(y) = rad(G)$. But then, as y is closer to c than v' is, vertex x cannot choose v' as its parent in T, since y is a better choice.

If $v' \in I(x,c)$ then, by the α_1-metric property, $d_G(c,v') + d_G(x,v) \leq d_G(v,c)$. As $d_G(c,v') + d_G(v',v) > d_G(v,c)$, we have $d_G(c,v') + d_G(x,v) = d_G(v,c)$. By Lemma 5, there must exist a neighbor x' of x in $I(x,v)$ and a neighbor v'' of v' in $I(v',c)$ with $d_G(x',v'') = 2$. Denote by w a common neighbor of x' and v''. We have $d_G(x,c) > d_G(w,c)$. Set $k := d_G(v,v') = d_G(v,C(G)) = ecc_G(v) - rad(G)$. Let $P_T := (x = a_1, \ldots, a_k = v)$ be the path in T between x and v. Let $P_G := (w = b_1, x' = b_2, \ldots, b_k = v)$ be a shortest path of G between w and v which shares a longest suffix with P_T, that is, $a_j = b_j$ for all $j > i$, $a_i \neq b_i$, and i is minimal under these conditions. Note that $i = 1$ and $a_2 = b_2 = v$ is possible. By Lemma 4, $ecc_G(a_i) = ecc_G(b_i) = rad(G) + i = ecc_G(a_{i+1}) - 1$.

Since v is a vertex closest to $C(G)$ fulfilling inequality $d_G(c,v') + d_G(v',v) > d_G(v,c)$, for vertex a_i ($i < k$), the equation $d_G(c,v')+d_G(v',a_i) = d_G(a_i,c)$ holds. Hence, $d_G(c,x)+d_G(x,a_i) = d_G(a_i,c)$. Also, by Lemma 5, $d_G(c,w) + d_G(w,b_i) = d_G(b_i,c)$. Consequently, $d_G(x,c) > d_G(w,c)$ and $d_G(x,a_i) = d_G(w,b_i)$ imply $d_G(a_i,c) > d_G(b_i,c)$. Therefore, vertex a_{i+1} cannot choose a_i as its parent in T, since b_i is a better choice.

The obtained contradictions prove that $d_G(c,v') + d_G(v',v) = d_G(v,c)$ and hence $d_G(v,c) = d_T(v,c)$. ☐

As a consequence of Lemma 4 and Lemma 6, we get the following result.

Lemma 7. *Let G be an (α_1, Δ)-metric graph whose eccentricity function is unimodal. Then, for a tree T constructed as described above and for every vertex v of G, $ecc_T(v) \leq ecc_G(v) + rad(C(G))$ holds.*

Proof. Let v be an arbitrary vertex of G, v' be a vertex of $C(G)$ closest to v in T, and u be a vertex most distant from v in T, i.e., $ecc_T(v) = d_T(v,u)$. By Lemma 4 and by the construction of T, $d_G(v,v') = d_T(v,v')$ and v' is a vertex of $C(G)$ closest to v in G. We have

$$ecc_T(v) = d_T(v,u) \leq d_T(v,v') + d_T(v',c) + d_T(c,u),$$

where $c \in C(C(G))$ is the root of the tree T (see the construction of T). Since $d_G(v,v') = d_T(v,v')$, $d_T(v',c) = d_G(v',c) \leq rad(C(G))$, and $d_T(c,u) = d_G(c,u) \leq rad(G)$ (by Lemma 6 and the fact that $c \in C(C(G))$), we obtain $ecc_T(v) \leq d_G(v,v') + rad(C(G)) + rad(G) = ecc_G(v) + rad(C(G))$, as $d_G(v,v') + rad(G) = d_G(v,C(G)) + rad(G) = ecc_G(v)$ by Lemma 4. ☐

4.2 Tree construction for eccentricity functions that are not unimodal

Consider now the case when the eccentricity function on G is not unimodal, that is, there is at least one vertex $v \notin C(G)$ in G which has no neighbor with smaller eccentricity. By Theorem 6, $ecc_G(v) = rad(G) + 1$, $diam(G) = 2rad(G)$ and v has a neighbor with the eccentricity equal to $ecc_G(v)$. We will need the following weaker version of Lemma 4.

Lemma 8. *Let $G = (V,E)$ be an (α_1, Δ)-metric graph. Let v be an arbitrary vertex of G and v' be an arbitrary vertex of $C(G)$ closest to v. Then,*

$$d_G(v,C(G)) + rad(G) - 1 \leq ecc_G(v) \leq d_G(v,C(G)) + rad(G).$$

Furthermore, there is a shortest path $P := (v' = x_0, x_1, \ldots, x_\ell = v)$, connecting v with v', for which the following holds:

*(a) if $ecc_G(v) = d_G(v, C(G)) + rad(G)$
then $ecc_G(x_i) = d_G(x_i, C(G)) + rad(G) = i + rad(G)$ for each $i \in \{0, \ldots, \ell\}$;*
*(b) if $ecc_G(v) = d_G(v, C(G)) + rad(G) - 1$
then $ecc_G(x_i) = d_G(x_i, C(G)) - 1 + rad(G) = i - 1 + rad(G)$ for each $i \in \{3, \ldots, \ell\}$ and $ecc_G(x_1) = ecc_G(x_2) = rad(G) + 1$.*

In particular, if $ecc_G(v) = rad(G) + 1$ then $d_G(v, C(G)) \leq 2$.

Proof. Let $r := rad(G)$ and $P := (v' = x_0, x_1, \ldots, x_\ell = v)$ be a shortest path connecting v with v' which minimizes $\sum_{i=0}^{\ell} ecc_G(x_i)$. Since v' is a vertex of $C(G)$ closest to v, $x_1 \notin C(G)$ and $ecc_G(x_1) = r + 1$. Consider a vertex $u \in F(x_1)$, i.e., $ecc_G(x_1) = d_G(x_1, u)$. We have $x_0 \in I(x_1, u)$ and $x_1 \in I(x_0, x_\ell)$. By the α_1-metric property,

$$d_G(x_\ell, u) \geq d_G(x_\ell, x_1) + d_G(x_0, u) = \ell - 1 + r = d_G(v, C(G)) - 1 + r.$$

Recall that for any two adjacent vertices z and y of any graph G, $|ecc_G(z) - ecc_G(y)| \leq 1$ holds. Therefore, if $ecc_G(v) \geq d_G(v, C(G)) + r$ then $ecc_G(x_i) = d_G(x_i, C(G)) + r = i + r$ holds for each $i \in \{0, \ldots, \ell\}$. If $ecc_G(v) = d_G(v, C(G)) - 1 + r$ then there must exist an index $i \in \{1, \ldots, \ell - 1\}$ such that $ecc_G(x_i) = ecc_G(x_{i+1}) = r + i$ and $ecc_G(x_j) = r + j$ for $j \leq i$. We claim that, by the minimality of $\sum_{i=0}^{\ell} ecc_G(x_i)$, $i = 1$ must hold.

Assume, by way of contradiction, that $i > 1$. By Theorem 6, necessarily, x_{i+1} has a neighbor t with $ecc_G(t) = ecc_G(x_{i+1}) - 1 = r + i - 1$. By Corollary 1, the set $\mathcal{C}_{\leq r+i-1}(G)$ of G is convex. Hence, vertices x_{i-1} and t with $ecc_G(x_{i-1}) = ecc_G(t) = r + i - 1$ must be connected in $\mathcal{C}_{\leq r+i-1}(G)$ by a shortest path of length at most 2 (since path $(x_{i-1}, x_i, x_{i+1}, t)$ has vertices of eccentricity larger than the eccentricities of the end-vertices x_{i-1} and t). Notice that vertices x_{i-1} and x_{i+1} cannot have a common neighbor z with $ecc_G(z) < r + i$ since, otherwise, by replacing vertex x_i with z in P, we get a shortest path between v' and v with a smaller sum of eccentricities of its vertices. Thus, x_{i-1} and t cannot be adjacent, i.e., $d_G(x_{i-1}, t) = 2$. Let w be a common neighbor of x_{i-1} and t. Necessarily, $ecc_G(w) \leq r + i - 1$ and hence $wx_{i+1} \notin E$. Since $I(x_{i-1}, t) \subset \mathcal{C}_{\leq r+i-1}(G)$, vertices t and x_i cannot be adjacent. To avoid an induced C_4, $wx_i \notin E$ as well. In an induced C_5 formed by x_{i-1}, x_i, x_{i+1}, t and w, by the triangle condition, there must exist a vertex w' which is adjacent to w, x_i, x_{i+1}, and, to avoid an induced C_4, to x_{i-1} and t. Again, from $w' \in I(x_{i-1}, t) \subset \mathcal{C}_{\leq r+i-1}(G)$ and $w'x_{i-1}, w'x_{i+1} \in E$, we obtain a contradiction with the minimality of $\sum_{i=0}^{\ell} ecc_G(x_i)$. $\square$

Now we are ready to construct an eccentricity approximating spanning tree T of G for the case when the eccentricity function is not unimodal. We know that $diam(G) = 2rad(G)$ in this case and, therefore, $C(G) \subseteq S_{rad(G)}(x, y)$ for any diametral pair of vertices x and y, i.e., for x, y with $d_G(x, y) = diam(G)$. By Theorem 8 and since $C(G)$ is convex, there is a vertex $c \in C(G)$ such that $C(G) \subseteq N[c]$. First we find such a vertex c in $C(G)$ and build a tree T' by making c adjacent with every other vertex of $C(G)$. Then, we expand this tree T' to a spanning tree T of G by identifying for every vertex $v \in V \setminus C(G)$ its parent vertex in the following way: if v has a neighbor with eccentricity less than $ecc_G(v)$, then among all such neighbors pick that vertex which is closest to c in G (break ties arbitrarily); if v has no neighbors with eccentricity less than $ecc_G(v)$ (i.e., $ecc_G(v) = rad(G) + 1$ by Theorem 6), then among all neighbors x of v with $ecc_G(x) = ecc_G(v) = rad(G) + 1$ pick again that vertex which is closest to c in G (break ties arbitrarily).

Lemma 9. *Let G be an (α_1, Δ)-metric graph whose eccentricity function is not unimodal. Then, for a tree T constructed as described above and every vertex v of G, $d_T(v, c) = d_G(v, c)$ holds.*

Proof. Assume, by way of contradiction, that $d_G(v,c) < k := d_T(v,c)$ and let v be a vertex with such a condition that has smallest eccentricity $ecc_G(v)$. We may assume that $ecc_G(v) > rad(G)+1$. Indeed, every v with $ecc_G(v) = rad(G)+1$ either has a neighbor in $C(G)$ or has a neighbor with a neighbor in $C(G)$ (see Lemma 8). Therefore, if $d_G(v,c) < d_T(v,c)$ then, by the construction of T, necessarily $d_G(v,c) = 2$, $d_T(v,c) = 3$ and the neighbor x of v on the path of T between v and c must have the eccentricity equal to $rad(G)+1 = ecc_G(v)$. But then, for a common neighbor w of v and c in G, $ecc_G(w) \leq rad(G)+1$ must hold and hence vertex v cannot choose x as its parent in T, since w is a better choice.

So, let $ecc_G(v) > rad(G)+1$. By Lemma 8, there must exist a shortest path in G between v and c such that the neighbor w of v on this path has eccentricity $ecc_G(w) = ecc_G(v)-1$. Hence, by the construction of T, $ecc_G(x) = ecc_G(v)-1$ must hold for the neighbor x of v on the path of T between v and c. By the minimality of $ecc_G(v)$, we have $d_G(x,c) = d_T(x,c) = k-1$. Since $d_G(w,c) = d_G(v,c)-1 < k-1$, a contradiction arises; again v cannot choose x as its parent in T, since w is a better choice. □

As a consequence of Lemma 8 and Lemma 9, we get the following result.

Lemma 10. *Let G be an (α_1,Δ)-metric graph with $diam(G) = 2rad(G)$. Then, for a tree T constructed as described above and every vertex v of G, $ecc_T(v) \leq ecc_G(v)+2$ holds.*

Proof. Let v be an arbitrary vertex of G and u be a vertex most distant from v in T, i.e., $ecc_T(v) = d_T(v,u)$. We have

$$ecc_T(v) = d_T(v,u) \leq d_T(v,c) + d_T(c,u) = d_G(v,c) + d_G(c,u) \leq d_G(v,c) + rad(G)$$

$$\leq d_G(v,C(G)) + 1 + rad(G) \leq ecc_G(v) + 2$$

since $d_G(c,u) \leq ecc_G(c) = rad(G)$, $d_G(v,c) \leq d_G(v,C(G))+1$ (recall that $C(G) \subseteq N[c]$), and $d_G(v,C(G)) - 1 + rad(G) \leq ecc_G(v)$ (by Lemma 8). □

Our main result is the following theorem. It combines Theorem 7, Lemma 7 and Lemma 10; the complexity follows straightforward.

Theorem 9. *Every (α_1,Δ)-metric graph $G=(V,E)$ has an eccentricity 2-approximating spanning tree. Furthermore, such a tree can be constructed in $\mathcal{O}(|V||E|)$ total time.*

As two consequences we have the following corollaries for two important subclasses of (α_1,Δ)-metric graphs.

Corollary 3. *If G is the underlying graph of a 7-systolic complex then G has an eccentricity 2-approximating spanning tree. In particular, every plane triangulation with inner vertices of degree at least 7 has an eccentricity 2-approximating spanning tree.*

Corollary 4 ([29]). *Every chordal graph has an eccentricity 2-approximating spanning tree.*

Note that the result of Corollary 4 (and hence of Theorem 9) is sharp as there are chordal graphs that do not have any eccentricity 1-approximating spanning tree [29].

5 Experimental results for some real-world networks

Here, we analyze if the eccentricity terrain of a network resembles the eccentricity terrain of a tree. Recall that in trees, the eccentricity of a vertex can range between $rad(T)$ and at least $2rad(T) - 1$ (as $diam(T) \geq 2rad(T) - 1$), every vertex $v \in V(T) \setminus C(T)$ has a neighbor u such that $ecc_T(v) = ecc_T(u) + 1$ (i.e., the eccentricity function on trees is unimodal), and the center $C(T)$ of a tree consists of one vertex or two adjacent vertices. We have seen that in (α_1, Δ)-metric graphs, the eccentricity function is almost unimodal, the eccentricity of a vertex can range between $rad(G)$ and at least $2rad(G) - 2$ (as $diam(G) \geq 2rad(G) - 2$), $diam(C(G)) \leq 3$, $rad(C(G)) \leq 2$, and the center $C(G)$ is convex and hence connected. Furthermore, every (α_1, Δ)-metric graph G admits an eccentricity 2-approximating spanning tree, which provides a strong evidence that the eccentricity terrain of G resembles the eccentricity terrain of a tree.

Table 1. Basic statistics of the analyzed networks: $|V|$ is the number of vertices; $|E|$ is the number of edges; $size(G) = |V| + |E|$; $\bar{d}$ is the average degree; $diam(G)$ is the diameter; $rad(G)$ is the radius. Most of the networks listed in this table are available at [1–4].

Network	Ref.	$\|V\|$	$\|E\|$	$\log_2(size(G))$	$\bar{d}$	$diam(G)$	$rad(G)$
Social Networks							
EMAIL	[17]	1133	5451	12.68	9.6	8	5
Facebook	[21]	4039	88 234	16.49	43.7	8	4
Dutch-Elite	[12]	3621	4310	12.95	2.4	22	12
JAZZ	[16]	198	2742	11.52	27.7	6	4
EVA	[27]	4475	4664	13.16	2.1	18	10
Internet Graphs							
AS-Graph-1	[1]	3015	5156	12.95	3.4	9	5
AS-Graph-2	[1]	4885	9276	13.79	3.8	11	6
AS-Graph-3	[1]	5357	10 328	13.94	3.9	9	5
Biological Networks							
E-coli-PI	[8]	126	581	9.47	9.2	5	3
Yeast-PI	[31]	1728	11 003	13.64	12.7	12	7
Macaque-brain-1	[25]	45	463	8.99	11.3	4	2
Macaque-brain-2	[23]	350	5198	12.44	29.7	4	3
E-coli-metabolic	[19]	242	376	9.27	3.1	16	9
C-elegans-metabolic	[14]	453	4596	12.3	8.9	7	4
Yeast-transcription	[22]	321	711	10.01	4.4	9	5
Other Networks							
US-Airlines	[6]	332	2126	11.26	12.8	6	3
POWER-Grid	[32]	4941	6594	13.49	2.7	46	23
Word-Adjacency	[26]	112	425	9.07	7.6	5	3
Food	[1]	135	596	9.51	8.8	4	3

In this section, we analyze vertex localities and centers in a collection of real-world networks/graphs coming from a number of different domains. Additionally, based on what we learned from (α_1, Δ)-metric graphs in Section 4, we propose two heuristics for constructing eccentricity approximating trees in general graphs and analyze their performance on our set of real-world networks. Some of those networks are not connected, but they usually have one very large connected component and a few very small components. In this case, we consider only a largest connected component. Note that all our networks are unweighted and we ignore directions of edges if a network is originally directed. A summary of basic statistical properties of largest connected components of the networks in our dataset is given in Table 1.

5.1 Dataset

First we describe the investigated networks.

Social networks.
EMAIL [2, 17]: This network represents the email interchanges between members of the university of Rovira i Virgili, Tarragona.
FACEBOOK [4, 21]: This network has 4039 users who belong to the ego networks (the network of friendship between a user's friends) of 10 people. Two vertices (users) are connected if they are Facebook friends.
DUTCH-ELITE [6, 12]: This is a network data on the administrative elite in the Netherlands, April 2006. Vertices represent persons and organizations that are most important to the Dutch government (2-mode network). An edge connects a person-vertex and an organization-vertex if the corresponding person belongs to the corresponding organization.
JAZZ [2, 16]: In this network, vertices represent different Jazz musicians and two vertices are connected if the two musicians have played together.
EVA [6, 27]: This network presents corporate ownership information as a social network. Two vertices are connected with an edge if one is the owner of the other.

Internet graphs.
AS-GRAPHs [1]: Those graphs represent the Autonomous Systems topology of the Internet. In each graph, a vertex represents an autonomous system, and two vertices are connected if the two autonomous systems share at least one physical connection. In this work, we use three AS graphs: AS-GRAPH-1, AS-GRAPH-2, and AS-GRAPH-3 for which the data were collected in November 1997, April 1999, and July 1999, respectively.

Biological networks.
Protein Interaction (PI) Networks: Generally, in a PI network, the vertices represent different proteins and the edges represent the connections between the interacting proteins. We consider the protein interaction networks of the Escherichia coli [8] and the Yeast [31].
Neural Networks: In those networks, neurons (vertices) are connected together through synapsis (edges). We analyze two different brain area networks of the Macaque monkey [23, 25].
Metabolic Networks: Metabolic networks are represented by metabolites (vertices) such as amino acids and biochemical reactions (directed edges). In this dataset, we have the Escherichia coli [19] and the Caenorhabditis elegans [14] metabolic networks.
Transcription Networks: Networks in which vertices are genes and edges represent different interactions (interrelationships) between genes. We analyze the Yeast transcription network [22].

Other networks.
US-AIRLINES [6]: The transportation network of airlines in the United States. The original graph from [6] is weighted. We ignored the weights in our experiments.
POWER-GRID [3, 32]: This network represents the topology of the Western States Power Grid of the United States.

WORD-ADJACENCY [3, 26]: The adjacency network of commonly occurring adjectives and nouns in the novel David Copperfield by Charles Dickens. An edge connects any adjacent pair of words.
FOOD [1]: This network represents the food predatory interactions among different species in the Ythan Eastuary environment. Vertices represent species, and a directed edge links two vertices if one species preys on the other.

5.2 Analysis of vertex localities and centers

Recall that the locality of a vertex $v \notin C(G)$ (with respect to the eccentricity function) is the distance from v to a closest vertex with smaller eccentricity:

$$loc(v) = \min\{d(v,x) : x \in V, ecc(x) < ecc(v)\}.$$

The locality of any central vertex $u \in C(G)$ is defined to be 1. Hence, the eccentricity function is unimodal on G if and only if all vertices of G have locality 1. Given a graph $G = (V, E)$, we can define its *eccentricity layering* $\mathcal{EL}(G) = (\mathcal{C}_0(G), \dots, \mathcal{C}_{diam(G)-rad(G)}(G))$ to be a partition of the vertex set V into layers $\mathcal{C}_k(G) = \{v \in V : ecc(v) = rad(G) + k\}$, $k = 0, 1, \dots, diam(G) - rad(G)$. Clearly, $\mathcal{C}_0(G) = C(G)$. The layer of each vertex u with respect to the eccentricity layering, denoted by $layer(u)$, is k if $u \in \mathcal{C}_k(G)$.

Table 2. Percent of vertices with localities equal to 1 and larger than 1 in each graph of the dataset.

Network	% of vertices with $k(\cdot) = 1$	% of vertices with $k(\cdot) > 1$
EMAIL	≈95%	≈5%
FACEBOOK	≈98%	≈2%
DUTCH-ELITE	≈97%	≈3%
JAZZ	100%	
EVA	≈99%	≈1%
AS-GRAPH-1	≈99%	≈1%
AS-GRAPH-2	≈98%	≈2%
AS-GRAPH-3	≈98%	≈2%
E-COLI-PI	≈90%	≈10%
YEAST-PI	≈95%	≈5%
MACAQUE-BRAIN-1	≈78%	≈22%
MACAQUE-BRAIN-2	100%	
E-COLI-METABOLIC	≈86%	≈14%
C-ELEGANS-METABOLIC	≈98%	≈2%
YEAST-TRANSCRIPTION	≈91%	≈9%
US-AIRLINES	100%	
POWER-GRID	≈99%	≈2%
WORD-ADJACENCY	≈77%	≈23%
FOOD	100%	

Theorem 6 from Section 3 says that if a vertex v with $loc(v) > 1$ exists in an (α_1, Δ)-metric graph G then $diam(G) = 2rad(G)$ and $ecc(v) = rad(G) + 1$, i.e., $layer(v) = 1$.

We analyzed vertex localities in the graphs from our dataset. It turns out that the eccentricity function is unimodal for the graphs Jazz, Macaque-brain-2, US-Airlines and Food. For all graphs, except Macaque-brain-1 (78%), E-coli-metabolic (86%) and Word-Adjacency (77%), at least 90% of vertices (in most cases close to 100%) have locality 1 (see Table 2).

In Fig. 3, we show also the distribution of vertices with $loc(\cdot) > 1$ over different layers of the eccentricity layering in each graph of the dataset. As in the case of (α_1, Δ)-metric graphs, in the graphs of our dataset (with the exception of POWER-Grid), the vertices with locality greater than 1 also tend to reside in the layers $\mathcal{C}_k(G)$ with smaller k.

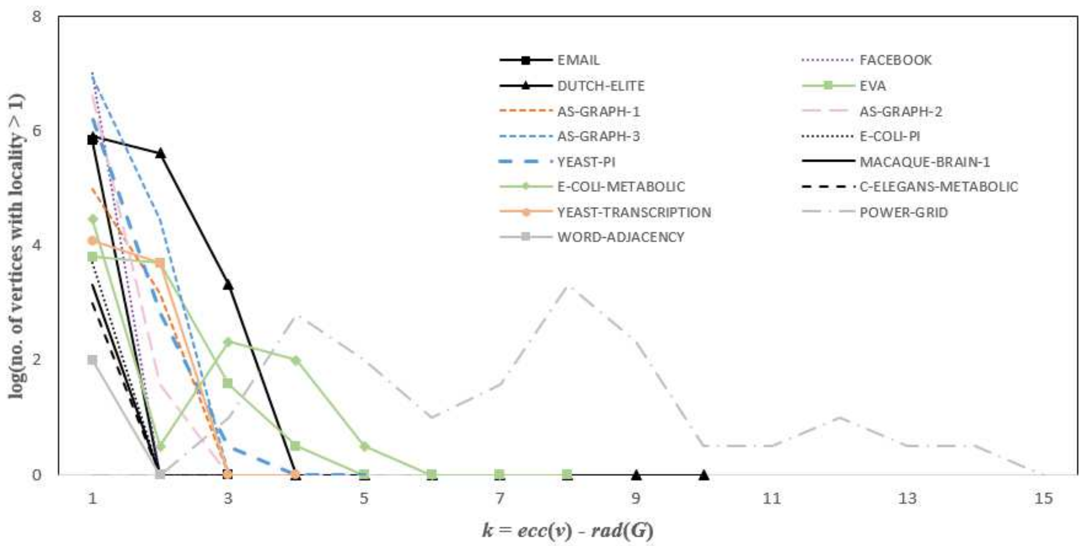

Fig. 3. Distribution of vertices with $loc(\cdot) > 1$ over different layers of the eccentricity layering in each graph of the dataset. Note that the graphs Jazz, Macaque-Brain-2, Us-Airlines, and Food are not included since they do not contain such vertices.

We know that $diam(G) \geq 2rad(G) - 2$ holds for every α_1-metric graph G [33]. From Table 1, we see that for all graphs in our dataset $2rad(G) - diam(G)$ is at most 2 as well. For graphs Facebook, Macaque-brain-1, US-Airlines and POWER-Grid, in fact, $diam(G) = 2rad(G)$ holds.

As we have mentioned earlier, the center $C(G)$ of any α_1-metric graph G is connected (in fact, it is convex and hence isometric). We analyzed centers of all graphs from our dataset (see Table 3). The centers of most of the graphs (except Dutch-Elite, AS-Graph-2, E-coli-metabolic and Yeast-transcription) turned out to be connected as well. By Theorem 7, we know that $diam(C(G)) \leq 3$ and $rad(C(G)) \leq 2$ holds for every (α_1, Δ)-metric graph G. As the centers of the graphs in our dataset are not necessarily isometric (distance-preserving) subgraphs, we used notions of *weak diameter* $\overline{diam}(C(G))$ and *weak radius* $\overline{rad}(C(G))$ to measure their centers, where $\overline{diam}(C(G)) = \max\{d_G(x, y) : x, y \in C(G)\}$ and $\overline{rad}(C(G)) = \min\{\max\{d_G(x, y) : y \in C(G)\} : x \in C(G)\}$.

Interestingly, the graphs with $diam(G) = 2rad(G)$ in our dataset (i.e., Facebook, Macaque-brain-1, US-Airlines and POWER-Grid) have single vertex centers. The centers of all graphs have small weak diameter and weak radius: the weak radius is 0 in 4 graphs, 1 in 2 graphs, 2 in 7 graphs, 3 in 5 graphs, and 4 in 1 graph (Dutch-Elite).

Table 3. The weak diameters and weak radii of the centers of each graph in the dataset.

Network	$\overline{diam}(C(G))$	$\overline{rad}(C(G))$	Connected?	$\frac{\lvert C(G)\rvert}{\lvert V\rvert}$
EMAIL	4	3	yes	215 / 1133
Facebook	0	0	yes	1 / 4039
Dutch-Elite	4	4	no	3 / 3621
Jazz	3	2	yes	56 / 198
EVA	3	2	yes	15 / 4475
AS-Graph-1	2	1	yes	32 / 3015
AS-Graph-2	4	3	no	531 / 4885
AS-Graph-3	2	2	yes	10 / 5357
E-coli-PI	2	2	yes	6 / 126
Yeast-PI	5	3	yes	53 / 1728
Macaque-brain-1	0	0	yes	1 / 45
Macaque-brain-2	3	2	yes	194 / 350
E-coli-metabolic	5	3	no	5 / 242
C-elegans-metabolic	4	2	yes	17 / 453
Yeast-transcription	3	3	no	3 / 321
US-Airlines	0	0	yes	1 / 332
POWER-Grid	0	0	yes	1 / 4941
Word-Adjacency	2	1	yes	4 / 112
Food	3	2	yes	53 / 135

5.3 Eccentricity approximating tree construction and analysis

We say that a tree T is an *eccentricity k-approximating tree* for a graph G if for every vertex v of G, $|ecc_T(v) - ecc_G(v)| \leq k$ holds. If T is a spanning tree, then $ecc_T(v) \geq ecc_G(v)$, for all $v \in V$, and this new definition agrees with the definition of an eccentricity k-approximating *spanning* tree.

Our goal in this section is to propose a heuristic for constructing an eccentricity k-approximating tree for general graphs such that the value of k is as small as possible. In our construction of an eccentricity 2-approximating spanning tree for an (α_1, Δ)-metric graph G, two main ingredients were crucial: 1. the eccentricity function on G is almost unimodal and the vertices with locality larger than 1 reside only in layer $\mathcal{C}_1(G)$; 2. the radius of the center $C(G)$ is at most 2. Our eccentricity 2-approximating spanning tree was a shortest-path-tree starting at a vertex $c \in C(C(G))$.

Although the weak radius of each graph in our dataset is relatively small (for 13 graphs it was at most 2, for 5 graphs at most 3, and only for Dutch-Elite it was 4; see Table 3), for some graphs, a small number of vertices with locality 2, 3 or 4 exists and those vertices may reside also at eccentricity layers $\mathcal{C}_k(G)$ with $k > 1$ (see Fig. 3).

Based on what we learned from (α_1, Δ)-metric graphs in Section 4 and on what we observed about vertex localities and centers in the graphs in our dataset, we propose two heuristics for constructing eccentricity approximating trees in general graphs. Both heuristics try to mimic the construction for (α_1, Δ)-metric graphs that we used in Section 4.

Our first heuristic, named *EAST*, constructs an *E*ccetricity *A*pproximating *S*panning *T*ree T_{EAST} as a shortest-path-tree starting at a vertex $c \in C(C(G))$. We identify an arbitrary vertex $c \in C(C(G))$ as the root of T_{EAST}, and for each other vertex v of G define its parent in T_{EAST} as follows: among all neighbors of v in $I(v, c)$ choose a vertex with minimum eccentricity (break ties arbitrarily). A formal description of this construction is given in Algorithm EAST.

Algorithm EAST. Eccentricity k-approximating spanning trees for general graphs.
Input. A graph $G = (V, E)$.
Output. An eccentricity k-approximating *spanning* tree $T = (V, U)$ of G.

---

1. $U \leftarrow \emptyset$
2. pick a vertex $c \in C(G)$ with the minimum distance to every other vertex in $C(G)$
3. **for** every $v \in V \setminus \{c\}$ **do**
4. among all neighbors of v in $I(v, c)$ choose a vertex u with minimum eccentricity
5. add edge uv to U
6. return $T = (V, U)$

Our second heuristic, named *EAT*, constructs for a graph G an *E*ccetricity *A*pproximating *T*ree T_{EAT} (not necessarily a spanning tree; it may have a few edges not present in graph G) as follows. We again identify an arbitrary vertex $c \in C(C(G))$ as the root of T_{EAT} and make it adjacent in T_{EAT} to all other vertices of $C(G)$ (clearly, some of these edges might not be contained in G). Then, for each vertex $v \in V \setminus C(G)$, we find a vertex u with $ecc_G(u) < ecc_G(v)$ which is closest to v, and if there is more than one such vertex, we pick the one which is closest to c. In other words, among all vertices $\{u \in V : d_G(u, v) = loc(v)$ and $ecc_G(u) < ecc_G(v)\}$, we choose a vertex u which is closest to c (break ties arbitrarily). Such a vertex u becomes the parent of v in T_{EAT}. Clearly, if $loc(v) > 1$ then edge uv of T_{EAT} is not present in G. A formal description of this construction is given in Algorithm EAT.

Algorithm EAT. Eccentricity k-approximating trees for general graphs.
Input. A graph $G = (V, E)$.
Output. An eccentricity k-approximating tree $T = (V, U)$ of G.

---

1. $U \leftarrow \emptyset$
2. pick a vertex $c \in C(G)$ with the minimum distance to every other vertex in $C(G)$
3. **for** every $u \in C(G) \setminus \{c\}$ **do**
4. add edge uc to U
5. **for** every $v \in V \setminus C(G)$ **do**
6. among all vertices $\{u \in V : d_G(u, v) = loc(v)$ and $ecc_G(u) < ecc_G(v)\}$
7. choose a vertex u which is closest to c
8. add edge uv to U
9. return $T = (V, U)$

We tested both heuristics on our set of real-world networks. Experimental results obtained are presented in Table 4 and Table 5.

Table 4 demonstrates the quality of the spanning tree T constructed by Algorithm EAST for each graph G in the dataset. Algorithm EAST was able to produce an eccentricity k-

Table 4. A spanning tree T constructed by Algorithm EAST: for each vertex $u \in V$, $k(u) = ecc_T(u) - ecc_G(u)$; $k_{max} = \max_{u \in V} k(u)$; $k_{avg} = \frac{1}{n} \sum_{u \in V} k(u)$.

Network	$diam(G)$	$diam(T)$	k_{max}	k_{avg}	% of vertices w. $k(\cdot) = 0$	% of vertices w. $k(\cdot) = 1$	% of vertices w. $k(\cdot) = 2$	% of vertices w. $k(\cdot) = 2$
EMAIL	8	10	3	1.774	≈0.79%	≈27.8 %	≈64.61%	≈6.8 %
Facebook	8	8	2	0.69	51.9 %	27.6 %	≈20.5 %	
Dutch-Elite	22	24	6	2.083	≈17.45%	≈0 %	≈61.28%	
Jazz	6	8	2	1.742	≈1.52%	≈22.72%	≈75.76%	
EVA	18	19	2	0.575	≈47.59%	≈47.26%	≈5.14%	
AS-Graph-1	9	10	2	0.64	≈35.78%	≈64.18%	≈0.03%	
AS-Graph-2	11	12	3	1.272	≈4.71%	≈63.4 %	≈31.85%	≈0.04%
AS-Graph-3	9	10	2	0.312	≈70.38%	≈28 %	≈1.62%	
E-coli-PI	5	6	2	0.769	≈34.92%	≈53.17%	≈11.9 %	
Yeast-PI	12	13	4	0.972	≈28 %	≈50.23%	≈18.5 %	≈2.89%
Macaque-brain-1	4	4	1	0.222	77.78%	22.22%		
Macaque-brain-2	4	6	2	1.489	≈1.71%	≈47.7 %	≈50.57%	
E-coli-metabolic	16	17	4	1.132	≈34.71%	≈33.1 %	≈17.77%	≈13.22%
C-elegans-metabolic	7	8	1	0.349	≈65.12%	≈34.88%		
Yeast-transcription	9	10	3	1.121	≈33.96%	≈26.79%	≈32.4 %	≈6.85%
US-Airlines	6	6	0	0	100 %			
POWER-Grid	46	46	4	1.409	≈46.35%	≈13.13%	≈12.61%	≈9.11%
Word-Adjacency	5	6	1	0.411	≈58.93%	≈41.07%		
Food	4	6	2	1.629	≈1.48%	≈34.07%	≈64.44%	

approximating spanning tree with $k = 0$ for 1 graph (US-Airlines), $k = 1$ for 3 graphs, $k = 2$ for 8 graphs, $k = 3$ for 3 graphs, $k = 4$ for 3 graphs, and $k = 6$ for 1 graph (Dutch-Elite). According to the criteria from [24] for the existence of eccentricity-preserving (i.e., eccentricity 0-approximating) spanning trees (see Introduction), graph US-Airlines has an eccentricity 0-approximating spanning tree and Algorithm EAST succeeded to construct such a spanning tree. Algorithm EAST succeeded to construct optimal spanning trees also for graphs Macaque-brain-1, C-elegans-metabolic and Word-Adjacency (those graphs do not satisfy the criteria for admitting eccentricity 0-approximating spanning trees and EAST constructs for them eccentricity 1-approximating spanning trees). For every graph $G = (V, E)$ in our dataset and for each corresponding constructed spanning tree T, we computed $k(u) = ecc_T(u) - ecc_G(u)$, for each vertex $u \in V$. Using this, for each graph G and spanning tree T we determined $k_{max} = \max_{u \in V} k(u)$ (the maximum difference between $ecc_T(u)$ and $ecc_G(u)$) and $k_{avg} = \frac{1}{n} \sum_{u \in V} k(u)$ (the average difference). Although k_{max} is greater than 2 for 7 graphs of the dataset, the average difference k_{avg} is smaller than 2 for all but one graph (Dutch-Elite) and is smaller than 1 for 10 graphs. Overall, the constructed trees preserve vertex eccentricities of the graphs with a high level of accuracy. If we consider, for example, graph AS-Graph-3 and its spanning tree constructed by EAST, we have $k_{max} = 2$ but about 70% of vertices preserved their eccentricities ($k(\cdot) = 0$), about 28% of vertices increased their eccentricity only by one ($k(\cdot) = 1$), and only the remaining 2% of vertices increased their eccentricity by two ($k(\cdot) = 2$); hence, the average difference k_{avg} is 0.312. If we consider the graph Dutch-Elite and its spanning tree constructed by EAST, we have $k_{max} = 6$ but about 79% of vertices increased their eccentricity only by two ($k(\cdot) \leq 2$), resulting in the average difference $k_{avg} = 2.083$, which is rather small even for (Dutch-Elite).

Table 5 demonstrates the quality of the (not necessarily spanning) tree T constructed by Algorithm EAT for each graph G in the dataset. The flexibility of being able to use edges in T that are not present in G allowed the algorithm to get even better approximations of vertex eccentricities in graphs by vertex eccentricities in trees. Algorithm EAT was able to produce an eccentricity k-approximating tree with $k = 0$ for 3 graphs (Facebook, Macaque-brain-1, US-Airlines), with $k = 1$ for all other graphs except POWER-Grid (which has $k = 3$). For every graph $G = (V, E)$ in our dataset and for each correspondingly constructed tree T, we computed $k(u) = ecc_T(u) - ecc_G(u)$, for each vertex $u \in V$, and then $k_{max} = \max_{u \in V} k(u)$, $k_{min} = \min_{u \in V} k(u)$, and $k_{avg} = \frac{1}{n} \sum_{u \in V} k(u)$. Interestingly, the difference between $ecc_T(u)$ and $ecc_G(u)$ falls in the range $[-1, 0]$ for 8 graphs, in the range $[0, 1]$ for 7 graphs, and only for one graph (POWER-Grid) it falls in the range $[-3, 0]$ (we excluded Facebook, Macaque-brain-1, US-Airlines in these counts as for them $k_{max} = k_{min} = 0$). For 7 graphs, more than 98% of the vertices preserved their eccentricities ($|k(\cdot)| = 0$). For POWER-Grid, more than 56% of vertices preserved their eccentricities.

Table 5. A tree T constructed by Algorithm EAT: for each vertex $u \in V$, $k(u) = ecc_T(u) - ecc_G(u)$; $k_{max} = \max_{u \in V} k(u)$; $k_{min} = \min_{u \in V} k(u)$; $k_{avg} = \frac{1}{n} \sum_{u \in V} k(u)$.

Network	$diam(G)$	$diam(T)$	$[k_{min}, k_{max}]$	k_{avg}	% of vertices with $k(\cdot) = 0$	% of vertices with $k(\cdot) = 1$
EMAIL	8	8	$[-1, 0]$	-0.0009	≈99.91%	≈0.09%
Facebook	8	8	$[0, 0]$	0	100 %	
Dutch-Elite	22	21	$[-1, 0]$	-0.771	≈22.92%	≈77.18%
Jazz	6	6	$[-1, 0]$	-0.015	≈98.48%	≈1.52%
EVA	18	18	$[-1, 0]$	-0.36	≈64.02%	≈35.98%
AS-Graph-1	9	10	$[0, 1]$	0.62	≈37.98%	≈62.02%
AS-Graph-2	11	12	$[0, 1]$	0.949	≈5.04%	≈94.96%
AS-Graph-3	9	10	$[0, 1]$	0.248	≈75.53%	≈24.47%
E-coli-PI	5	6	$[0, 1]$	0.595	≈40.48%	≈59.52%
Yeast-PI	12	12	$[-1, 0]$	-0.168	≈83.22%	≈16.78%
Macaque-brain-1	4	4	$[0, 0]$	0	100 %	
Macaque-brain-2	4	4	$[-1, 0]$	-0.003	≈99.71%	≈0.29%
E-coli-metabolic	16	15	$[-1, 0]$	-0.624	≈37.6 %	≈62.4 %
C-elegans-metabolic	7	8	$[0, 1]$	0.342	≈65.78%	≈34.22%
Yeast-transcription	9	9	$[0, 1]$	0.019	≈98.13%	≈1.87%
US-Airlines	6	6	$[0, 0]$	0	100 %	
POWER-Grid	46	43	$[-3, 0]$	-1.309	≈56.34%	0 %
Word-Adjacency	5	6	$[0, 1]$	0.152	≈84.82%	≈15.18%
Food	4	4	$[-1, 0]$	-0.015	≈98.52%	≈1.48%

6 Concluding remarks

We conclude the paper with some immediate questions building off our results.

1. Can our result on eccentricity 2-approximating spanning trees for (α_1, Δ)-metric graphs be extended to arbitrary α_1-metric graphs?
2. If we drop the requirement for a tree to be a spanning tree, does every (α_1, Δ)-metric graph (in particular, every chordal graph) admit an eccentricity k-approximating tree with $k < 2$?

More generally, we are interested in the following questions.

3. What is the complexity of the problem: Given a graph G and an integer k, check if G admits an eccentricity k-approximating (spanning) tree?

We suspect that this problem is NP-complete. So, it is natural to ask:

4. Can this problem be efficiently approximated? Is a constant factor approximation possible?
5. Do our heuristics for general graphs provide any provable good approximation?

References

1. http://web.archive.org/web/20060506132945/http://www.cosin.org/extra/data/data.html
2. http://deim.urv.cat/∼alexandre.arenas/data/welcome.htm.
3. http://www-personal.umich.edu/ mejn/netdata/.
4. https://snap.stanford.edu/data/.
5. H.-J. Bandelt, V. Chepoi, 1-Hyperbolic graphs, *SIAM J. Discr. Math.* 16 (2003), 323–334.
6. V. Batagelj and A. Mrvar, Pajek datasets, (2006). Available at http://vlado.fmf.uni-lj.si/pub/networks/data/.
7. A. Brandstädt, V. Chepoi, F.F. Dragan, Distance approximating trees for chordal and dually chordal graphs, *J. Algorithms* 30 (1999) 166–184.
8. G. Butland, J. Peregrin-Alvarez, J. Li, W. Yang, X. Yang, V. Canadien, A. Starostine, D. Richards, B. Beattie, N. Krogan, M. Davey1, J. Parkinson, J. Greenblatt, A. Emili. Interaction network containing conserved and essential protein complexes in Escherichia coli, *Nature,* 433 (2005), 531-537.
9. V. Chepoi, Some d-convexity properties in triangulated graphs, in *Mathematical Research*, vol. 87, Ştiinţa, Chişinău, 1986, pp. 164–177 (Russian).
10. V. Chepoi, Centers of triangulated graphs, *Math. Notes* 43 (1988), 143–151.
11. V. Chepoi, Graphs of some CAT(0) complexes, *Adv. Appl. Math.* 24 (2000) 125–179.
12. W. de Nooy. The network data on the administrative elite in the netherlands in April-June 2006.
13. F.F. Dragan, Centers of graphs and the Helly property (in Russian), Ph.D. Thesis, Moldova State University, (1989).
14. J. Duch, A. Alex, Community detection in complex networks using extremal optimization. *Physical review E,* 72 (2005), 027104.
15. M. Farber, R.E. Jamison, On local convexity in graphs, *Discr. Math.* 66 (1987), 231–247.
16. P. M. Gleiser and L. Danon, Community structure in jazz. *Advances in complex systems,* 6 (2003), 565-573.
17. R. Guimera, L. Danon, A. Diaz-Guilera, F. Giralt, A. Arenas, Self-similar community structure in a network of human interactions. *Physical review E,* 68 (2003), 065103.
18. D. Kratsch, H.-O. Le, H. Müller, E. Prisner and D. Wagner, Additive tree spanners, *SIAM J. Discrete Math.* 17 (2003), 332–340.
19. H. Ma, AP. Zeng, Reconstruction of metabolic networks from genome data and analysis of their global structure for various organisms. *Bioinformatics,* 19 (2003), 270-277.
20. M.S. Madanlal, G. Vankatesan, C. Pandu Rangan, Tree 3-spanners on interval, permutation and regularbipartite graphs, *Inform. Process. Lett.* 59 (1996), 97–102.
21. J. Mcauley, J. Leskovec, Learning to discover social circles in ego networks. *Advances in neural information processing systems*, 2012, 539-547.
22. R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, U. Alon. Network motifs: simple building blocks of complex networks, *Science,* 298 (2002), 824-827.
23. D. Modha, R. Singh, Network architecture of the long-distance pathways in the macaque brain. *Proceedings of the National Academy of Sciences,* 107 (2010), 13485–13490.
24. R. Nandakumar and K. Parthasarathy, Eccentricity-preserving spanning trees. *J. Math. Phys. Sci.* 24(1990), 33–36.

25. L. Négyessy, T. Nepusz, L. Kocsis, F. Bazsó, Prediction of the main cortical areas and connections involved in the tactile function of the visual cortex by network analysis. *European Journal of Neuroscience,* 23 (2006), 1919-1930.
26. M. E. Newman, Finding community structure in networks using the eigenvectors of matrices. *Physical review E,* 74 (2006), 036104.
27. K. Norlen, G. Lucas, M. Gebbie, J. Chuang, EVA: Extraction, visualization and analysis of the telecommunications and media ownership network. *Proceedings of International Telecommunications Society 14th Biennial Conference (ITS2002), Seoul Korea,* (2002).
28. E. Prisner, Distance approximating spanning trees, *Proceedings of the Symposium on Theoretical Aspects of Computer Science* (STACS'97), *Lecture Notes on Computer Science* 1200, 1997, pp. 499–510.
29. E. Prisner, Eccentricity-approximating trees in chordal graphs, *Discrete Mathematics* 220(2000), 263–269.
30. V.P. Soltan, V.D. Chepoi, Conditions for invariance of set diameters under d-convexification in a graph, *Cybernetics* 19 (1983), 750–756 (Russian, English transl.).
31. C. Von Mering, R. Krause, B. Snel, M. Cornell, S. G. Oliver, S. Fields, P. Bork, Comparative assessment of large-scale data sets of protein-protein interactions. *Nature,* 417 (2002), 399-403.
32. D. J. Watts, S. H. Strogatz, Collective dynamics of 'small-world' networks. *Nature,* 393 (1998), 440-442.
33. S.V. Yushmanov, V. Chepoi, A general method of investigation of metric graph properties related to the eccentricity, in *Mathematical Problems in Cybernetics*, vol. 3, Nauka, Moscow, 1991, pp. 217–232 (Russian).