Model of Computation and Runtime Analysis

Model of Computation

Model of Computation

Specifies

- Set of operations
- Cost of operations (not necessarily time)

Examples

- Turing Machine
- Random Access Machine (RAM)
- PRAM
- Map Reduce(?)

Random Access Machine

Word

- Group of constant number of bits (e. g. byte)
- $\geq \log(\text{input size})$
- Usually integers or floats

Memory

- Big array of words
- Access by address

Operations

- Read and write a word from or into memory
- Arithmetic $+$, $-$, $*$, $/$, mod, $\lfloor\,\rfloor$
- Logic (can be bitwise) $\wedge$, $\vee$, xor, $\neg$
- Comparison based decisions

Each operation cost 1 unit of time.

Asymptotic Complexity

Asymptotic Complexity

Goal

- Determine runtime of an algorithm.

Depends on

- Input
- Hardware
- Programming language, compiler, and runtime environment

Solution

- Asymptotic Complexity
- How does the runtime behave based on the input size n?

Asymptotic Complexity

Hardware

- Raspberry Pi 2B 0.9 GHz
- Nexus 5 2.3 GHz
- Intel i7 4.0 GHz
- Same for other components (e.g. memory)

Runtime Environment

- Machine code (e. g. C++)
- Managed code (e. g. C# / Java)
- Interpreted code (e. g. Python)
- Virtual Machines (e. g. VirtualBox)

Conclusion

- Ignore constant factors.

Asymptotic Complexity

Consider two algorithms

- $T_1(n) = n^2 + 5n + 5$
- $T_2(n) = n^2$

n	4	16	64	256	1024	4096
$T_1(n)$	41	341	4,421	66,821	1,053,701	16,797,701
$T_2(n)$	16	256	4,096	65,536	1,048,576	16,777,216
T_1/T_2	2.5625	1.332	1.0793	1.0196	1.0049	1.0012

Conclusion

- Only keep strongest part. (n^2 in this case)

Example

Consider two algorithms and two computers

- Fast computer and slow algorithm
  10^7 operations per second $\quad T_1(n) = n^2$
- Slow computer and fast algorithm
  10^4 operations per second $\quad T_2(n) = n \lceil \log_2 n \rceil$
- Input size: 10^6

Runtime

- $T_1 = \dfrac{(10^6)^2}{10^7}\,\mathrm{s} = 10^5\,\mathrm{s} \approx 27.8\,\mathrm{h}$
- $T_2 = \dfrac{10^6 \lceil \log_2 10^6 \rceil}{10^4}\,\mathrm{s} = 2{,}000\,\mathrm{s} \approx 33.3\,\mathrm{min}$

Conclusion

- First lower complexity, then constant factors.

Big-O Notation

Based on complexity, $3n^2 - \log_2 n$, and $n^2 + 5n + 5$ are the same as n^2.

How do we write this?

Big-O Notation

- $\mathcal{O}(g) = \{ f : \mathbb{N} \to \mathbb{N} \mid \exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : f(n) \leq c \cdot g(n) \}$
- $f \in \mathcal{O}(g)$ means g is an upper bound for f.
- $\mathcal{O}(3n^2 - \log n) = \mathcal{O}(n^2 + 5n + 5) = \mathcal{O}(n^2)$

If an algorithm has runtime $n^2 + 5n + 5$, we say it runs in $\mathcal{O}(n^2)$ time.

Note that $\mathcal{O}(n) \subset \mathcal{O}(n \log n) \subset \mathcal{O}(n^2)$

Common Examples

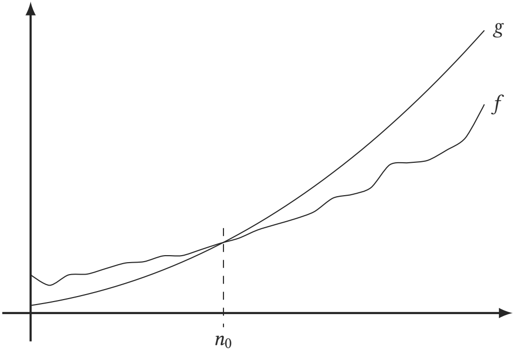
Complexity		Example
$\mathcal{O}(1)$	constant	basic operations
$\mathcal{O}(\log n)$	logarithmic	binary search
$\mathcal{O}(n)$	linear	counting, linear search, DFS/BFS
$\mathcal{O}(n \log n)$		sorting, finding doubles, convex hull
$\mathcal{O}(n^2)$	quadratic	checking all pairs
$\mathcal{O}\left(2^{\log^c n}\right)$	quasi polynomial	Graph Isomorphism†
$\mathcal{O}(2^n)$	exponential	SAT
$\mathcal{O}(n!)$		checking all permutations

† Preliminary result, not peer reviewed yet.

Big-O Notation — $f \in \mathcal{O}(g)$

$f \in \mathcal{O}(g)$: g is an upper bound for f.

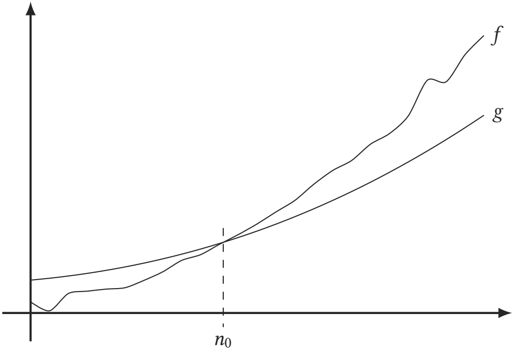
- $\exists c > 0\ \exists n_0 > 0\ \forall n \geq n_0 : f(n) \leq c \cdot g(n)$



Big-O Notation — $f \in \Omega(g)$

$f \in \Omega(g)$: g is a lower bound for f.

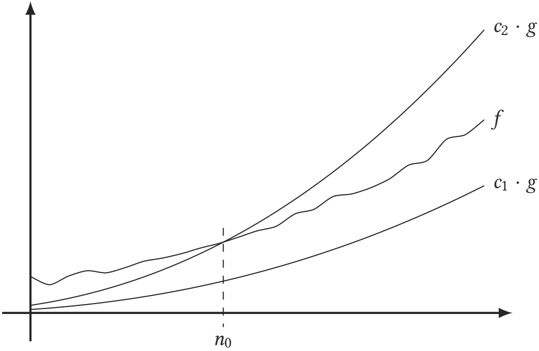
- $\exists c > 0\ \exists n_0 > 0\ \forall n \geq n_0 : f(n) \geq c \cdot g(n)$
- $f \in \Omega(g) \leftrightarrow g \in \mathcal{O}(f)$



Big-O Notation — $f \in \Theta(g)$

$f \in \Theta(g)$

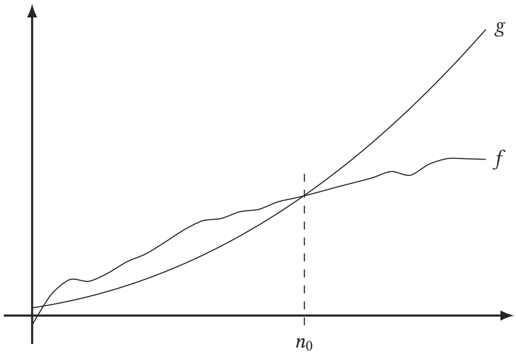
- $\exists c_1, c_2 > 0\ \exists n_0 > 0\ \forall n \geq n_0 : c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$
- $\Theta(g) = \mathcal{O}(g) \cap \Omega(g)$



Big-O Notation — $f \in o(g)$

$f \in o(g)$: f is dominated by g.

- $\forall c > 0\ \exists n_0 > 0\ \forall n \geq n_0 : f(n) \leq c \cdot g(n)$
  (This includes $c \leq 1$.)



Big-O Notation

$f \in \mathcal{O}(g)$: g is an upper bound for f.

- $\exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : f(n) \leq c \cdot g(n)$

$f \in \Omega(g)$: g is a lower bound for f.

- $\exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : f(n) \geq c \cdot g(n)$
- $f \in \Omega(g) \leftrightarrow g \in \mathcal{O}(f)$

$f \in \Theta(g)$

- $\exists c_1, c_2 > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : \; c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$
- $\Theta(g) = \mathcal{O}(g) \cap \Omega(g)$

$f \in o(g)$: f is dominated by g.

- $\forall c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : f(n) \leq c \cdot g(n)$
  (This includes $c \leq 1$.)

Questions

True or False? Explain your answer.

a) $f \in \mathcal{O}(g)$ implies $g \in \mathcal{O}(f)$

b) $f + g \in \Theta(\min(f, g))$

c) $f \in \mathcal{O}(g)$ implies $\log f \in \mathcal{O}(\log g)$

d) $f \in \mathcal{O}(g)$ implies $2^f \in \mathcal{O}(2^g)$

e) $f \in \mathcal{O}(f^2)$

f) $f \in \mathcal{O}(g)$ implies $g \in \Omega(f)$

g) $f(n) \in \Theta(f(n/2))$

h) $g \in o(f)$ implies $f + g \in \Theta(f)$

Questions

Rank the following functions by order of growth. Partition your list into equivalence classes such that functions f_i and f_j are in the same class if and only if $f_i \in \Theta(f_j)$.

2^{2^n}	$n^{1/\log n}$	$\log\log n$	$n \cdot 2^n$	$\log n$
$n^{\log\log n}$	n^3	1	$2^{\log n}$	$(\log n)^{\log n}$

Questions

Joe claims he can prove that $2^n \in \mathcal{O}(1)$. His proof goes by induction on n.

Base case

- $2^1 = 2$, *i. e.,* $2^1 \in \mathcal{O}(1)$.

Inductive step

- *Assume now that* $2^{n-1} \in \mathcal{O}(1)$ *(Inductive Hypothesis).*
- $2^n = 2 \cdot 2^{n-1}$
- *Because* $2f(n) \in \mathcal{O}(f(n))$, $2^n \in \mathcal{O}(1)$.

What is wrong with Joe's "proof"?

Runtime Analysis for Recurrences

Divide and Conquer

Idea

- Split problem into smaller sub-problems.
- Solve sub-problems recursively.
- Combine solutions of sub-problems to solve original problem.

Examples

- Binary Search
- Merge sort, Quicksort
- Matrix multiplication
- Drawing binary trees

Runtime of Divide and Conquer

General Formula

$$T(n) = \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & \text{if } n > 1 \end{cases}$$

For simplicity, we ignore the case $n = 1$.

Binary Search

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

Merge sort

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Solving Recurrence

Substitution Method

- Guess a (upper or lower) bound
- Prove it using induction

Recursion Tree

- Convert recurrence to tree.
- Each node represents a function call.
- Add cost of each layer and of all layers.

Master Theorem

- General solution (for some cases)

Substitution Method

Example: $T(n) = 2T\left(\frac{n}{2}\right) + n$

Hypothesis: $T(n) \in \mathcal{O}(n \log n)$, i. e. $T(n) \leq c\, n \log n$

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&\leq 2c\,(n/2)\log(n/2) + n \\
&= c\,n \log n - c\,n \log 2 + n \\
&= c\,n \log n - n(c \log 2 - 1) \\
&\leq c\,n \log n
\end{aligned}
$$

□

Substitution Method

Example: $T(n) = 4T\left(\frac{n}{2}\right) + n$

Hypothesis: $T(n) \in \mathcal{O}(n^2)$, i. e. $T(n) \leq c\, n^2$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4c\,(n^2/4) + n \\
&= c\,n^2 + n
\end{aligned}
$$

Does not work.

General advise for induction: Make your hypothesis stronger.

Substitution Method

Example: $T(n) = 4T\left(\frac{n}{2}\right) + n$

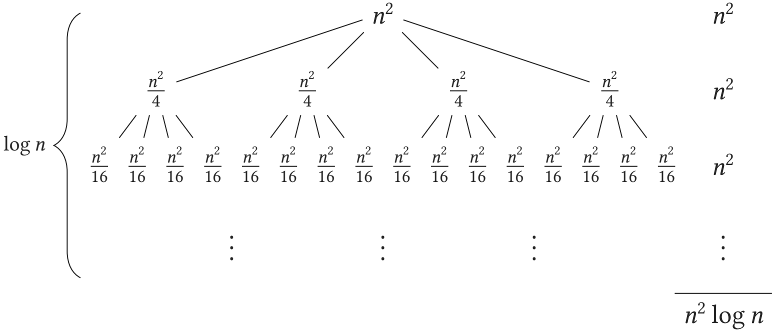
Hypothesis: $T(n) \leq c\,n^2 - n$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c\left(n^2/4\right) - 4(n/2) + n \\ &= c\,n^2 - 2n + n \\ &= c\,n^2 - n \end{aligned}$$

$\square$

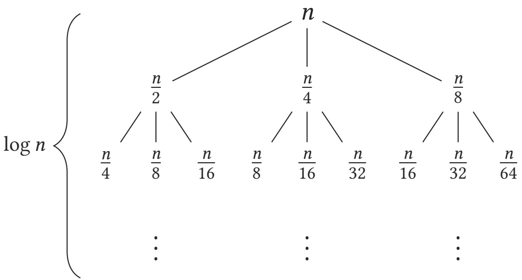
Recursion Tree

Example: $T(n) = 4T\left(\frac{n}{2}\right) + n^2$



Recursion Tree

Example: $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$



n

$\frac{7}{8} n$

$\left(\frac{7}{8}\right)^2 n$

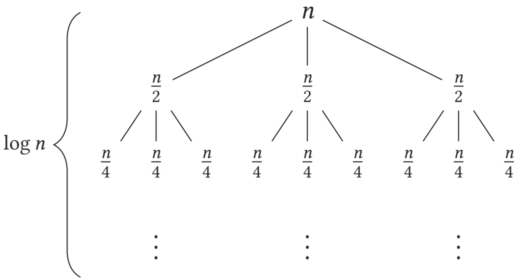
$\vdots$

$$\sum_{i=0}^{\log n} \left(\frac{7}{8}\right)^i n \leq 8n$$

$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $0 < x < 1$

Recursion Tree

Example: $T(n) = 3T\left(\frac{n}{2}\right) + n$



$$n$$

$$\frac{3}{2}n$$

$$\frac{9}{4}n$$

$$\vdots$$

$$\sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^i n$$

Example: $T(n) = 3T(n/2) + n$

We know, $\sum_{i=0}^{m} r^i = \dfrac{r^{m+1} - 1}{r - 1}$

Thus,

$$
\begin{aligned}
n \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i &= n \, \frac{1.5 \cdot 1.5^{\log n} - 1}{1.5 - 1} \\
&= 3n \cdot 1.5^{\log n} - 2n \\
&= 3n \cdot \left(2^{\log 1.5}\right)^{\log n} - 2n \\
&= 3n \cdot \left(2^{\log n}\right)^{\log 1.5} - 2n \\
&\approx 3n \cdot n^{0.58} - 2n \\
&= 3n^{1.58} - 2n \\
&\in \Theta\left(n^{1.58}\right)
\end{aligned}
$$

Master Theorem

Consider a recurrence in the form (with $a \geq 1$, $b > 1$)

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

(1) $\quad f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon}) \Rightarrow T(n) \in \Theta(n^{\log_b a})$

(2) $\quad f(n) \in \Theta(n^{\log_b a}) \Rightarrow T(n) \in \Theta(n^{\log_b a} \log n)$

(3) $\quad f(n) \in \Omega(n^{\log_b a + \varepsilon}) \Rightarrow T(n) \in \Theta(f(n))$

For (1) and (3), $\varepsilon > 0$.
For (3), $0 < c < 1$ and $af(\frac{n}{b}) \leq cf(n)$.

Master Theorem

$T(n) = 3T\left(\frac{n}{2}\right) + n$ (from recursion tree: $\Theta(n^{1.58})$)

- $a = 3$, $b = 2$
- $\log_b a = \log_2 3 \approx 1.58$
- $f(n) = n$, $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$ (Case 1)
- $T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{1.58})$

$T(n) = 4T\left(\frac{n}{2}\right) + n^2$ (from recursion tree: $\Theta(n^2 \log n)$)

- $a = 4$, $b = 2$
- $\log_b a = \log_2 4 = 2$
- $f(n) = n^2$, $f(n) \in \Theta(n^{\log_b a})$ (Case 2)
- $T(n) \in \Theta(n^{\log_b a} \log n) = \Theta(n^2 \log n)$

Master Theorem

$T(n) = 2T\left(\frac{n}{2}\right) + n^2$

- $a = 2$, $b = 2$
- $\log_b a = \log_2 2 = 1$
- $f(n) = n^2$, $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ (Case 3)
- $T(n) \in \Theta(f(n)) = \Theta(n^2)$