Balanced Search Trees

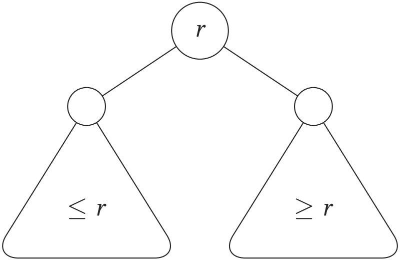
Binary Search Trees

Binary Search Tree

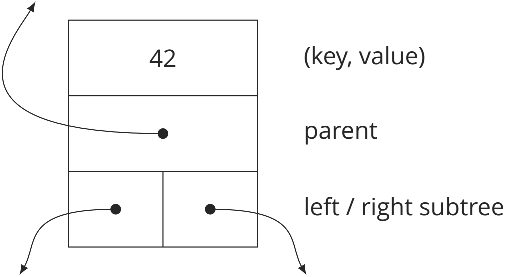
Binary Search Tree

A binary tree is a *binary search tree* if

- each element in the left subtree is smaller than the root,
- each element in the right subtree is larger than the root, and
- the left and the right subtree are binary search trees.



Implementation



Dictionary

Dictionary

A *dictionary* is an abstract data type which stores key-value pairs hand has the following operations:

- Insert(k, v)
- Find(k)
- Delete(k)

Insert(k, v)

- Inserts a key-value pair (k, v) into the dictionary.

Find(k)

- Returns a value with the key k.

Delete(k)

- Deletes a key-value pair with the key k.

BST – Insert(k, v)

Idea

- Find a a free spot in the tree and add a node which stores (k, v).

Strategy

- Start at root r.
- If $k < \mathrm{key}(r)$, continue in left subtree.
- If $k > \mathrm{key}(r)$, continue in right subtree.

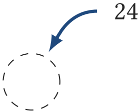
What if $k = \mathrm{key}(r)$?

Runtime

- $\mathcal{O}(h)$ (h is the height of the tree.)

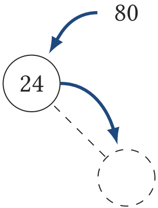
BST – Insert Example

Insert the numbers 22, 80, 18, 9, 90, 24.



BST – Insert Example

Insert the numbers $22, 80, 18, 9, 90, 24$.



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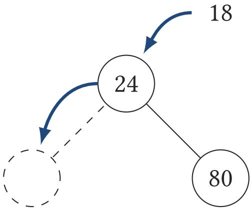

BST – Insert Example

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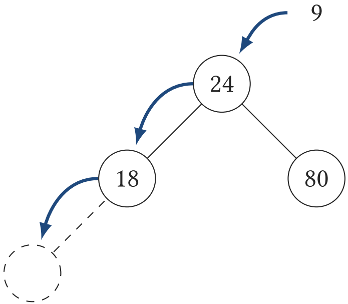
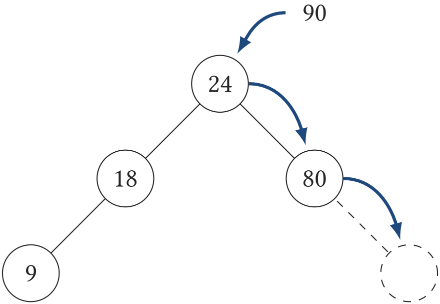


Insert the numbers 22, 80, 18, 9, 90, 24.



BST – Insert Example

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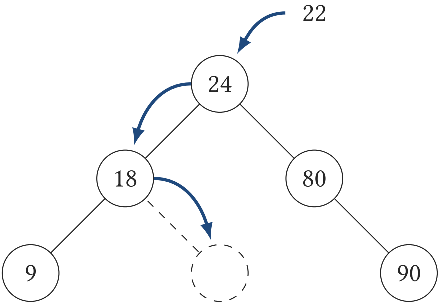
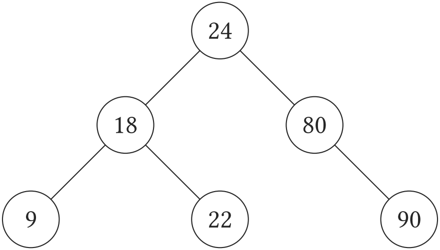


BST – Insert Example

Insert the numbers 22, 80, 18, 9, 90, 24.



BST – Find(k)

Find the node with key k.

Strategy

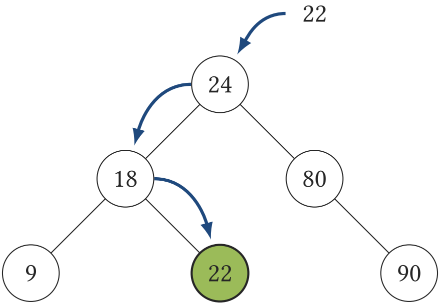
- Start at root r.
- If $k = \mathrm{key}(r)$, return r.
- If $k < \mathrm{key}(r)$, continue in left subtree.
- If $k > \mathrm{key}(r)$, continue in right subtree.

Runtime

- $\mathcal{O}(h)$ (h is the height of the tree.)

BST – Find Example

Find the number 22.



BST – Delete(k)

Delete the node with key k.

Strategy

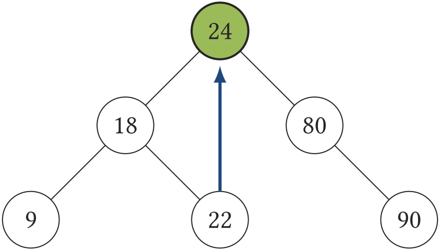
- $n :=$ Find(k)
- Let m be the node in the left subtree with the largest key or the node in the right subtree with the smallest key.
- Replace n with m.

Runtime

- $\mathcal{O}(h)$ (h is the height of the tree.)

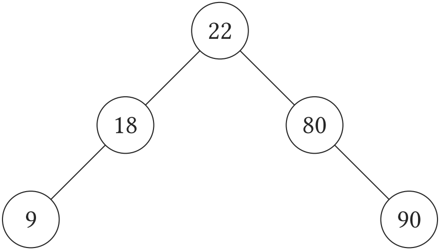
BST – Delete Example

Delete the number 24.



BST – Delete Example

Delete the number 24.

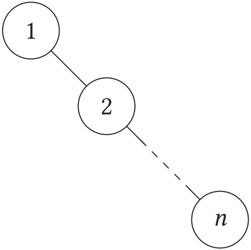


BST as Dictionary

Runtime of all operations is $\mathcal{O}(h)$.

- What is h in the worst case?

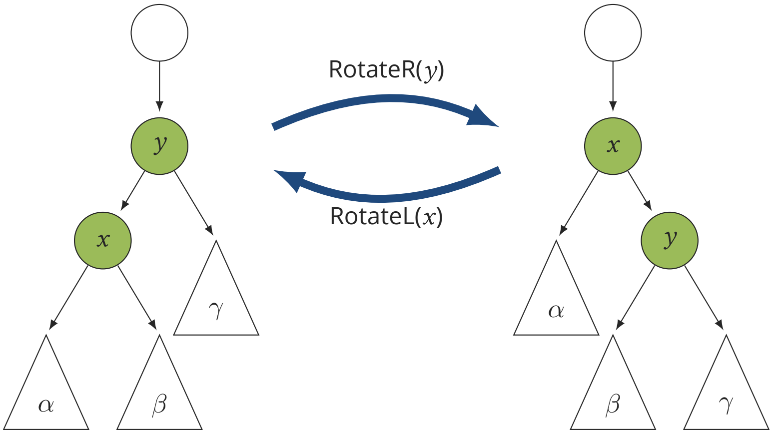
Consider inserting the sequence $1, 2, \ldots, n-1, n$



Thus, worst case height $h \in \mathcal{O}(n)$.

- How do we keep the tree balanced?

Rotation



How do we use this to keep a tree balanced?

Red-Black Trees

Red-Black Tree

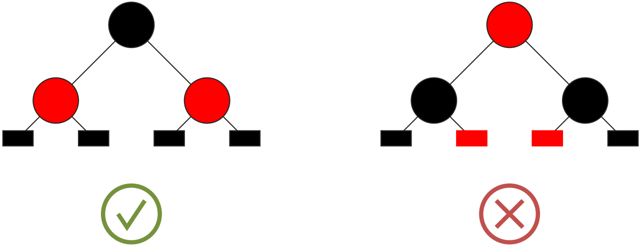
Red-Black Tree

A *red-black tree* is a binary search tree with the following properties:

0. The root is black.
1. A node is either red or black.
2. All `Null`-pointers are black.
3. If a node is red, then both its children are black.
4. Every path from a given node n to any of its descendant `Null`-pointers contains the same number of black nodes. This number is called black-height of n.

Red-Black Tree – Example

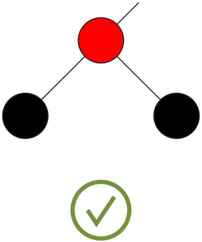
The tree on the right validates property (0), (1), and (2).

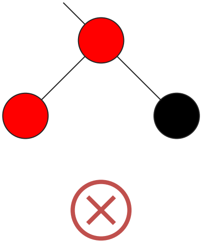


(We will ignore `Null`-pointers from here.)

Red-Black Tree – Example

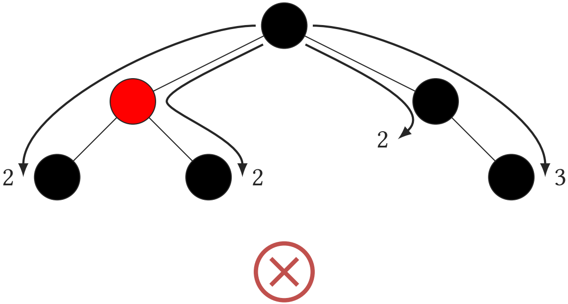
The tree on the right validates property (3).



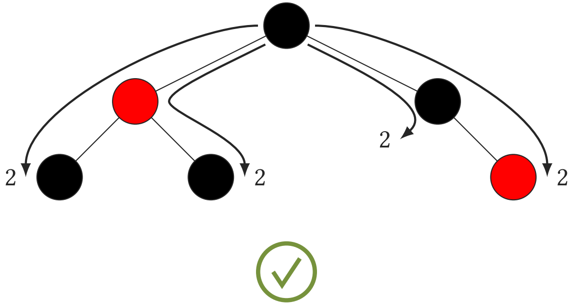


Red-Black Tree – Example

Validation of property (3).



Red-Black Tree – Example

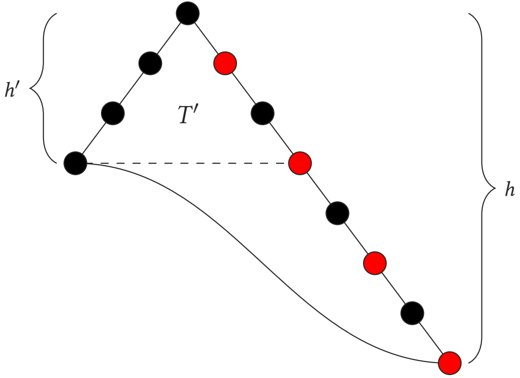


Red-Black Tree – Height

Theorem

A red-black tree with n nodes has a height of at most $\mathcal{O}(\log n)$.

Red-Black Tree – Height



T' is full. Thus, $h' \leq \log n$.

Because $h \leq 2h'$, $h \leq 2 \log n \in \mathcal{O}(\log n)$

Red-Black Tree – Insert and Delete

Basic Strategy

- Use Insert(k, v) and Delete(k) as defined for BSTs.
- New added nodes are red.
- Problem: The resulting tree may violate some properties of a red-black tree.

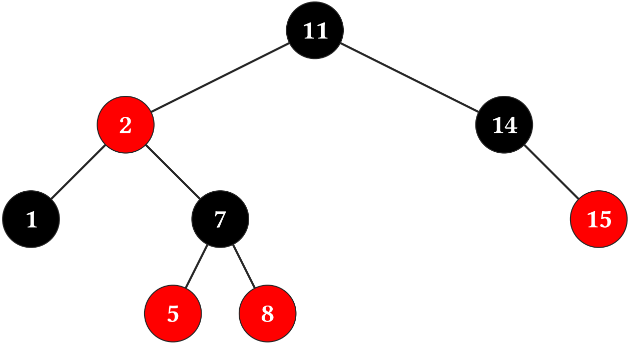
Restoring Red-Black Property

- Done by rotation and recolouring.
- There are five cases for insertion and six for removal. We will not discuss them here.
- General idea: Restore properties for the current layer, move the “incorrectness" to an upper layer, and repeat this on the upper layer.

Runtime

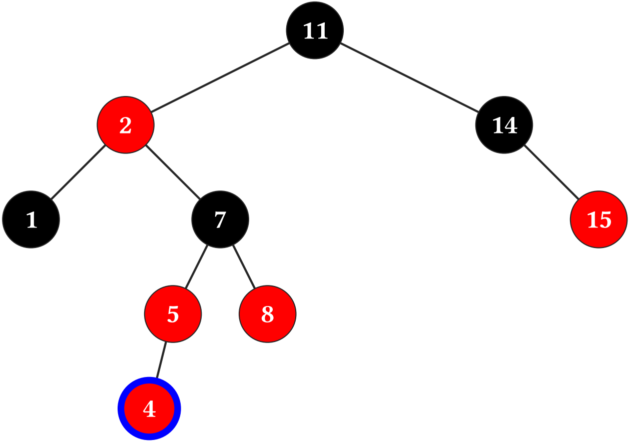
- $\mathcal{O}(\log n)$ for both operations

Given this red-black tree. We want to insert 4.



Red-Black Tree – Insertion Example

Given this red-black tree. We want to insert 4.



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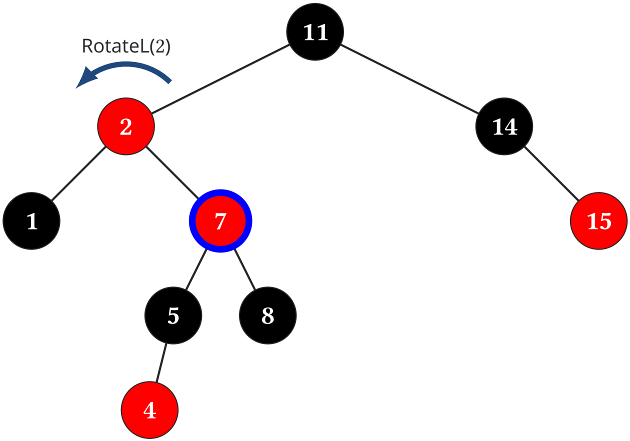

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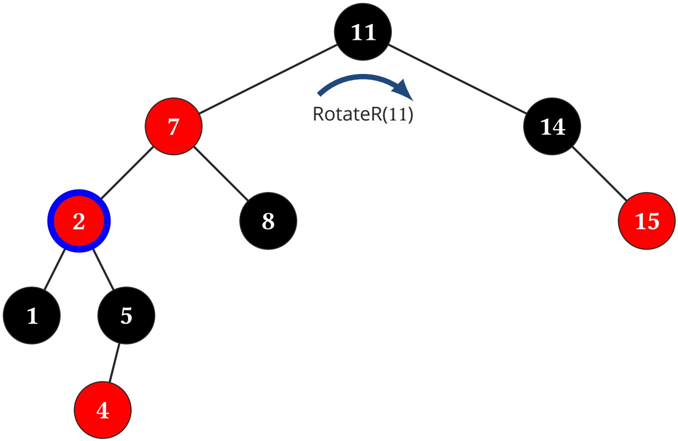

Red-Black Tree – Insertion Example

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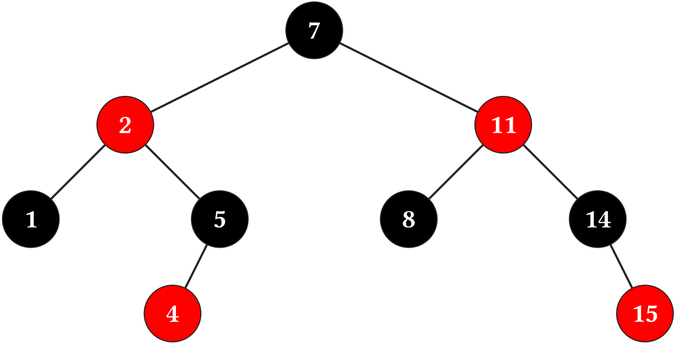


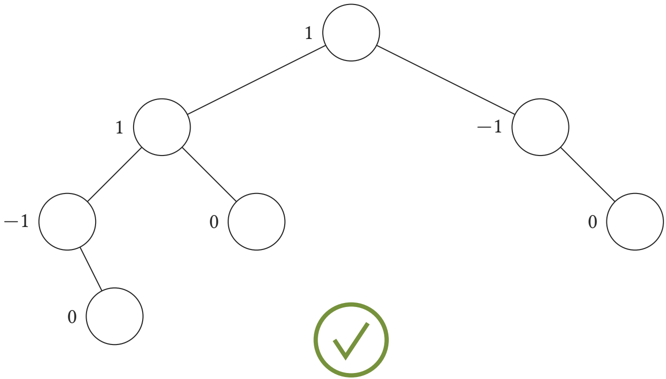
AVL Trees

AVL Tree

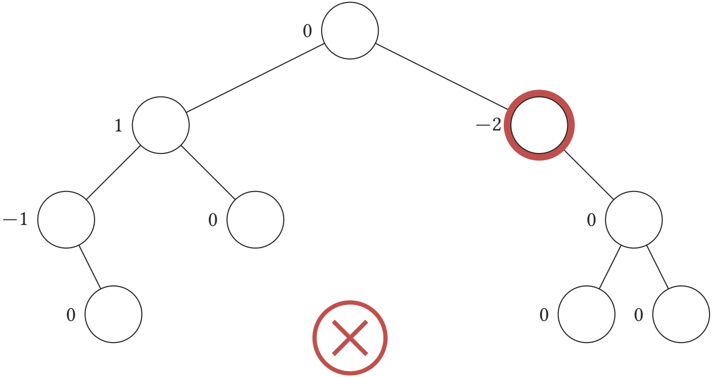
AVL Tree

A binary tree is an *AVL tree* if, for each node, the height of the left and right subtree differ by at most one.

AVL Tree – Example



AVL Tree – Example



AVL Tree – Height

Theorem

An AVL tree with n nodes has a height of at most $\mathcal{O}(\log n)$.

Proof. Let N_h be the min. number of nodes in an AVL tree of height h.

$$\begin{aligned} N_h &= 1 + N_{h-1} + N_{h-2} \\ &\geq 2 \cdot N_{h-2} \\ &\geq 2^{h/2} \end{aligned}$$

Thus, $h \leq 2 \log_2 N_h$, i. e., $h \in \mathcal{O}(\log n)$. $\square$

AVL Tree – Insert and Delete

Basic Strategy (similar to red-black trees)

- Use Insert(k, v) and Delete(k) as defined for BSTs.
- Problem: The resulting tree may violate some properties of an AVL tree.

Restoring AVL Property

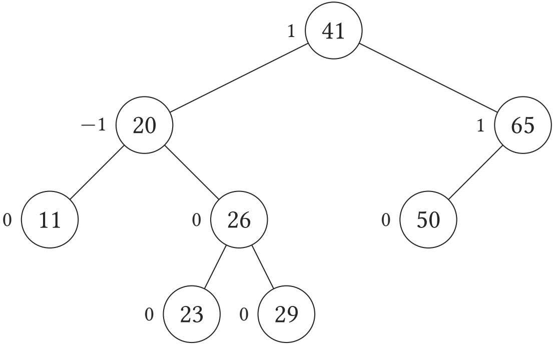
- Done by rotation.
- General idea: Restore properties for the current layer and repeat this on the upper layer.
- We will not discuss the details here.

Runtime

- $\mathcal{O}(\log n)$ for both operations

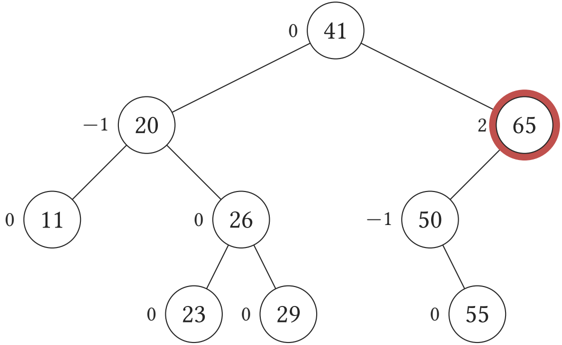
AVL Tree – Insertion Example

Insert(55)



AVL Tree – Insertion Example

Insert(55)



AVL Tree – Insertion Example

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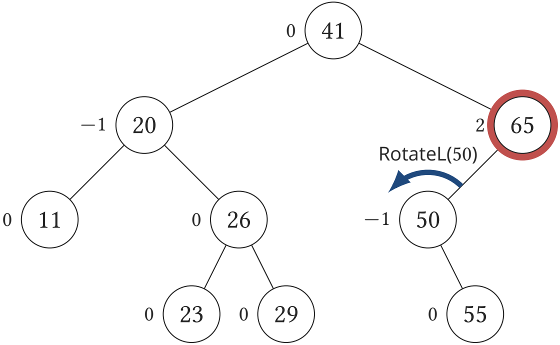

AVL Tree – Insertion Example

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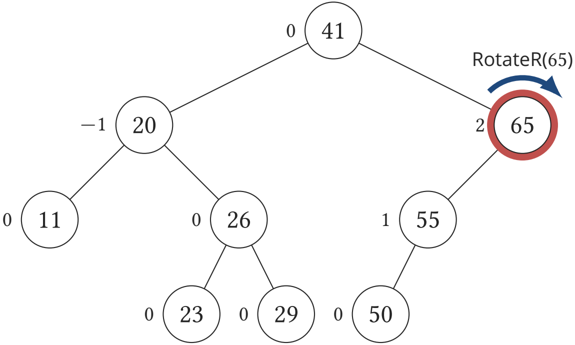
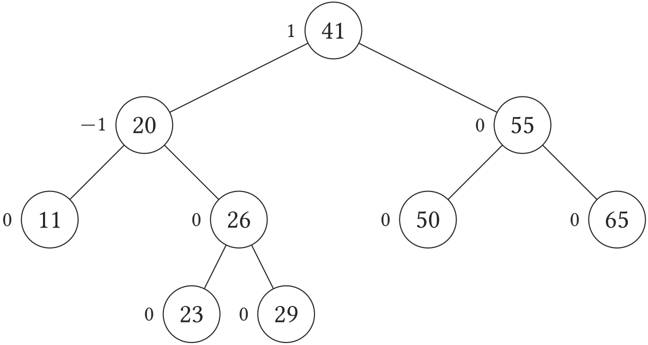


AVL Tree – Insertion Example

Insert(55)



B-Trees

B-Tree

B-Tree

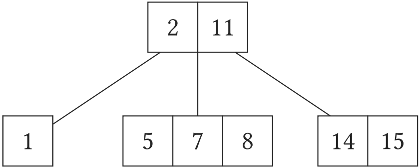
A *B-Tree* is a search tree such that, for some constant $t \geq 2$,

(1) each node n stores $|n|$ sorted keys ($t - 1 \leq |n| \leq 2t - 1$),

(2) each node which is not a leaf has $|n| + 1$ subtrees, and

(3) all leaves are on the same layer.

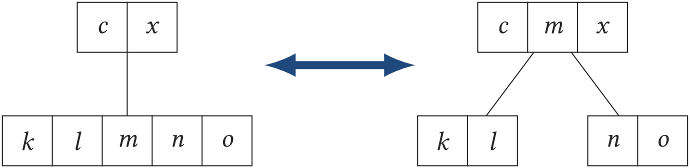
The root r is excluded from property (1). Instead, $1 \leq |r| \leq 2t - 1$.



B-Tree – Splitting and Merging

Full nodes (with $2t - 1$ keys) can be slitted.

- Remove middle key.
- Include it into parent node.

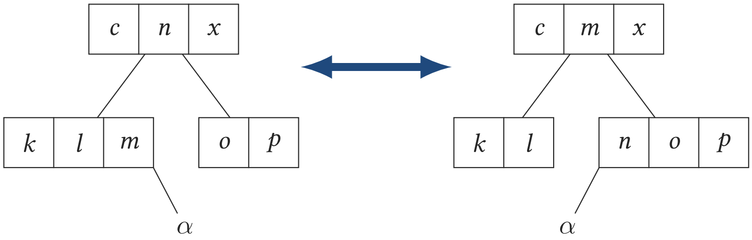


Neighbouring nodes with $t - 1$ keys can be merged.

- Remove separating key from parent node.
- Add it in middle of new node.

B-Tree – Shifting Keys

Keys can be shifted to decrease the size of a node and increase the size of its neighbour.



B-Tree – Insertion

Idea

- Similar to BSTs, find leaf which would contain the key and add it.

Problem

- What if leaf is full (stores $2t - 1$ keys)?
- What if leaf cannot be split because parent is full too?

Solution

- When searching for leaf, split every full node on the path.

Runtime: $\mathcal{O}(t \cdot \log_t n)$

- $\mathcal{O}(t)$ for splitting nodes.
- $\mathcal{O}(\log_t n)$ for the path from root to leaf.

B-Tree – Deletion

Strategy

- Search key in tree.
- For every node on path, ensure at least t keys are in the node (using merging and shifting).

Case 1: Key is in leaf.

- Simply delete key.

Case 2: Key is not in leaf.

- Replace key by k', the largest key in left child or smallest key in right child.
- Recursively delete k'.

Runtime: $\mathcal{O}(t \cdot \log_t n)$

- $\mathcal{O}(t)$ for merging nodes.
- $\mathcal{O}(\log_t n)$ for the path from root to leaf.