New Graph Classes of Bounded Clique-Width *

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Abstract

Clique-width of graphs is a major new concept with respect to efficiency of graph algorithms; it is known that every problem expressible in a certain kind of Monadic Second Order Logic called *LinEMSOL*($\tau_{1,L}$) by Courcelle, Makowsky and Rotics, is linear-time solvable on any graph class with bounded clique-width for which a k-expression for the input graph can be constructed in linear time. The notion of clique-width extends the one of treewidth since bounded treewidth implies bounded clique-width. We give a complete classification of all graph classes defined by forbidden one-vertex extensions of the P_4 (i.e. the path with four vertices a, b, c, d and three edges ab, bc, cd) with respect to bounded clique-width. Our results extend and improve recently published structural and complexity results in a systematic way.
Keywords: Clique-width of graphs; cograph generalizations; linear-time algorithms.

1 Introduction

Recently, in connection with graph grammars, Courcelle, Engelfriet and Rozenberg in [17] introduced the notion of clique-width of a graph which is closely related to modular decomposition. The concept of clique-width of a graph has attracted much attention due to the fact that, in [18], Courcelle, Makowsky and Rotics have shown that every graph problem definable in LinEMSOL($\tau_{1,L}$) (a variant of Monadic Second Order Logic) is linear-time solvable on graphs with bounded clique-width if not only the input graph G but also a k-expression of G is given. (The time complexity may be

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sublinear with respect to the size of the input graph G if the input is just a k-expression of G). The problems Vertex Cover, Maximum Weight Stable Set (MWS), Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL($\tau_{1,L}$) definable problems.

It is known that the class of P_4-free graphs (also called *cographs*) is exactly the class of graphs having clique-width at most 2, and a 2-expression can be found in linear time along the cotree of a cograph. Due to the basic importance of cographs, it is of interest to consider graph classes defined by forbidden one-vertex extensions of P_4 - see Figure 1 - which are natural generalizations of cographs.

The aim of this paper is to investigate the structure and clique-width of all such graph classes in a systematic way. This leads to a complete classification of these classes with respect to bounded clique-width. Part of the motivation comes from known examples such as the (P_5,co-P_5,bull)-free graphs studied by Fouquet in [22] (see Theorem 11) and the (P_5,co-P_5,chair)-free graphs studied by Fouquet and Giakoumakis in [23] (see Theorem 8). Moreover, there are papers such as [32, 37] dealing with (chair,co-P,gem)-free graphs and [26] dealing with (P_5,P,gem)-free graphs where it is shown that the MWS problem can be solved in polynomial time on these classes. Our results imply bounded clique-width and linear time for any LinEMSOL($\tau_{1,L}$) definable problem including MWS for these classes as well as for many other examples. This also continues research done in [2, 4, 5, 6, 7, 9, 10, 11, 12] and is partially based on some of the results of these papers.

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. For a vertex $v \in V$, let $N(v) = \{u : uv \in E\}$ denote the *neighborhood* of v in G, and for a subset $U \subseteq V, v \notin U$, let $N_U(v) = \{u : u \in U, uv \in E\}$ denote the *neighborhood* of v with respect to U. The edge set between two disjoint vertex sets X, Y forms a *join*, denoted by $X \textcircled{1} Y$ (*co-join*, denoted by $X \textcircled{0} Y$) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds. Subsequently, we will use these symbols also as operation symbols i.e. the join operation between disjoint vertex sets X, Y adds all edges between them whereas the co-join operation for X and Y is the disjoint union of the subgraphs induced by X and Y (without edges between them).

A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e., every vertex $v \in V \setminus M$ has either a join or a co-join to M. A module is *trivial* if it is either the empty set, a one-vertex set or the entire vertex set V. Nontrivial modules are called *homogeneous sets*. A graph is *prime* if it contains only trivial modules. The notion of modules plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms - see e.g. [35] for modular decomposition of discrete structures and its algorithmic use.

For $U \subseteq V$ let $G(U)$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. A vertex set $U \subseteq V$ is *stable* (or *independent*) in G if the vertices in U are pairwise nonadjacent. Let co-$G = \overline{G} = (V, \overline{E})$ denote the *complement graph* of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable

set in $\overline{G}$.
For $k \geq 1$, let P_k denote a chordless path with k vertices and $k-1$ edges, and for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *hole* in G is an induced subgraph of G isomorphic to C_k, $k \geq 5$. An *odd hole* is a hole with odd number of vertices. Note that the P_4 is the smallest nontrivial prime graph with at least three vertices and the complement of a P_4 is a P_4 itself. The *house* is the $\overline{P_5}$. The *domino* is the graph consisting of a P_5 with vertices a, b, c, d, e and edges ab, bc, cd, de plus an additional vertex f being adjacent to a, c and e.
For a subgraph H of G, a vertex not in H is a *k-vertex with respect to H* if it has exactly k neighbors in H. *H has no k-vertex* if there is no k-vertex with respect to H.

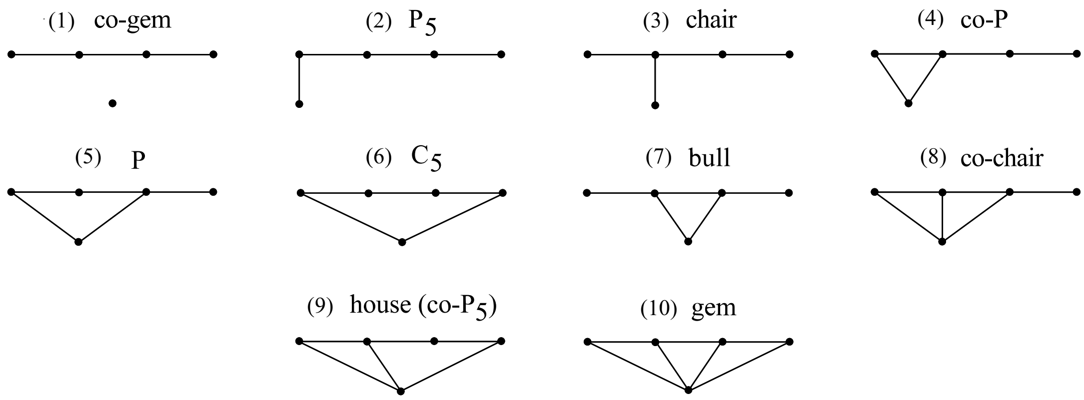

Figure 1: All one-vertex extensions of a P_4

See Figure 1 for the definition of chair, P, bull, gem and their complements. Note that the complement of a bull is a bull itself. The *diamond* is the $K_4 - e$, i.e., a four vertex clique minus one edge. The $2K_2$ is the complement of C_4 i.e. it consists of four vertices a, b, c, d and two edges ab and cd.
Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$. A graph is a *split graph* if G is partitionable into a clique and a stable set.

Lemma 1 ([21]) *G is a split graph if and only if G is ($2K_2$,C_4,C_5)-free.*

In what follows, we need the following classes of (prime) graphs:

- G is a *thin spider* if its vertex set is partitionable into a clique C and a stable set S with $|C| = |S|$ or $|C| = |S| + 1$ such that the edges between C and S are a matching and at most one vertex in C is not covered by the matching (an unmatched vertex is called the *head of the spider*). The thin spider with 6 vertices is also called *net*.

- A graph is a *thick spider* if it is the complement of a thin spider.

- G is *matched co-bipartite* if its vertex set is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching.

- G is *co-matched bipartite* if G is the complement graph of a matched co-bipartite graph.
- A bipartite graph $B = (X, Y, E)$ is a *chain graph* [36] if there is an ordering $x_1, x_2, \ldots, x_k$ of all vertices in X such that $N(x_i) \subseteq N(x_j)$ for all $1 \leq i < j \leq k$. (Note that then also the neighborhoods of the vertices from Y are linearly ordered by set inclusion.) If moreover, $|X| = |Y| = k$ and $N(x_i) = \{y_1, \ldots, y_i\}$ for all $1 \leq i \leq k$ then B is prime.
- G is a *co-bipartite chain graph* if it is the complement of a bipartite chain graph.
- G is an *enhanced co-bipartite chain graph* if it is partitionable into a co-bipartite chain graph with cliques C_1, C_2 and three additional vertices a, b, c (a and c optional) such that $N(a) = C_1 \cup C_2$, $N(b) = C_1$, and $N(c) = C_2$, and there are no other edges in G.
- G is an *enhanced bipartite chain graph* if it is the complement of an enhanced co-bipartite chain graph.

2 Cographs, clique-width and logical expressibility of problems

The P_4-free graphs (also called *cographs*) play a fundamental role in graph decomposition; see [16] for linear time recognition of cographs, [14, 15, 16] for more information on P_4-free graphs and [8] for a survey on this graph class and related ones.
For a cograph G, either G or its complement is disconnected, and the *cotree* of G expresses how the graph is recursively generated from single vertices by repeatedly applying join and co-join operations.
Note that the cographs are those graphs whose modular decomposition tree contains only join and co-join nodes as internal nodes.

Based on the following operations on vertex-labeled graphs, namely

(i) creation of a vertex labeled by integer l,

(ii) disjoint union (i.e., co-join),

(iii) join between all vertices with label i and all vertices with label j for $i \neq j$, and

(iv) relabeling all vertices of label i by label j,

the notion of *clique-width* $cwd(G)$ of a graph G is defined in [17] as the minimum number of labels which are necessary to generate G by using these operations. Cographs are exactly the graphs whose clique-width is at most two.
A *k-expression* for a graph G of clique-width k describes the recursive generation of G by repeatedly applying these operations using at most k different labels.

Proposition 1 ([18, 19]) *The clique-width $cwd(G)$ of a graph G is the maximum of the clique-width of its prime subgraphs, and the clique-width of the complement graph $\overline{G}$ is at most $2 \cdot cwd(G)$.*

Recently, the concept of clique-width of a graph attracted much attention since it gives a unified approach to the efficient solution of many problems on graph classes of bounded clique-width via the expressibility of the problems in terms of logical expressions.
In [18], it is shown that every problem definable in a certain kind of Monadic Second Order Logic, called *LinEMSOL*($\tau_{1,L}$) in [18], is linear-time solvable on any graph class with bounded clique-width for which a k-expression can be constructed in linear time. Hereby, in [18], it is mentioned that, roughly speaking, MSOL(τ_1) is Monadic Second Order Logic with quantification over subsets of vertices but not of edges; MSOL($\tau_{1,L}$) is the restriction of MSOL(τ_1) with the addition of labels added to the vertices, and LinEMSOL($\tau_{1,L}$) is the restriction of MSOL($\tau_{1,L}$) which allows to search for sets of vertices which are optimal with respect to some linear evaluation functions.
The problems Vertex Cover, Maximum Weight Stable Set, Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL($\tau_{1,L}$) definable problems.

Theorem 1 ([18]) *Let $\mathcal{C}$ be a class of graphs of clique-width at most k such that there is an $\mathcal{O}(f(|E|,|V|))$ algorithm, which for each graph G in $\mathcal{C}$, constructs a k-expression defining it. Then for every LinEMSOL($\tau_{1,L}$) problem on $\mathcal{C}$, there is an algorithm solving this problem in time $\mathcal{O}(f(|E|,|V|))$.*

As an application, it was shown in [18] that P_4-sparse graphs and some variants of them have bounded clique-width. Hereby, a graph is *P_4-sparse* if no set of five vertices in G induces at least two distinct P_4's [27, 28]. From the definition, it is obvious that a graph is P_4-sparse if and only if it contains no C_5, P_5, $\overline{P_5}$, P, $\overline{P}$, chair, co-chair (see Figure 1). See [12] for a systematic investigation of superclasses of P_4-sparse graphs.
In [27], it was shown that the prime P_4-sparse graphs are the spiders (which were called *turtles* in [27]), and according to Proposition 1 and the fact that the clique-width of thin spiders is at most 4 (which is easy to see), it follows that P_4-sparse graphs have bounded clique-width.
It is straightforward to see that the clique-width of matched co-bipartite (co-matched bipartite) graphs, bipartite chain (co-bipartite chain) graphs as well as the clique-width of induced paths and cycles is at most 4, and corresponding k-expressions can be determined in linear time.
A graph G is *distance-hereditary* [30] if for every connected induced subgraph H of G, the distance between any two vertices in H is the same as their distance in G.

Theorem 2 ([1]) *A graph is distance-hereditary if and only if it is house-, hole-, domino-, and gem-free.*

See also [8] for further information on distance-hereditary graphs. In [25], Golumbic and Rotics have shown that the clique width of distance-hereditary graphs is at most 3 and corresponding k-expressions can be determined in linear time.

3 Classification strategy

Let $\mathcal{F}(P_4)$ denote the set of the 10 one-vertex extensions of the P_4 (see Figure 1). For $\mathcal{F}' \subseteq \mathcal{F}(P_4)$, there are 1024 classes of $\mathcal{F}'$-free graphs. The two extreme cases are:

(i) For $\mathcal{F}' = \emptyset$, the class of $\mathcal{F}'$-free graphs is the class of all graphs.

(ii) Assume that a graph has at least 5 vertices. Then it is a cograph if and only if it contains none of the 10 possible one-vertex extensions of a P_4. Thus, for $\mathcal{F}' = \mathcal{F}(P_4)$ (and for graphs with at least five vertices), the class of $\mathcal{F}'$-free graphs is the class of cographs.

Our aim is to give a complete classification of all the other classes $\mathcal{F}'$-free graphs with respect to their clique-width. Hereby, in Figure 2 we denote graph classes by the numbers of their forbidden subgraphs as they are enumerated in Figure 1; for example, the (P_5,gem)-free graphs are the class denoted by (2,10), and the class of complements of these graphs is the class of (co-P_5,co-gem)-free graphs, i.e., the class denoted by $(1, 9)$. Recall that by Proposition 1, a class has bounded clique-width if and only if the class of its complement graphs has bounded clique-width. Thus, it is enough to mention the class with lexicographically smaller name (in the example, (1,9) and not (2,10)).

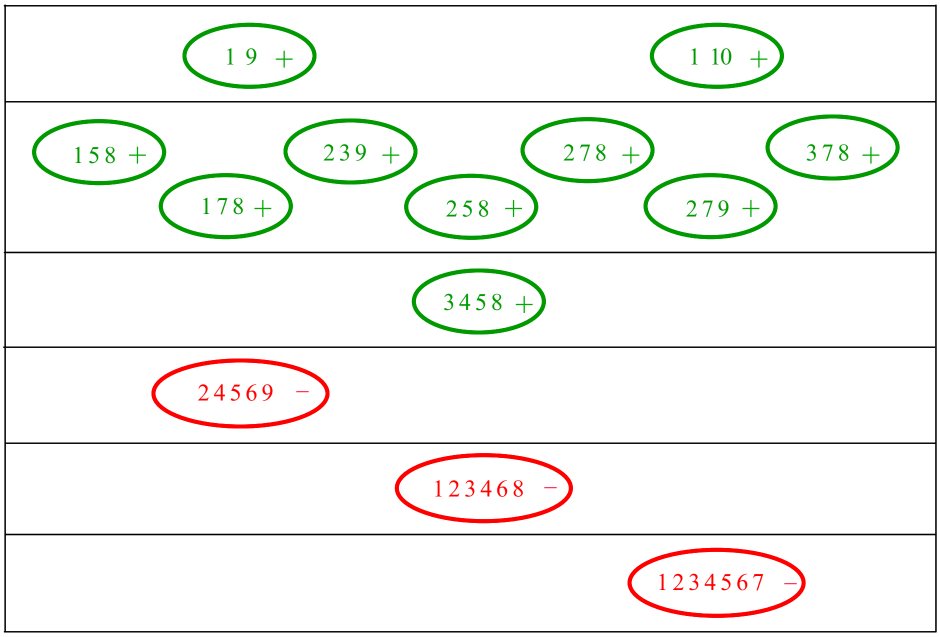

Figure 2: Essential classes for all combinations of forbidden 1-vertex P_4 extensions; + (−) denotes bounded (unbounded) clique-width

Summarizing, Figure 2 describes all inclusion-minimal classes of unbounded clique-width and all inclusion-maximal classes of bounded clique-width. Note that any subclass of bounded clique-width has bounded clique-width as well, whereas any superclass of unbounded clique-width has unbounded clique-width as well. Thus, the classification is obtained by two types of results:

- bounded clique-width of some inclusion-maximal classes;
- unbounded clique-width of all other classes.

3.1 Unbounded clique-width

All facts on unbounded clique-width used in this paper are based on the following results by Makowsky and Rotics [34]:

Theorem 3 ([34])

(i) *The following grid types (see Figure 3) have unbounded clique-width:*

- *the F_n grid;*
- *the $H_{n,q}$ grid*

(ii) *Split graphs have unbounded clique-width.*

Hereby the F_n grid and the $H_{n,q}$ grid are constructed from an $n \times n$ square grid G_n as follows (see [34]): Let $n \geq 4$ and $q \geq 2$.

(i) F_n is the result of the following operations:

- Replace every edge of G_n by a simple path with four edges, introducing three new vertices which are the internal vertices of the path. Let G'_n denote the resulted graph.
- For all vertices v of degree 3 in G'_n do the following: For the two neighbors u and w of v such that u, v and w are in the same row or column in G'_n, add an edge connecting u to w. Let G''_n denote the resulted graph.
- For all vertices v of degree 4 in G''_n do the following: For the four neighbors u_1, u_2, u_3, u_4 of v, omit v from G''_n and add all edges connecting u_i and u_j, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ such that u_1, u_2, u_3, u_4 will form a clique. Let F_n denote the resulted graph.

(ii) $H_{n,q}$ is the result of the following operations:

- Replace every edge of G_n by a simple path with three edges, introducing two new vertices which are the internal vertices of the path. Let G'_n denote the resulted graph.
- For all vertices v of degree 4 in G'_n do the following: For the four clockwise neighbors u_1, u_2, u_3, u_4 of v, omit v from G'_n and add the edges $u_i u_{i+1}$ (index arithmetic modulo 4) such that u_1, u_2, u_3, u_4 induce a C_4 in G'_n. Let G''_n denote the resulted graph.

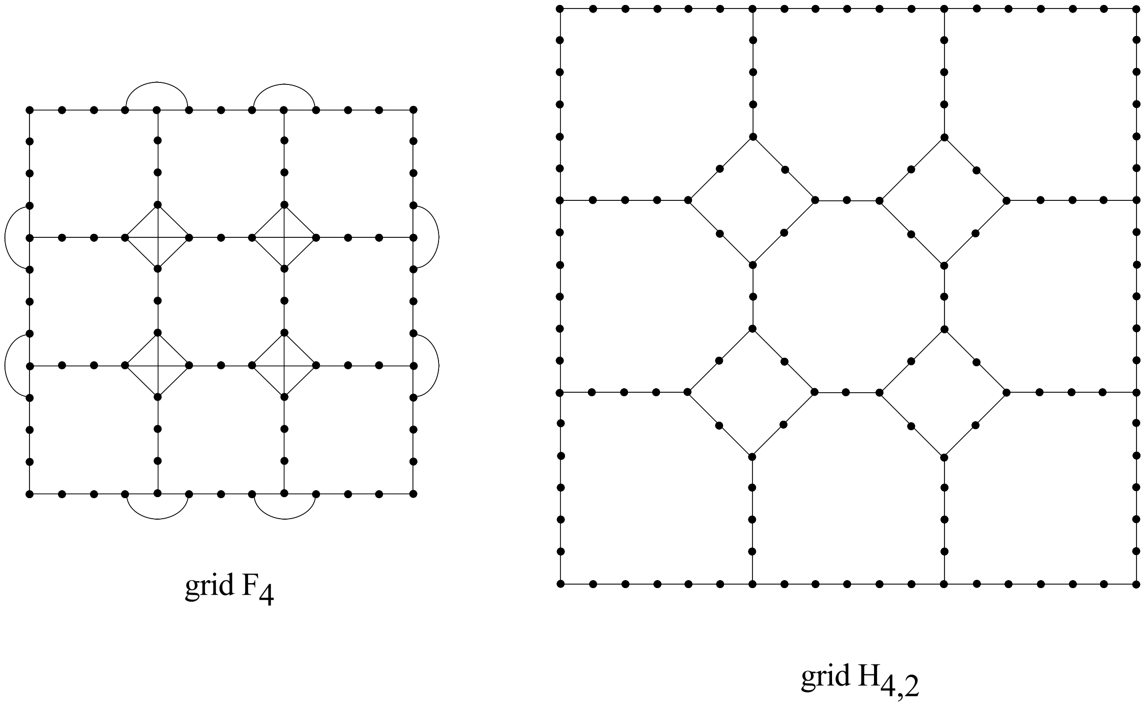

Figure 3: The grids F_4 and $H_{4,2}$

– Replace every edge of G''_n by a simple path with q edges, introducing $q-1$ new vertices which are the internal vertices of the path. Let $H_{n,q}$ denote the resulted graph.

By the definitions of F_n and $H_{n,q}$, F_n grids do not contain chair, P, C_5, co-chair, house, and gem, and $H_{n,q}$ grids do not contain co-P, P, C_5, bull, co-chair, house, and gem. Moreover, by Lemma 1, split graphs do not contain P_5, co-P, P, C_5, and house. This proves

Lemma 2 *The following graph classes have unbounded clique-width:*

(*i*) $(1, 2, 3, 4, 6, 8)$*-free graphs;*

(*ii*) $(1, 2, 3, 4, 5, 6, 7)$*-free graphs;*

(*iii*) $(2, 4, 5, 6, 9)$*-free graphs.*

This implies unbounded clique-width for all classes with label – in Figure 2 and in Figure 6.

3.2 Bounded clique-width

Proposition 1 shows that in order to obtain an upper bound on the clique-width of a graph class, it is enough to study its prime graphs. This is done extensively here and in other papers, and Theorem 4 is a good example:

Theorem 4 ([10]) *If G is a prime (diamond,co-diamond)-free graph then G or $\overline{G}$ is either a matched co-bipartite graph or G has at most 9 vertices.*

Using linear-time modular decomposition, this gives a linear-time construction for corresponding 9-expressions (the bound 9 can be slightly improved).

Other basic results on bounded clique-width used here are the subsequent Theorems 5 - 13:

Theorem 5 ((1,9), [6]) *The clique-width of (P_5,gem)-free graphs is at most 5.*

The proof given in [6] is long and technically involved and does not provide a linear-time construction of corresponding 5-expressions.

Theorem 6 ((1,10), [7]) *The clique-width of (gem,co-gem)-free graphs is at most 16.*

The proof of Theorem 6 is technically very involved and does not give linear-time algorithms for constructing corresponding k-expressions.

Theorem 7 ((1,5,8), [9]) *If G is a prime (chair, co-P, gem)-free graph then G fulfills one of the following conditions:*

(*i*) *G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*

(*ii*) *G is a thin spider;*

(*iii*) *G is a co-matched bipartite graph;*

(*iv*) *G has at most 9 vertices.*

Theorem 8 ((2,3,9), [23]) *If G is a prime (P_5,$\overline{P_5}$,chair)-free graph then G is either a co-bipartite chain graph or a spider or C_5.*

Theorem 9 ((2,5,8), [12]) *If G is a prime (chair,co-P,house)-free graph then G fulfills one of the following conditions:*

(*i*) *G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*

(*ii*) *G is a co-matched bipartite graph;*

(*iii*) *G is a spider.*

Theorem 10 ((2,7,8), [4]) *If G is a prime ($\overline{P_5}$,bull,chair)-free graph then G is either a co-matched bipartite graph or an induced path or cycle or $\overline{G}$ is (P_5,diamond)-free.*

Theorem 11 ((2,7,9), [22]) *If G is a prime (P_5,$\overline{P_5}$,bull)-free graph then G or $\overline{G}$ is a bipartite chain graph or a C_5.*

Theorem 12 ((3,4,5,8),[12]) *If G is a prime (chair,co-chair,P,co-P)-free graph then one of the following conditions holds:*

(i) *G or $\overline{G}$ is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*

(ii) *G is a spider;*

(iii) *G has at most 9 vertices.*

Theorem 13 ((3,7,8), [4]) *If G is a prime (bull,chair,co-chair)-free graph then G or $\overline{G}$ is either a co-matched bipartite graph or an induced path or cycle.*

In section 4, we will derive structure results for some of the subclasses of (1,9) and (1,10), namely (1,2,9), (1,2,10), (1,4,10), (1,3,9), (1,7,9), (1,8,9), (1,3,10), (1,4,9), and we will give a detailed structure description for the class (1,7,8) and a shorter proof for Theorem 7 on (1,5,8). These results imply linear-time algorithms for constructing corresponding k-expressions.

Finally, a straightforward case analysis gives the complete classification described in Figure 2.

3.3 Further tools

In this subsection we collect some further helpful facts which are needed in section 4:

Lemma 3 ([29]) *If a prime graph contains an induced C_4 (induced $2K_2$) then it contains an induced co-P_5 or A or domino (induced P_5 or co-A or co-domino).*

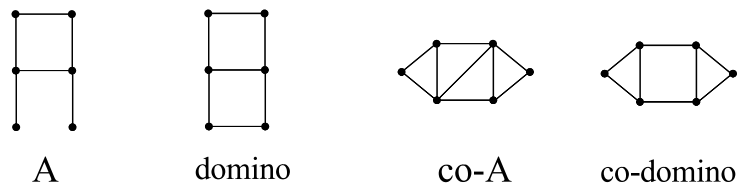

Figure 4: The A and domino and their complements

The proof of Lemma 3 can be extended in a straightforward way to the case of a diamond instead of a C_4. For this purpose let us call *d-A* the graph resulting from an A graph by adding an additional diagonal edge in the C_4, and *d-domino* the graph resulting from a domino graph by adding an additional diagonal edge in one of the C_4's - see Figure 5.

Lemma 4 *If a prime graph contains an induced diamond (co-diamond) then it contains an induced gem or d-A or d-domino (co-gem or co-d-A or co-d-domino).*

See e.g. [2] for a proof of Lemma 4.

Theorem 14 ([2]) *Prime (P_5,diamond)-free graphs are either matched co-bipartite or a thin spider or an enhanced bipartite chain graph or have at most 9 vertices.*

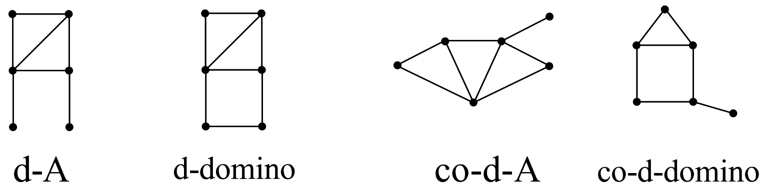

Figure 5: The *d-A* and *d*-domino and their complements

Lemma 5 ([5]) *Prime (P_5,gem)-free graphs containing a co-domino are matched co-bipartite.*

Lemma 6 ([4]) *Prime chair-free bipartite graphs are either co-matched bipartite or an induced path or cycle.*

Lemma 7 ([11]) *Prime chair-free split graphs are spiders.*

The following lemma is a useful result by De Simone:

Lemma 8 ([20]) *Every prime graph is either bipartite or co-diamond-free or is an odd hole or contains an induced subgraph from the following family F of graphs:*

1. *an odd hole plus a new vertex adjacent to exactly one of the hole vertices (called an* odd apple *in [20]);*
2. *F_1: thin spider with 6 vertices (i.e. net);*
3. *F_2: a bull plus a new vertex adjacent to one of the bull vertices of degree 3;*
4. *F_3: the d-A (see Figure 5);*
5. *F_4: the co-d-A (see Figure 5);*
6. *F_5: the co-d-domino (see Figure 5);*
7. *F_6: a house plus a new vertex adjacent to one of the house vertices of degree 3.*

Lemma 9 ([4, 20]) *Prime (bull, chair)-free graphs containing a co-diamond are either co-matched bipartite or an induced path or cycle.*

Lemma 10 *Prime co-gem-free bipartite graphs are co-matched bipartite.*

Proof. Let $B = (X, Y, E)$ be a prime co-gem-free bipartite graph. Assume first that B is co-diamond-free. If B is not co-matched then let $a \in X$, $b, c \in Y$, $b \neq c$, such that $ab \notin E$ and $ac \notin E$. Since B, as a prime graph, is connected, there is a vertex $d \in Y$ with $ad \in E$ but now a, b, c, d induce a co-diamond - contradiction. Assume now that the prime bipartite co-gem-free graph B contains a co-diamond. Then, by Lemma 4, it contains a co-d-A or co-d-domino but both of them are not bipartite - contradiction. $\square$

Lemma 11 *Prime (co-diamond,gem)-free graphs containing a diamond have at most 11 vertices.*

Proof. Let H be an induced subgraph of a graph G and assume that $V(H) = \{v_1, \ldots, v_h\}$. We use the following notion: For a set $S \subseteq \{1, \ldots, 5\}$, V_S denotes the set of all vertices $v \in G - H$ such that $N_H(v) = \{v_i \mid i \in S\}$. We also write V_{123} for $V_{\{1,2,3\}}$ and so on. Recall that a vertex not in H is a *k-vertex* for H if it has exactly k neighbors in H.
By Lemma 4 and since G is co-diamond-free, G contains a d-domino D. Let D consist of the P_5 Q with vertices $1, 2, 3, 4, 5$ and edges $12, 23, 34, 45$ and a 4-vertex a for Q being nonadjacent to 2.

Now, since G is co-diamond- and gem-free, the following conditions hold:

Claim 3.1 *Q has no 0-vertex, no 1-vertex and no 5-vertex.*

Claim 3.2 *For all 2-vertices v, $v \in V_{24}$. Moreover, V_{24} is a stable set.*

Proof. Obviously, the only possible 2-vertices for Q are adjacent to 2 and 4 or to 1 and 5 but in the last case, if x is a 2-vertex adjacent to 1 and 5, since $a4, 2, x$ is no co-diamond, $ax \in E$ holds but then $345xa$ is a gem - contradiction. Thus, every 2-vertex for Q is adjacent to 2 and 4, and V_{24} is a stable set since G has no co-diamond. $\square$

Claim 3.3 $V_{124} = V_{234} = V_{134} = V_{135} = V_{145} = V_{245} = \emptyset$.

Proof. Since for $x \in V_{124}$, $1x, 3, 5$ is a co-diamond, $V_{124} = \emptyset$, and analogously, $V_{245} = \emptyset$. Since for $x \in V_{234}$, $x3, 1, 5$ is a co-diamond, $V_{234} = \emptyset$. Now let $x \in V_{134}$. Since $x3a54$ is no gem, $ax \in E$ holds but now $1x45a$ is a gem - contradiction. Thus, $V_{134} = \emptyset$. Now let $x \in V_{135}$. Since $a4, 2, x$ is no co-diamond, $ax \in E$ holds but now $1x34a$ is a gem - contradiction. Thus, $V_{135} = \emptyset$. Now let $x \in V_{145}$. Since $3a5x4$ is no gem, $ax \in E$ holds but now $1x43a$ is a gem - contradiction. Thus, $V_{145} = \emptyset$. $\square$

Note that $V_{1345} \neq \emptyset$ since $a \in V_{1345}$, and $V_{1234} = V_{2345} = \emptyset$ since G is gem-free.

Claim 3.4 $V_{1245} = \emptyset$

Proof. If $x \in V_{1245}$ then, since $3a5x4$ is no gem, $ax \in E$ holds but now, $1x43a$ is a gem - contradiction. $\square$

Claim 3.5 $V_{1345} \textcircled{0} (V_{123} \cup V_{125} \cup V_{235} \cup V_{1235})$.

Proof. Let $y \in V_{1345}$. If $x \in V_{123}$ then, since $x345y$ is no gem, $xy \notin E$ holds. If $x \in V_{125}$ then, since $345xy$ is no gem, $xy \notin E$ holds. If $x \in V_{235}$ or $x \in V_{1235}$ then, since $2xy43$ is no gem, $xy \notin E$ holds. $\square$

Claim 3.6 $V_{1235} \textcircled{0} V_{345}$.

Proof. If $x \in V_{1235}$ and $y \in V_{345}$ then, since $123yx$ is no gem, $xy \notin E$ holds. $\square$

Claim 3.7 *V_{24}①$(V_{123} \cup V_{125} \cup V_{235} \cup V_{345})$.*

Proof. Let $x \in V_{24}$. If $y \in V_{123}$ then, since $1y, x, 5$ is no co-diamond, $xy \in E$ holds. If $y \in V_{125}$ then, since $1y, 3, x$ is no co-diamond, $xy \in E$ holds. If $y \in V_{235}$ then, since $3y, 1, x$ is no co-diamond, $xy \in E$ holds. If $y \in V_{345}$ then, since $5y, 1, x$ is no co-diamond, $xy \in E$ holds. □

Claim 3.8 *$V_{125} = \emptyset$ or $V_{235} = \emptyset$.*

Proof. Assume that $x \in V_{125}$ and $y \in V_{235}$. Then, if $xy \notin E$, $1x, 4, y$ is a co-diamond, and if $xy \in E$, $12y5x$ is a gem - contradiction. □

Claim 3.9 *V_{123}①$(V_{125} \cup V_{235})$.*

Proof. Let $x \in V_{123}$. If $y \in V_{125}$ then, since $y1x32$ is no gem, $xy \in E$ holds. If $y \in V_{235}$ then, since $1x, 4, y$ is no co-diamond, $xy \in E$ holds. □

Claim 3.10 *V_{345}⓪$(V_{123} \cup V_{125} \cup V_{235})$.*

Proof. Let $x \in V_{345}$. If $y \in V_{123}$ then, since $123xy$ is no gem, $xy \notin E$ holds. If $y \in V_{125}$ then, since $345yx$ is no gem, $xy \notin E$ holds. If $y \in V_{235}$ then, since $23x5y$ is no gem, $xy \notin E$ holds. □

Claim 3.11 *If $V_{1235} \neq \emptyset$ then $V_{125} = V_{235} = \emptyset$.*

Proof. Let $x \in V_{1235}$. If $y \in V_{235}$ then, since $1x3y2$ is no gem, $xy \in E$ holds but now, $12y5x$ is a gem - contradiction. If $y \in V_{125}$ then, since $y1x32$ is no gem, $xy \in E$ holds but now, $32y5x$ is a gem - contradiction. □

Claim 3.12 *$V_{1345} = \{a\}$.*

Proof. Suppose that $|V_{1345}| \geq 2$. Since G is prime, there are vertices $a_1, a_2 \in V_{1345}$ and $z \notin V_{1345}$ being adjacent to a_1 and nonadjacent to a_2. By Claims 3.1-3.5, $z \in V_{24}$ or $z \in V_{345}$. If $z \in V_{24}$, then, since $1, a_2, 4, z, a_1$ is no gem, a_1 and a_2 are nonadjacent, but now $za_1 5a_2 4$ is a gem - contradiction. If $z \in V_{345}$, then, since $2, a_2, a_1 z$ is no co-diamond, a_1 and a_2 are adjacent, but now $1a_2 5za_1$ is a gem - contradiction. □

Claim 3.13 *$|V_{1235}| \leq 1$.*

Proof. Suppose that $|V_{1235}| \geq 2$. Since G is prime, there are vertices $a_1, a_2 \in V_{1235}$ and $z \notin V_{1235}$ being adjacent to a_1 and nonadjacent to a_2. By Claim 3.5, $z \notin V_{1345}$, and by Claim 3.11, $V_{235} = V_{125} = \emptyset$. Thus, $z \in V_{123} \cup V_{345} \cup V_{24}$. If $z \in V_{123}$, then, since $z3a_2 5a_1$ is no gem, $a_1 a_2 \notin E$ holds but now, $za_1, a_2, 4$ is a co-diamond - contradiction. By Claim 3.6, $z \notin V_{345}$. If $z \in V_{24}$ then, since $za_1 1a_2 2$ is no gem, $a_1 a_2 \in E$ holds but now $z2a_2 5a_1$ is a gem - contradiction. □

Claim 3.14 *$|V_{123}| \leq 1$.*

Proof. Suppose $|V_{123}| \geq 2$. Since G is prime, there are vertices $a_1, a_2 \in V_{123}$ and $z \notin V_{123}$ being adjacent to a_1 and nonadjacent to a_2. By Claim 3.7, $z \notin V_{24}$. By Claim 3.9 and Claim 3.10, z is not a 3-vertex. By Claim 3.5, z is not the 4-vertex a. Thus, $z \in V_{1235}$. But then by Claim 3.5, $\{z, a_2, a, 4\}$ induces a co-diamond - contradiction. □

Claim 3.15 $|V_{125}| \leq 1$.

Proof. Suppose $|V_{125}| \geq 2$. Since G is prime, there are vertices $a_1, a_2 \in V_{125}$ and $z \notin V_{125}$ being adjacent to a_1 and nonadjacent to a_2.
By Claim 3.5, $z \neq a$. By Claim 3.7, $z \notin V_{24}$. By Claims 3.8, 3.9 and 3.10, z is not a 3-vertex. Thus $z \in V_{1235}$ but then by Claim 3.5, $z, a_2, a, 4$ is a co-diamond - contradiction. □

Claim 3.16 $|V_{235}| \leq 1$.

Proof. Note first that V_{235} is a clique since otherwise, by Claim 3.5, two nonadjacent vertices in V_{235} together with the edge $a4$ would induce a co-diamond. Now, if $|V_{235}| \geq 2$ then 1, 4 and any two vertices in V_{235} induce a co-diamond - contradiction. □

Claim 3.17 $|V_{345}| \leq 1$.

Proof. Note first that V_{345} is a clique since otherwise, two nonadjacent vertices in V_{345} together with the edge 12 would induce a co-diamond. Suppose that $|V_{345}| \geq 2$. Since G is prime, there are vertices $a_1, a_2 \in V_{345}$ and $z \notin V_{345}$ being adjacent to a_1 and nonadjacent to a_2. We are going to show that $z = a$ holds.
By Claim 3.6, $z \notin V_{1235}$, by Claim 3.7, $z \notin V_{24}$, and by Claim 3.10, $z \notin V_{123} \cup V_{125} \cup V_{235}$. Thus, the only possibility is $z \in V_{1345}$,i.e., $z = a$ due to Claim 3.12. Now, if $|V_{345}| \geq 2$ and a is distinguishing two vertices in V_{345} then a has a neighbor $x \in V_{345}$ but now $\{x, 4\}$ is a homogeneous set in G: V_{345} is a clique, by Claim 3.6, $V_{1235}⓪(V_{345} \cup \{4\})$, by Claim 3.7, $V_{24}①(V_{345} \cup \{4\})$ and by Claim 3.10, $(V_{123} \cup V_{125} \cup V_{235})⓪(V_{345} \cup \{4\})$. Thus, no vertex in G can distinguish x and 4 - contradiction. □

Claim 3.18 $|V_{24}| \leq 2$.

Proof. Observe that

at most one vertex in V_{24} is nonadjacent to a, (1)

otherwise by Claim 3.2 two vertices in V_{24} nonadjacent to a together with a and 3 would induce a co-diamond. Similarly,

if b is the vertex of V_{1235} then at most one vertex in V_{24} is nonadjacent to b. (2)

Recall that V_{24} is a stable set, i.e., V_{24} vertices cannot distinguish each other. Moreover, by Claim 3.7, a vertex $z \notin V_{24}$ distinguishing V_{24} vertices is a 4-vertex, i.e., $z \in$

$V_{1235} \cup V_{1345}$. We distinguish between two cases.
Case 1. $V_{1235} = \emptyset$
Suppose to the contrary that $|V_{24}| \geq 3$. By (1) there are two vertices $x, x' \in V_{24}$ such that x and x' both are adjacent to a. Now it is easy to see that $N(a) \cap V_{24}$ is a homogeneous set in G - contradiction.
Case 2. $V_{1235} \neq \emptyset$
Let b be the vertex of V_{1235}. In this case, by Claim 3.11, $V_{125} = V_{235} = \emptyset$. Thus, the only 3-vertices are in $V_{123} \cup V_{345}$, and by Claim 3.7, V_{24}①$(V_{123} \cup V_{345})$ holds.
We first show that

$$\text{every vertex of } V_{24} \text{ is nonadjacent to } a \text{ or nonadjacent to } b. \tag{3}$$

Proof. Suppose to the contrary that $x \in V_{24}$ is adjacent to both a and b. Then it is easy to see that $\{x, 3\}$ is a homogeneous set in G since vertices in $V_{123} \cup V_{345}$ are also adjacent to 3, and other vertices cannot distinguish them.

Now (1), (2) and (3) imply that $|V_{24}| \leq 2$, proving Claim 3.18. □

Finally, by Claims 3.8 and 3.11 – 3.18, G has at most 11 vertices. □

4 Structure and clique-width results

Figure 6 contains all combinations of three forbidden P_4 extensions (enumeration according to Figure 1 similarly as in Figure 2). Again for complementary pairs of classes such as (1,2,9) and (2,9,10), we take only the lexicographically smaller one, namely (1,2,9).

Theorem 15 ((1,2,9)) *If G is a prime (P_5,co-P_5,gem)-free graph then G is distance-hereditary or a C_5.*

Proof. Let G be a prime (P_5,co-P_5,gem)-free graph. Assume first that G contains a C_5 C with vertices $1, 2, 3, 4, 5$ and edges $\{i, i+1\}$, $i \in \{1, \ldots, 5\}$ (index arithmetic modulo 5). Since G is P_5-, house- and gem-free, C has no k-vertex for $k \in \{1, 4, 5\}$, 3-vertices for C have consecutive neighbors in C, and 2-vertices for C have non-consecutive neighbors in C. Moreover, the C_5 dominates G i.e. C has no 0-vertex. Now let $N_C(v_i)$ denote the set of vertices being adjacent to v_{i-1} and v_{i+1} and possibly to v_i (index arithmetic modulo 5). It is easy to see that there is a join between $N_C(v_i)$ and $N_C(v_{i+1})$, and there is a co-join between $N_C(v_i)$ and $N_C(v_{i+2})$. Thus, these vertex sets are modules, and therefore, $G = C$.

Now let G be a (C_5,P_5,co-P_5,gem)-free graph. Then G is (house,hole,domino,gem)-free which means that, by Theorem 2, G is distance-hereditary. □

The subsequent Theorems 16 and 17 are a simple consequence of Lemma 4.

Theorem 16 ((1,2,10),(1,4,10)) *If G is a prime (P_5, gem, co-gem)-free or (gem, co-P, co-gem)-free graph then G is (diamond,co-diamond)-free.*

1 2 3 −	1 2 4 −	1 2 5 −	1 2 6 −	1 2 7 −	1 2 8 −	1 2 9 +	1 2 10 +
1 3 4 −	1 3 5 −	1 3 6 −	1 3 7 −	1 3 8 −	1 3 9 +	1 3 10 +	1 4 5 −
1 4 6 −	1 4 7 −	1 4 8 −	1 4 9 +	1 4 10 +	1 5 6 −	1 5 7 −	1 5 8 +
1 5 9 +	1 6 7 −	1 6 8 −	1 6 9 +	1 6 10 +	1 7 8 +	1 7 9 +	1 7 10 +
1 8 9 +	2 3 4 −	2 3 5 −	2 3 6 −	2 3 7 −	2 3 8 −	2 3 9 +	2 4 5 −
2 4 6 −	2 4 7 −	2 4 8 −	2 4 9 −	2 5 6 −	2 5 7 −	2 5 8 +	2 6 7 −
2 6 8 −	2 6 9 −	2 7 8 +	2 7 9 +	3 4 5 −	3 4 6 −	3 4 7 −	3 4 8 −
3 5 6 −	3 5 7 −	3 6 7 −	3 6 8 −	3 7 8 +	4 5 6 −	4 5 7 −	4 6 7 −

Figure 6: All combinations of three forbidden P_4 extensions; $+$ $(-)$ denotes bounded (unbounded) clique-width

Theorem 17 ((1,3,9),(1,7,9),(1,8,9)) *If G is a prime (P_5, gem, co-chair)-free or (P_5, gem, bull)-free or (P_5, gem, chair)-free graph then G is (P_5, diamond)-free.*

Thus, the structure of the classes considered in Theorems 16 and 17 is described in Theorems 4, 14 respectively.

Theorem 18 ((1,3,10)) *If G is a prime (co-gem,chair,gem)-free graph then G or $\overline{G}$ is a matched co-bipartite graph or G has at most 11 vertices.*

Proof. By Lemma 8, every prime graph is either bipartite or co-diamond-free or is an odd hole or contains an induced subgraph from the family F of graphs described in Lemma 8. Since an odd apple as well as the graphs F_3, F_4, F_5 and F_6 contain a chair and F_1 and F_2 contain a co-gem and since G is prime (co-gem, chair, gem)-free, it follows that G is F-free and contains no odd hole of length at least 7. Thus G is either a C_5 or bipartite or co-diamond-free.
If G is bipartite, then by Lemma 10, G is co-matched bipartite. So let us assume that G is (co-diamond, gem)-free. If G contains no diamond then by Theorem 4, G or $\overline{G}$ is a matched co-bipartite graph or G has at most 9 vertices and we are done. If G contains a diamond then by Lemma 11, G has at most 11 vertices. Theorem 18 follows. □

Theorem 19 ((1,4,9)) *If G is a prime (P_5,P,gem)-free graph then G is matched co-bipartite or a gem-free split graph or a C_5.*

Proof. Let G be a prime (P_5,P,gem)-free graph. If G contains a co-domino then we are done by Lemma 5. Thus, by Lemma 3, assume that from now on, G is $2K_2$-free. Furthermore, if G contains no house then, by Theorem 15, G is a distance-hereditary graph or a C_5. Furthermore, if G is distance-hereditary and $2K_2$-free then it is $(2K_2, C_4, C_5)$-free and thus a gem-free split graph.
Now assume that G contains a house with vertices $v_1, \ldots, v_5$. We call the house vertices of degree 3 the *inner vertices*, the house vertex adjacent to both inner vertices the *top vertex* and the two other vertices the *bottom vertices* of the house. Assume that v_1 and v_5 are the bottom vertices, $v_2(v_4)$ is the inner vertex adjacent to $v_1(v_5)$ and v_3 is the top vertex.

Claim 4.1 *If H is a house in G and v is a vertex not in H but being adjacent to H then only the following neighborhood types are possible:*

(1) v is adjacent to the inner vertices;

(2) v is adjacent to the top and bottom vertices;

(3) v is nonadjacent to exactly one bottom vertex.

Proof. By a straightforward case analysis, one can see that all other cases lead to a $2K_2$, P or gem. □

Claim 4.2 *Every house in G dominates G.*

Proof. If a vertex y at distance two of a house H would exist then a common neighbor x of H and y leads to a $2K_2$ if x has adjacency type (1) and to a P if x has adjacency type (2) or (3) - contradiction. □

Note that $v_2 \in V_{134}$ and $v_4 \in V_{235}$ (V_{134} and V_{235} are defined as in the proof of Lemma 11).

Claim 4.3 *V_{134} and V_{235} are modules in G (and thus have size 1).*

Proof. Without loss of generality, assume that V_{134} is no module, i.e., there are $x, y \in V_{134}$ and $z \notin V_{134}$ with $xz \in E$, $yz \notin E$. Note that V_{134} is a clique since G is P-free.

Since $zxv_1v_4v_5$ is no P, $zv_1 \in E$ or $zv_4 \in E$ or $zv_5 \in E$. If only one of zv_1, zv_4, zv_5 is an edge then $zv_1v_4v_5y$ is a P - contradiction. Thus, at least two of these pairs are edges.

Case 1. $zv_1 \in E$, $zv_4 \in E$, $zv_5 \notin E$. Then, since $v_1zv_4v_5v_3$ is no P, $zv_3 \in E$ follows but now $z \in V_{134}$ - contradiction.

Case 2. $zv_1 \in E$, $zv_5 \in E$, $zv_4 \notin E$. Then $yxzv_5v_1$ is a gem - contradiction.

Case 3. $zv_4 \in E$, $zv_5 \in E$, $zv_1 \notin E$. Then $yxzv_5v_4$ is a gem - contradiction.

Case 4. $zv_1 \in E$, $zv_5 \in E$ and $zv_4 \in E$. Then $yxzv_5v_1$ is a gem - contradiction.

This proves Claim 4.3. □

Claims 4.1, 4.2 and 4.3 imply that every vertex outside $\{v_1, v_2, v_3, v_4, v_5\}$ is either adjacent to v_2 and v_4 and nonadjacent to v_1 and v_5 or vice versa. Let

$L := \{v : v \in V$ and v is adjacent to v_2 and v_4 and nonadjacent to v_1 and $v_5\}$ and

$R := \{v : v \in V$ and v is nonadjacent to v_2 and v_4 and adjacent to v_1 and $v_5\}$.

Note that $v_3 \in L$.

Claim 4.4 *L and R are cliques.*

Proof. Assume that there are $x, x' \in L$ with $xx' \notin E$. Then consider the co-connected component $C(x, x')$ in L containing x and x'. Since this cannot be a module, there must be a vertex u in R distinguishing two nonadjacent vertices in $C(x, x')$. Without loss of generality, say, $ux \in E$ and $ux' \notin E$ but now xv_2v_1ux' is a P - contradiction. In a completely analogous way, one can show that R is a clique. □

Claim 4.5 *The edges between L and R form a matching.*

Proof. Assume that there are $x, x' \in L$ having a common neighbor y in R. Then there must be a vertex $y' \in R$ distinguishing x and x'; w.l.o.g., say $x'y' \in E$ and $xy' \notin E$ but now $v_4xyy'x'$ is a gem - contradiction. In a completely analogous way, one can show that there are no $y, y' \in R$ having a common neighbor in L. □

Claim 4.5 shows that G is a matched co-bipartite graph if G contains a house. Theorem 19 follows. □

Now we will give a shorter proof of Theorem 7; recall that it claims the following:

If G is a prime (chair, co-P, gem)-free graph then G fulfills one of the following conditions:

(*i*) G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;

(*ii*) G is a thin spider;

(*iii*) G is a co-matched bipartite graph;

(*iv*) G has at most 9 vertices.

Proof of Theorem 7. Let G be a prime (chair, co-P, gem)-free graph. Then, by Lemma 4, G is diamond-free. Now we apply Lemma 8; the only graph in F not containing a chair, co-P or gem is the net (i.e., the thin spider with 6 vertices). Since G is prime (chair, co-P, gem)-free, it follows that G is bipartite or co-diamond-free or is an odd hole or G contains an induced net.

If G is bipartite then we are done by Lemma 6. If G is co-diamond-free then, by Theorem 4, G or $\overline{G}$ is a matched co-bipartite graph or G has at most 9 vertices. A matched co-bipartite graph with at least 8 vertices, however, contains a co-P, and thus, G is a co-matched bipartite graph or has at most 9 vertices.

It remains to consider the case when G contains a net H, say with vertices v_1, v_2, v_3, v_4, v_5, v_6 and edges v_1v_2, v_2v_3, v_3v_1, v_1v_4, v_2v_5 and v_3v_6. We are going to show that in this case, G is a thin spider.

Using the fact that G is (chair, co-P, diamond)-free it is a matter of routine to see that only $V_0 :=$ the set of vertices having no neighbor in H and V_{123} can be nonempty.

Since G is diamond-free,

$$V_{123} \text{ induces a clique} \tag{4}$$

Since G is (co-P,chair)-free,

$$\text{each } v \in V_{123} \text{ has at most one neighbor } w \in V_0 \tag{5}$$

Since G is co-P-free and G is connected, it follows from (5) that

$$V_0 \text{ is a stable set} \tag{6}$$

By (4) and (6), G is a (diamond,chair)-free split graph and thus, by Lemma 7 and the fact that thick spiders contain a diamond, G is a thin spider. □

Theorem 20 ((1,7,8)) *If G is a prime (bull,chair,gem)-free graph then G fulfills one of the following conditions:*

(i) *G or $\overline{G}$ is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*

(ii) *G or $\overline{G}$ is a co-matched bipartite graph;*

(iii) *G has at most 11 vertices.*

Proof. By Lemma 9, we may assume that G is co-diamond-free. If G is diamond-free then we are done by Theorem 4. If G contains a diamond then by Lemma 11, G has at most 11 vertices (a more detailed analysis shows that G has at most 9 vertices). □

Corollary 1 *The following classes (and their complement classes) have bounded clique-width and corresponding k-expressions can be determined in linear time: (1,2,9), (1,2,10), (1,3,9), (1,3,10), (1,4,9), (1,4,10), (1,5,8), (1,7,8), (1,7,9), (1,8,9), (2,3,9), (2,5,8), (2,7,8), (2,7,9), (3,4,5,8), (3,7,8). Thus, every LinEMSOL($\tau_{1,L}$) definable problem is solvable in linear time on these graph classes.*

By Theorem 5, the classes (1,5,9) and (1,6,9) have bounded clique-width and by Theorem 6, the classes (1,6,10) and (1,7,10) have bounded clique-width but their structure is more complicated than the previous examples, and so far, we do not know any linear time algorithm for determining k-expressions for these graphs.

5 Conclusion

As before, let $\mathcal{F}(P_4)$ denote the set of the 10 one-vertex extensions of the P_4 (see Figure 1). For $\mathcal{F}' \subseteq \mathcal{F}(P_4)$, there are 1024 classes of $\mathcal{F}'$-free graphs. Recall that Figure 2 shows all inclusion-minimal classes of unbounded clique-width and all inclusion-maximal classes of bounded clique-width. This classification is obtained from the previous results by a straightforward case analysis which shows the following (note that again we consider pairs of graph classes namely a class together with the class of its complement graphs):

For $|\mathcal{F}'| \in \{9, 8\}$, all classes of $\mathcal{F}'$-free graphs have bounded clique-width. For $|\mathcal{F}'| = 7$, (1,2,3,4,5,6,7) is the only inclusion-minimal class of unbounded clique-width (enumeration with respect to Figure 1) whereas all the other classes (as subclasses of some class of bounded clique-width) have bounded clique-width, and similarly for $|\mathcal{F}'| = 6$ and $|\mathcal{F}'| = 5$. For $|\mathcal{F}'| = 4$ there is exactly one inclusion-maximal class of bounded clique-width namely (3,4,5,8). For $|\mathcal{F}'| = 3$, the inclusion-maximal classes of bounded clique-width are (1,5,8), (1,7,8), (2,3,9), (2,5,8), (2,7,8), (2,7,9), (3,7,8). For $|\mathcal{F}'| = 2$, the only classes of bounded clique-width are (1,9) and (1,10), and for $|\mathcal{F}'| = 1$, all classes have unbounded clique-width.

Thus, we obtained a complete classification of graph classes defined by forbidden one-vertex extensions of the P_4 with respect to their clique-width. All these classes generalize cographs (which are the P_4-free graphs) having clique-width at most 2. In this way, we find new important graph classes of bounded clique-width such as the classes mentioned in Corollary 1, and using linear-time modular decomposition, we obtain linear-time algorithms for constructing corresponding k-expressions. By the results of Courcelle, Makowsky and Rotics [18], this provides linear-time algorithms on these classes for all problems which can be formulated in a certain Monadic Second Order Logic called LinEMSOL($\tau_{1,L}$) in [18]. This improves some recently published results of other authors:

- (*i*) Theorem 7 implies that every LinEMSOL($\tau_{1,L}$) definable problem can be solved in linear time on (chair,co-P,gem)-free graphs which drastically improves the results of [32, 37] for the MWS problem on this class.
- (*ii*) Theorem 19 implies that every LinEMSOL($\tau_{1,L}$) definable problem can be solved in linear time on (co-gem,co-P,house)-free graphs and on (P_5,P,gem)-free graphs which drastically improves the result of [26] for the MWS problem on this class.

The main open problems seem to be the following:

Question 1. Can one find a simple condition like planarity for minors and treewidth, that would give bounded clique-width if induced subgraph exclusion is considered?

This seems to be a very ambitious task but our results could eventually be a small step towards such a result. Moreover, our results could eventually help for solving the other big challenge on clique-width:

Question 2. What is the complexity of determining whether the input graph has clique-width at most k (for fixed k)?

This problem is solvable in polynomial time for $k = 3$ (see [13]) and open for $k \geq 4$.

Further open problems.

> Is there a linear time algorithm for determining a k-expression with constant k for the classes (1,9) and (1,10) (and the corresponding subclasses (1,5,9), (1,6,9), (1,6,10), (1,7,10)?

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References

[1] H.-J. Bandelt, H.M. Mulder, Distance-hereditary graphs, *J. Combin. Theory (B)* 41 (1986) 182-208

[2] A. Brandstädt, (P_5,Diamond)-Free Graphs Revisited: Structure and Linear Time Optimization, *Manuscript* 2000; to appear in *Discrete Applied Math.*

[3] A. Brandstädt, F.F. Dragan, H.-O. Le, R. Mosca, New graph classes of bounded clique-width, extended abstract in: *Conference Proceedings of WG'2002*, LNCS 2573 (2003) 57-67

[4] A. Brandstädt, C.T. Hoàng, V.B. Le, Stability Number of Bull- and Chair-Free Graphs Revisited, *Manuscript* 2001; to appear in *Discrete Applied Math.*

[5] A. Brandstädt, D. Kratsch, On the structure of (P_5,gem)-free graphs, *Manuscript* 2001; tentatively accepted for *Discrete Applied Math.*

[6] A. Brandstädt, H.-O. Le, R. Mosca, Chordal co-gem-free and (P_5,gem)-free graphs have bounded clique-width, *Manuscript* 2002; tentatively accepted for *Discrete Applied Math.*

[7] A. Brandstädt, H.-O. Le, R. Mosca, Gem- and co-gem-free graphs have bounded clique-width, *Manuscript* 2002;

[8] A. Brandstädt, V.B. Le, J. Spinrad, Graph Classes: A Survey, *SIAM Monographs on Discrete Math. Appl., Vol.* 3, SIAM, Philadelphia (1999)

[9] A. Brandstädt, H.-O. Le, J.-M. Vanherpe, Structure and Stability Number of (Chair, Co-P, Gem)-Free Graphs, *Information Processing Letters* 86 (2003) 161-167

[10] A. Brandstädt, S. Mahfud, Maximum Weight Stable Set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time, *Information Processing Letters* 84 (2002) 251-259

[11] A. Brandstädt, R. Mosca, On the Structure and Stability Number of P_5- and Co-Chair-Free Graphs, *Manuscript* 2001; accepted for *Discrete Applied Math.*

[12] A. Brandstädt, R. Mosca, On Variations of P_4-Sparse Graphs, *Manuscript* 2001; to appear in *Discrete Applied Math.*

[13] D.G. Corneil, M. Habib, J. Lanlignel, B. Reed, U. Rotics, Polynomial time recognition of clique-width ≤ 3 graphs, Extended abstract in: *Conference Proceedings of LATIN 2000*, LNCS 1726 (2000) 126-134

[14] D.G. Corneil, H. Lerchs, L. Stewart-Burlingham, Complement reducible graphs, *Discrete Applied Math.* 3 (1981) 163-174

[15] D.G. Corneil, Y. Perl, L.K. Stewart, Cographs: recognition, applications, and algorithms, *Congressus Numer.* 43 (1984) 249-258

[16] D.G. Corneil, Y. Perl, L.K. Stewart, A linear recognition algorithm for cographs, *SIAM J. Computing* 14 (1985) 926-934

[17] B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraph grammars, *J. Comput. Syst. Sciences*, 46 (1993) 218-270

[18] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique width, extended abstract in: *Conf. Proc. WG'98*, LNCS 1517 (1998) 1-16; *Theory of Computing Systems* 33 (2000) 125-150

[19] B. Courcelle, S. Olariu, Upper bounds to the clique-width of graphs, *Discrete Applied Math.* 101 (2000) 77-114

[20] C. De Simone, On the vertex packing problem, *Graphs and Combinatorics* 9 (1993) 19-30

[21] S. Földes, P.L. Hammer, Split graphs, *Congressus Numer.* 19 (1977), 311-315

[22] J.-L. Fouquet, A decomposition for a class of (P_5, $\overline{P_5}$)-free graphs, *Discrete Math.* 121 (1993) 75-83

[23] J.-L. Fouquet, V. Giakoumakis, On semi-P_4-sparse graphs, *Discrete Math.* 165-166 (1997) 267-290

[24] J.-L. Fouquet, V. Giakoumakis, H. Thuillier, F. Maire, On graphs without P_5 and $\overline{P_5}$, *Discrete Math.* 146 (1995) 33-44

[25] M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, *Internat. Journal of Foundations of Computer Science* 11 (2000) 423-443

[26] A. Hertz, On a graph transformation which preserves the stability number, *Yugoslav Journal of Oper. Res.*, to appear,

[27] C.T. Hoàng, *A Class of Perfect Graphs*, *Ms. Sc. Thesis*, School of Computer Science, McGill University, Montreal (1983)

[28] C.T. Hoàng, *Perfect Graphs*, *Ph. D. Thesis*, School of Computer Science, McGill University, Montreal (1985)

[29] C.T. Hoàng, B. Reed, Some classes of perfectly orderable graphs, *J. Graph Theory* 13 (1989) 445-463

[30] E. Howorka, A characterization of distance-hereditary graphs, *Quart. J. Math. Oxford* Ser. 2, 28 (1977), 417-420

[31] B. Jamison, S. Olariu, A unique tree representation for P_4-sparse graphs, *Discrete Applied Math.* 35 (1992), 115-129

[32] V.V. Lozin, Conic reduction of graphs for the stable set problem, *Discrete Math.* 222 (2000) 199-211

[33] N.V.R. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, *Annals of Discrete Mathematics* 56 (1995)

[34] J.A. Makowsky, U. Rotics, On the clique-width of graphs with few P_4's,
Internat. J. of Foundations of Computer Science 3 (1999) 329-348

[35] R.H. Möhring, F.J. Radermacher, Substitution decomposition for discrete structures and connections with combinatorial optimization, *Annals of Discrete Math.* 19 (1984) 257-356

[36] M. Yannakakis, The complexity of the partial order dimension problem, *SIAM J. Algebraic and Discrete Methods* 3 (1982) 351-358

[37] I.E. Zverovich, I.I. Zverovich, Extended (P_5,$\overline{P_5}$)-free graphs, *Rutcor Research Report* RRR 22-2001 (2001)
`http://rutcor.rutgers.edu/~rrr`