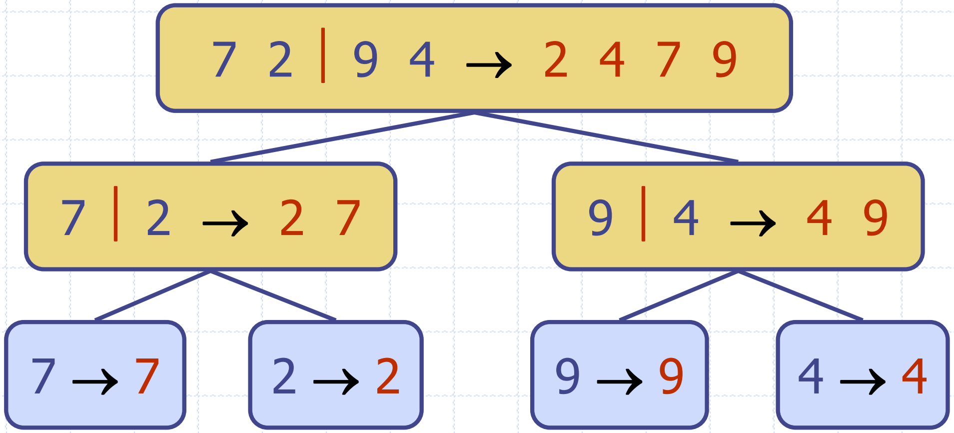
Divide-and-Conquer

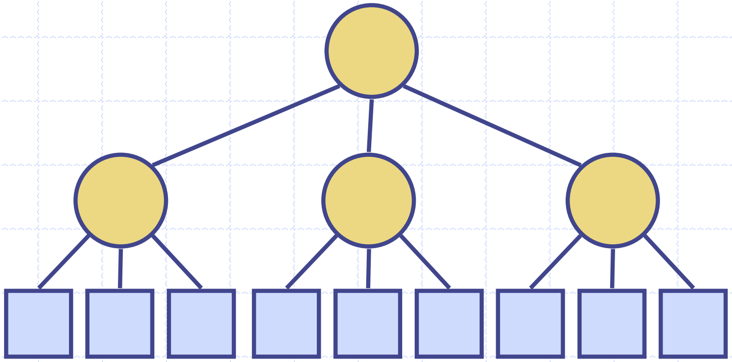


Outline and Reading

- Divide-and-conquer paradigm (§5.2)
- Review Merge-sort (§4.1.1)
- Recurrence Equations (§5.2.1)
  - Iterative substitution
  - Recursion trees
  - Guess-and-test
  - The master method
- Integer Multiplication (§5.2.2)

Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S in two or more disjoint subsets S_1, S_2, …
  - Recur: solve the subproblems recursively
  - Conquer: combine the solutions for S_1, S_2, …, into a solution for S
- The base case for the recursion are subproblems of constant size
- Analysis can be done using **recurrence equations**



Merge-Sort Review

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S, C)
    Input sequence S with n elements, comparator C
    Output sequence S sorted according to C
    if $S.size() > 1$
        $(S_1, S_2) \leftarrow partition(S, n/2)$
        $mergeSort(S_1, C)$
        $mergeSort(S_2, C)$
        $S \leftarrow merge(S_1, S_2)$

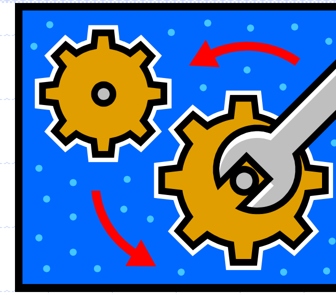
Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b.
- Likewise, the basis case ($n < 2$) will take at b most steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- We can therefore analyze the running time of merge-sort by finding a **closed form solution** to the above equation.
  - That is, a solution that has $T(n)$ only on the left-hand side.

Iterative Substitution



- In the iterative substitution, or "plug-and-chug," technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern:

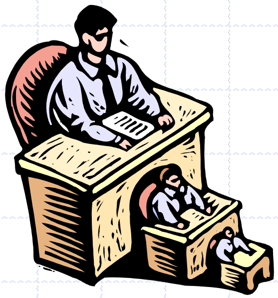
$$
\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(2T(n/2^2)) + b(n/2)) + bn \\
&= 2^2 T(n/2^2) + 2bn \\
&= 2^3 T(n/2^3) + 3bn \\
&= 2^4 T(n/2^4) + 4bn \\
&= \ldots \\
&= 2^i T(n/2^i) + ibn
\end{aligned}
$$

- Note that base, T(n)=b, case occurs when 2^i=n. That is, i = log n.
- So,

$$T(n) = bn + bn\log n$$

- Thus, T(n) is O(n log n).

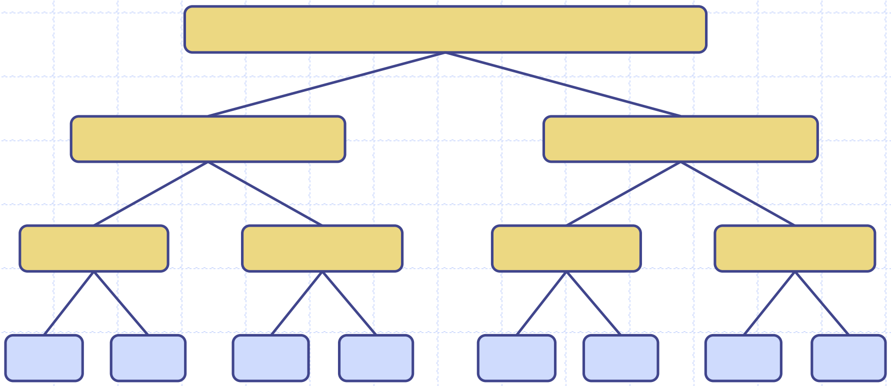
The Recursion Tree



- Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

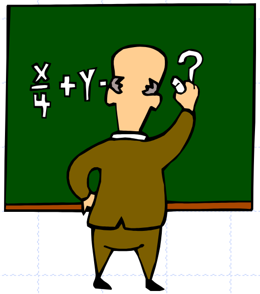
depth	T's	size	time
0	1	n	bn
1	2	$n/2$	bn
i	2^i	$n/2^i$	bn
…	…	…	…



Total time = $bn + bn \log n$

(last level plus all previous levels)

Guess-and-Test Method



- In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

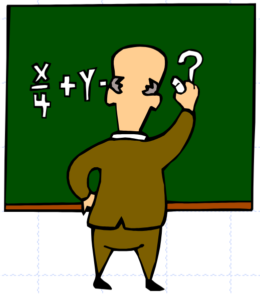
$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- Guess: T(n) < cn log n.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &= 2(c(n/2)\log(n/2)) + bn\log n \\ &= cn(\log n - \log 2) + bn\log n \\ &= cn\log n - cn + bn\log n \end{aligned}$$

- Wrong: we cannot make this last line be less than cn log n

Guess-and-Test Method, Part 2



- Recall the recurrence equation:

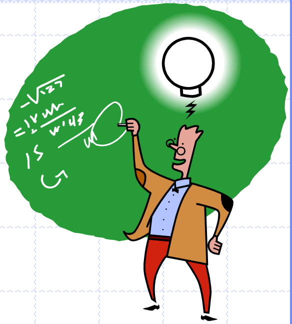
$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- Guess #2: T(n) < cn log^2 n.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &= 2(c(n/2)\log^2(n/2)) + bn\log n \\ &= cn(\log n - \log 2)^2 + bn\log n \\ &= cn\log^2 n - 2cn\log n + cn + bn\log n \\ &\leq cn\log^2 n \end{aligned}$$

  - if c > b.
- So, T(n) is O(n log^2 n).
- In general, to use this method, you need to have a good guess and you need to be good at induction proofs.

Master Method



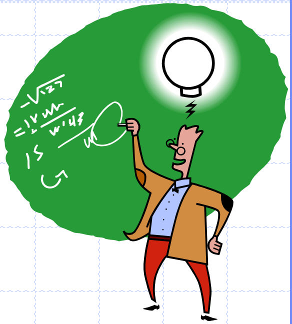
- Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

- The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

Master Method, Example 1



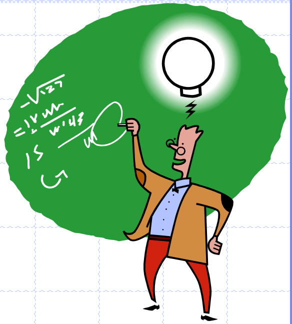
- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
- The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

- Example: $$T(n) = 4T(n/2) + n$$

Solution: $\log_b a=2$, so case 1 says T(n) is $O(n^2)$.

Master Method, Example 2



- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$
- The Master Theorem:

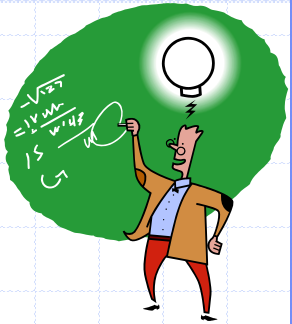
$$\begin{aligned}
&1.\ \text{if } f(n) \text{ is } O(n^{\log_b a-\varepsilon}), \text{ then } T(n) \text{ is } \Theta(n^{\log_b a}) \\
&2.\ \text{if } f(n) \text{ is } \Theta(n^{\log_b a}\log^k n), \text{ then } T(n) \text{ is } \Theta(n^{\log_b a}\log^{k+1} n) \\
&3.\ \text{if } f(n) \text{ is } \Omega(n^{\log_b a+\varepsilon}), \text{ then } T(n) \text{ is } \Theta(f(n)), \\
&\quad \text{provided } af(n/b) \le \delta f(n) \text{ for some } \delta < 1.
\end{aligned}$$

- Example:

$$T(n) = 2T(n/2) + n\log n$$

Solution: $\log_b a=1$, so case 2 says T(n) is O(n $\log^2$ n).

Master Method, Example 3



- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

- The Master Theorem:

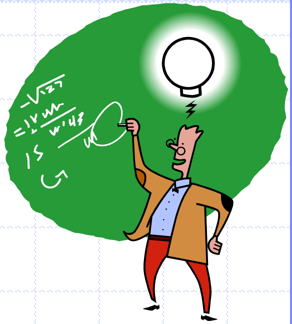
  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
     provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

- Example:

$$T(n) = T(n/3) + n\log n$$

Solution: $\log_b a = 0$, so case 3 says T(n) is O(n log n).

Master Method, Example 4

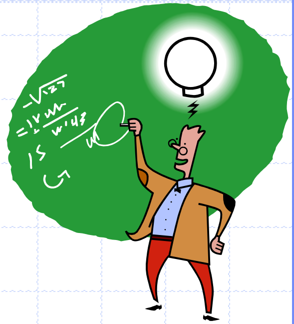


- The form: $$T(n)=\begin{cases} c & \text{if } n<d \\ aT(n/b)+f(n) & \text{if } n\ge d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b)\le \delta f(n)$ for some $\delta<1$.
- Example:

$$T(n)=8T(n/2)+n^2$$

Solution: $\log_b a=3$, so case 1 says $T(n)$ is $O(n^3)$.

Master Method, Example 5

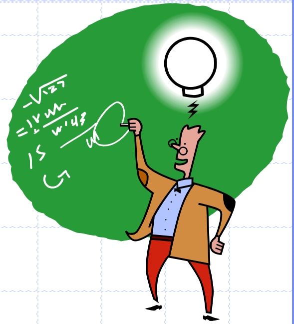


- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a = 2$, so case 3 says T(n) is O(n^3).

Master Method, Example 6

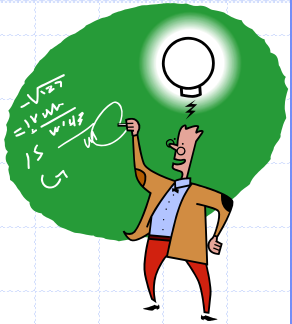


- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = T(n/2) + 1$$ (binary search)

Solution: $\log_b a = 0$, so case 2 says T(n) is O(log n).

Master Method, Example 7



- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$
- The Master Theorem:

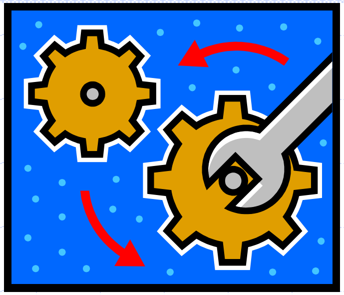
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
     provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

- Example:

$$T(n) = 2T(n/2) + \log n$$ (heap construction)

Solution: $\log_b a=1$, so case 1 says T(n) is O(n).

Iterative "Proof" of the Master Theorem

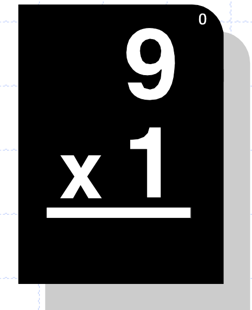


- Using iterative substitution, let us see if we can find a pattern:

$$
\begin{aligned}
T(n) &= aT(n/b) + f(n) \\
&= a(aT(n/b^2)) + f(n/b)) + bn \\
&= a^2T(n/b^2) + af(n/b) + f(n) \\
&= a^3T(n/b^3) + a^2 f(n/b^2) + af(n/b) + f(n) \\
&= \ldots \\
&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\
&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)
\end{aligned}
$$

- We then distinguish the three cases as
  - The first term is dominant
  - Each part of the summation is equally dominant
  - The summation is a geometric series

Integer Multiplication



- Algorithm: Multiply two n-bit integers I and J.
  - Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

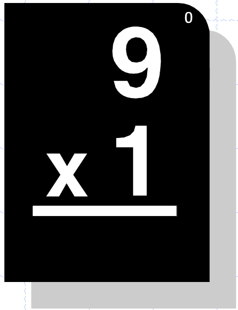
$$J = J_h 2^{n/2} + J_l$$

  - We can then define I*J by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

  - So, $T(n) = 4T(n/2) + n$, which implies $T(n)$ is $O(n^2)$.
  - But that is no better than the algorithm we learned in grade school.

An Improved Integer Multiplication Algorithm



- Algorithm: Multiply two n-bit integers I and J.
  - Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

  - Observe that there is a different way to multiply parts:

$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

  - So, $T(n) = 3T(n/2) + n$, which implies $T(n)$ is $O(n^{\log_2 3})$, by the Master Theorem.
  - Thus, $T(n)$ is $O(n^{1.585})$.