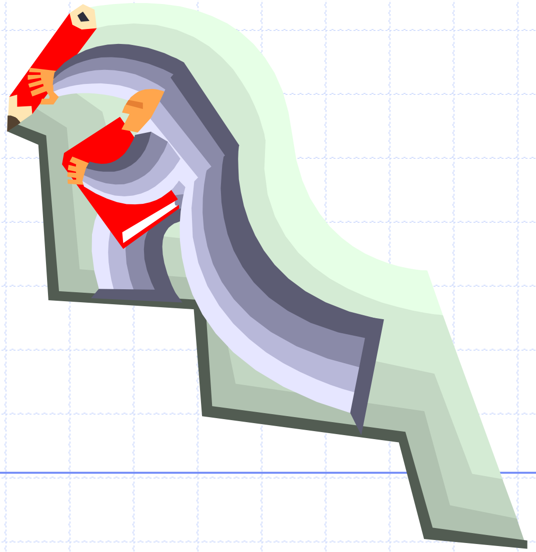
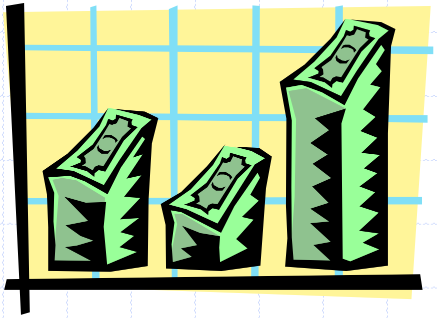
The Greedy Method



Outline and Reading

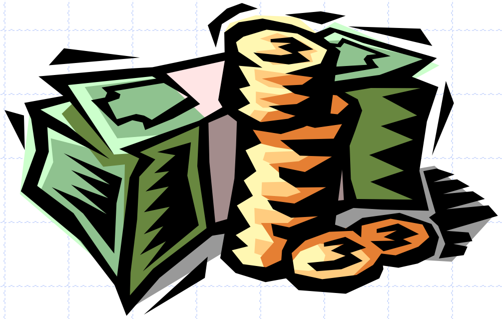
- The Greedy Method Technique (§5.1)
- Fractional Knapsack Problem (§5.1.1)
- Task Scheduling (§5.1.2)
- Minimum Spanning Trees (§7.3) [future lecture]

The Greedy Method Technique



- **The greedy method** is a general algorithm design paradigm, built on the following elements:
  - **configurations**: different choices, collections, or values to find
  - **objective function**: a score assigned to configurations, which we want to either maximize or minimize
- It works best when applied to problems with the **greedy-choice** property:
  - a globally-optimal solution can always be found by a series of local improvements from a starting configuration.

Making Change



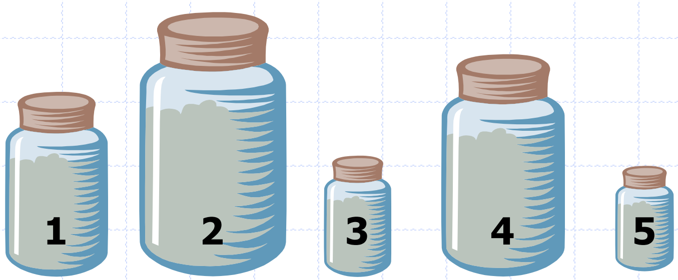
- **Problem:** A dollar amount to reach and a collection of coin amounts to use to get there.
- **Configuration:** A dollar amount yet to return to a customer plus the coins already returned
- **Objective function:** Minimize number of coins returned.
- **Greedy solution:** Always return the largest coin you can
- **Example 1:** Coins are valued $.32, $.08, $.01
  - Has the greedy-choice property, since no amount over $.32 can be made with a minimum number of coins by omitting a $.32 coin (similarly for amounts over $.08, but under $.32).
- **Example 2:** Coins are valued $.30, $.20, $.05, $.01
  - Does not have greedy-choice property, since $.40 is best made with two $.20's, but the greedy solution will pick three coins (which ones?)

The Fractional Knapsack Problem

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are allowed to take fractional amounts, then this is the **fractional knapsack problem**.
  - In this case, we let x_i denote the amount we take of item i
  - Objective: maximize $$\sum_{i \in S} b_i (x_i / w_i)$$
  - Constraint: $$\sum_{i \in S} x_i \leq W$$

Example

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.



Items:	1	2	3	4	5
Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Benefit:	$12	$32	$40	$30	$50
Value: ($ per ml)	3	4	20	5	50

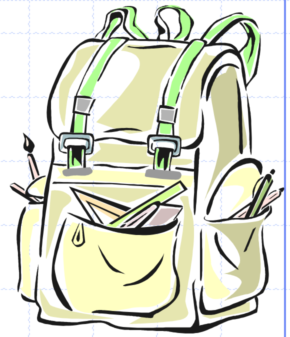
“knapsack”

10 ml

Solution:

- 1 ml of 5
- 2 ml of 3
- 6 ml of 4
- 1 ml of 2

The Fractional Knapsack Algorithm

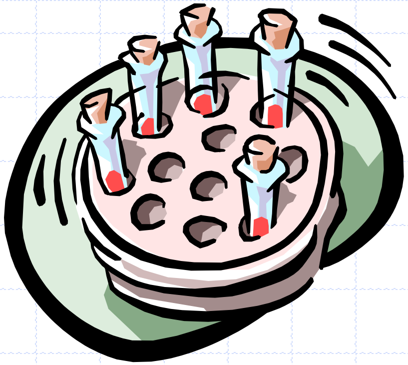


- Greedy choice: Keep taking item with highest **value** (benefit to weight ratio)
  - Since $\sum_{i \in S} b_i (x_i / w_i) = \sum_{i \in S} (b_i / w_i) x_i$
  - Run time: O(n log n). Why?
- Correctness: Suppose there is a better solution
  - there is an item i with higher value than a chosen item j (i.e., $v_i < v_j$) but $x_i < w_i$ and $x_j > 0$ If we substitute some i with j, we get a better solution
  - How much of i: $\min\{w_i - x_i, x_j\}$
  - Thus, there is no better solution than the greedy one

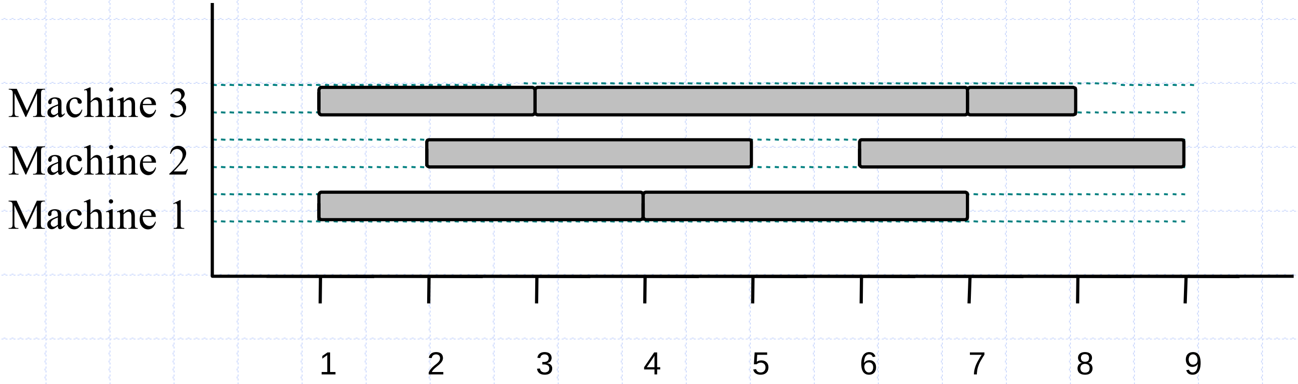
Algorithm *fractionalKnapsack*(*S*, *W*)
  Input: set S of items w/ benefit b_i and weight w_i; max. weight W
  Output: amount x_i of each item i to maximize benefit with weight at most W

for *each item i in S*
  $x_i \leftarrow 0$
  $v_i \leftarrow b_i / w_i$ {value}
$w \leftarrow 0$ {total weight}
while $w < W$
  remove item i with highest v_i
  $x_i \leftarrow \min\{w_i, W - w\}$
  $w \leftarrow w + \min\{w_i, W - w\}$

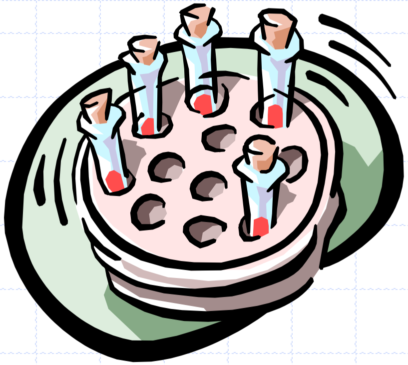
Task Scheduling



- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
- Goal: Perform all the tasks using a minimum number of "machines."



Task Scheduling Algorithm



- Greedy choice: consider tasks by their start time and use as few machines as possible with this order.
  - Run time: O(n log n). Why?
- Correctness: Suppose there is a better schedule.
  - We can use k-1 machines
  - The algorithm uses k
  - Let i be first task scheduled on machine k
  - Machine i must conflict with k-1 other tasks
  - But that means there is no non-conflicting schedule using k-1 machines

Algorithm *taskSchedule*(T)

Input: set T of tasks w/ start time s_i and finish time f_i

Output: non-conflicting schedule with minimum number of machines

$m \leftarrow 0$ {no. of machines}

while *T is not empty*

    remove task i w/ smallest s_i

    if *there's a machine j for i* **then**

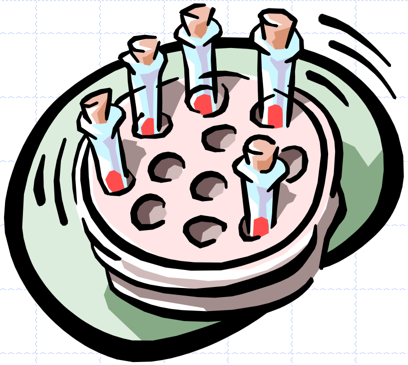
        schedule i on machine j

    else

        $m \leftarrow m + 1$

        schedule i on machine m

Example



- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
  - [1,4], [1,3], [2,5], [3,7], [4,7], [6,9], [7,8] (ordered by start)
- Goal: Perform all tasks on min. number of machines