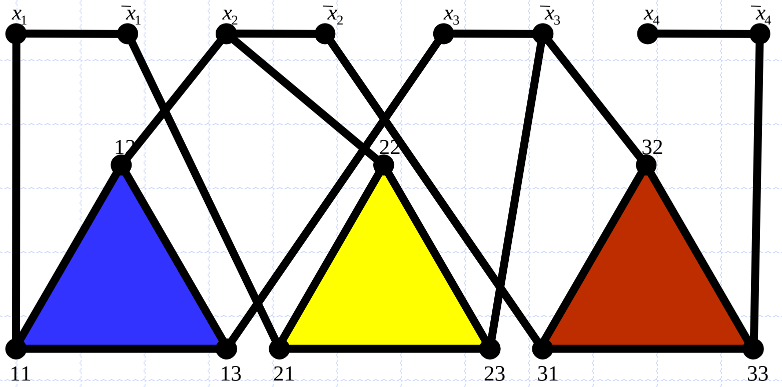
NP-Completeness

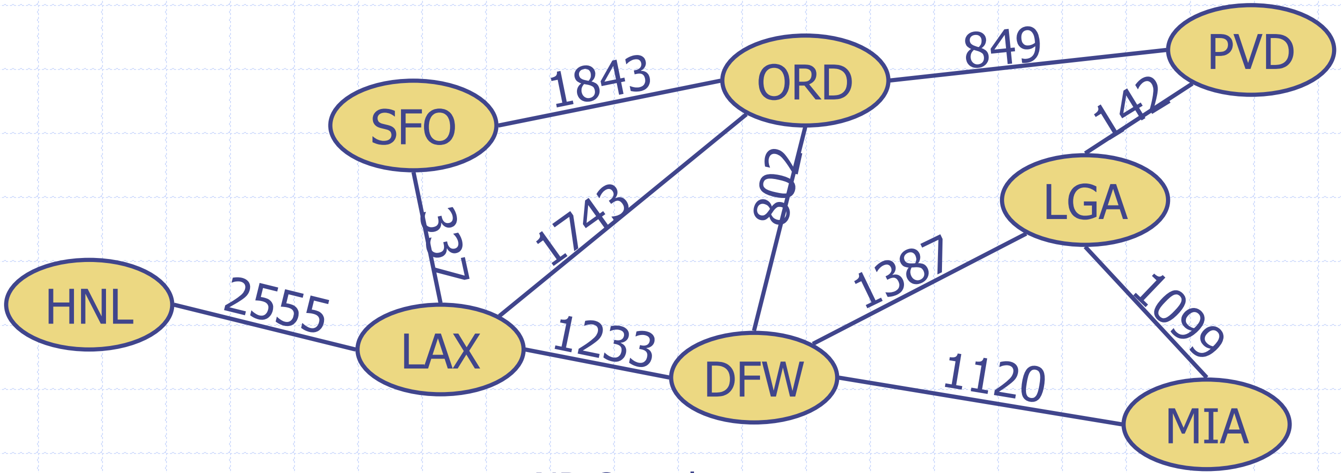


Outline and Reading

- P and NP (§13.1)
  - Definition of P
  - Definition of NP
  - Alternate definition of NP
- NP-completeness (§13.2)
  - Definition of NP-hard and NP-complete
  - The Cook-Levin Theorem

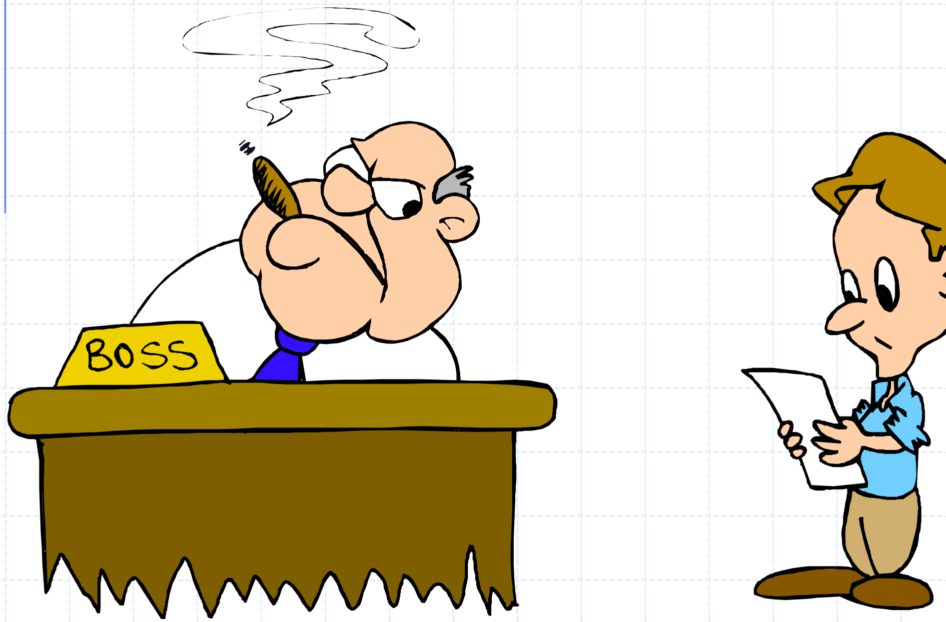
Running Time Revisited

- Input size, ***n***
  - To be exact, let ***n*** denote the number of **bits** in a nonunary encoding of the input
- All the polynomial-time algorithms studied so far in this course run in polynomial time using this definition of input size.
  - Exception: any pseudo-polynomial time algorithm



Dealing with Hard Problems

- What to do when we find a problem that looks hard…

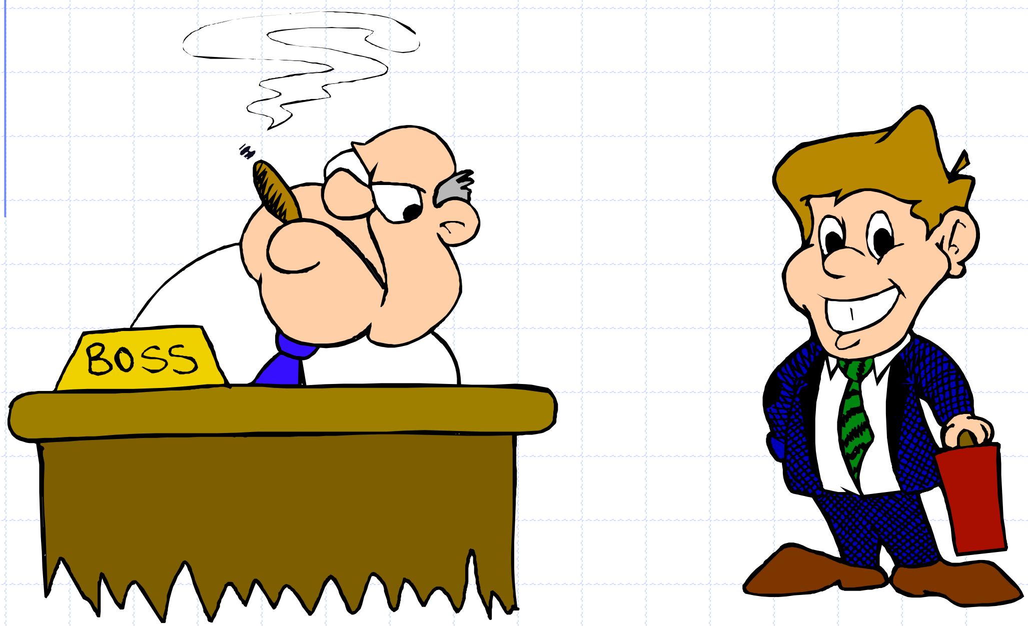


I couldn't find a polynomial-time algorithm;
I guess I'm too dumb.

 (cartoon inspired by [Garey-Johnson, 79])

Dealing with Hard Problems

- Sometimes we can prove a strong lower bound… (but not usually)

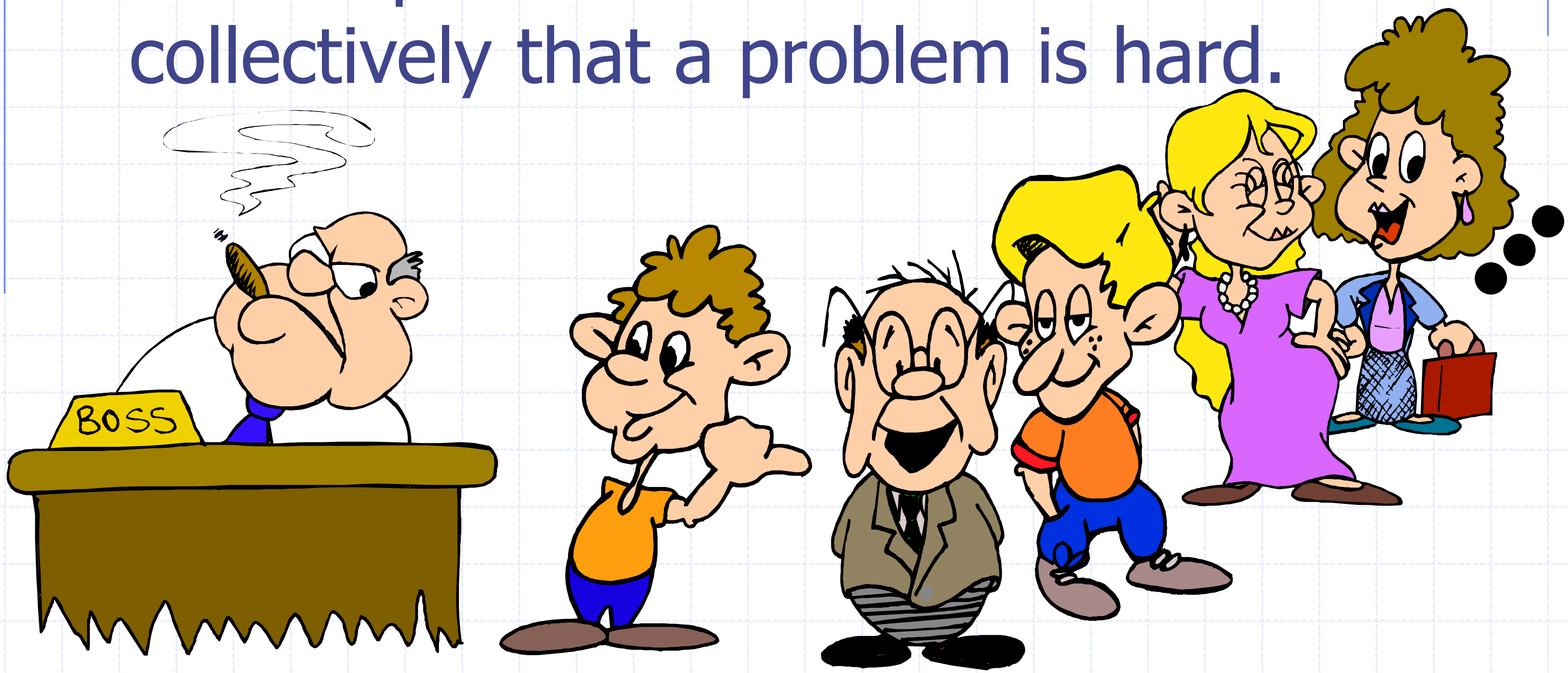


I couldn't find a polynomial-time algorithm, because no such algorithm exists!

 (cartoon inspired by [Garey-Johnson, 79])

Dealing with Hard Problems

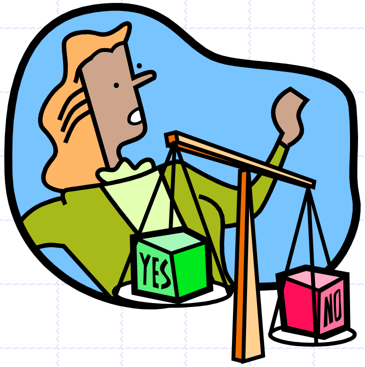
- NP-completeness let's us show collectively that a problem is hard.



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people.

 (cartoon inspired by [Garey-Johnson, 79])

Polynomial-Time Decision Problems



- To simplify the notion of "hardness," we will focus on the following:
  - Polynomial-time as the cut-off for efficiency
  - Decision problems: output is 1 or 0 ("yes" or "no")
    - Examples:
    - Does a given graph G have an Euler tour?
    - Does a text T contain a pattern P?
    - Does an instance of 0/1 Knapsack have a solution with benefit at least K?
    - Does a graph G have an MST with weight at most K?

Problems and Languages

- A **language** L is a set of strings defined over some alphabet Σ
- Every decision algorithm A defines a language L
  - L is the set consisting of every string x such that A outputs "yes" on input x.
  - We say "A **accepts** x" in this case
    - Example:
    - If A determines whether or not a given graph G has an Euler tour, then the language L for A is all graphs with Euler tours.

The Complexity Class P

- A **complexity class** is a collection of languages
- P is the complexity class consisting of all languages that are accepted by **polynomial-time** algorithms
- For each language L in P there is a polynomial-time decision algorithm A for L.
  - If n=|x|, for x in L, then A runs in p(n) time on input x.
  - The function p(n) is some polynomial

The Complexity Class NP

- We say that an algorithm is non-deterministic if it uses the following operation:
  - Choose(b): chooses a bit b
  - Can be used to choose an entire string y (with |y| choices)
- We say that a non-deterministic algorithm A **accepts** a string x if there exists some sequence of choose operations that causes A to output "yes" on input x.
- NP is the complexity class consisting of all languages accepted by **polynomial-time non-deterministic** algorithms.

NP example

- Problem: Decide if a graph has an MST of weight K

- Algorithm:
  1. Non-deterministically choose a set T of n-1 edges
  2. Test that T forms a spanning tree
  3. Test that T has weight at most K

- Analysis: Testing takes O(n+m) time, so this algorithm runs in polynomial time.

The Complexity Class NP Alternate Definition

- We say that an algorithm B **verfies** the acceptance of a language L if and only if, for any x in L, there exists a certificate y such that B outputs "yes" on input (x,y).
- NP is the complexity class consisting of all languages verified by **polynomial-time** algorithms.

- We know: P is a subset of NP.
- Major open question: P=NP?
- Most researchers believe that P and NP are different.

NP example (2)

- Problem: Decide if a graph has an MST of weight K

- Verification Algorithm:
  1. Use as a certificate, y, a set T of n-1 edges
  2. Test that T forms a spanning tree
  3. Test that T has weight at most K

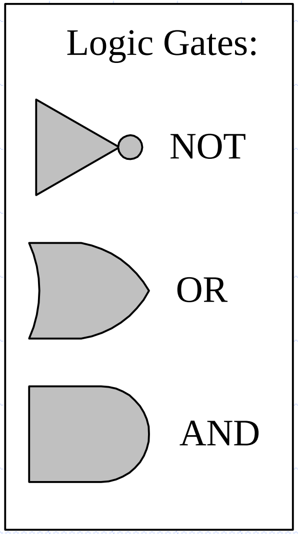
- Analysis: Verification takes O(n+m) time, so this algorithm runs in polynomial time.

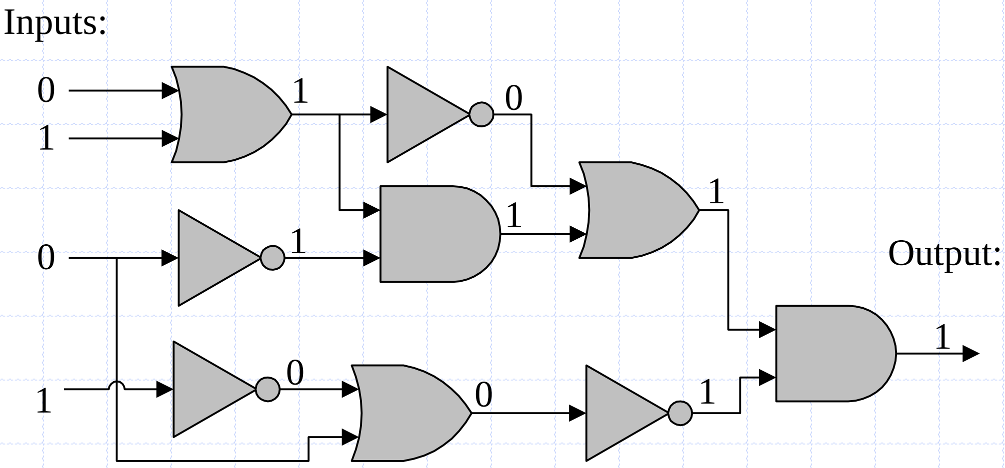
Equivalence of the Two Definitions

- Suppose A is a non-deterministic algorithm
  - Let y be a certificate consisting of all the outcomes of the choose steps that A uses
  - We can create a verification algorithm that uses y instead of A's choose steps
  - If A accepts on x, then there is a certificate y that allows us to verify this (namely, the choose steps A made)
  - If A runs in polynomial-time, so does this verification algorithm
- Suppose B is a verification algorithm
  - Non-deterministically choose a certificate y
  - Run B on y
  - If B runs in polynomial-time, so does this non-deterministic algorithm

An Interesting Problem

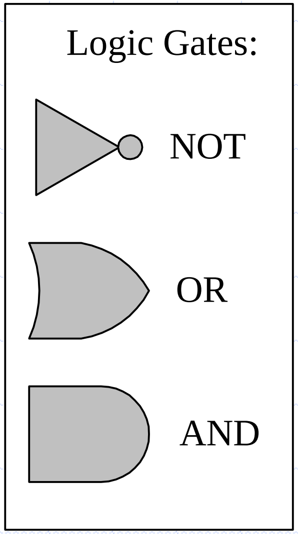
- A Boolean circuit is a circuit of AND, OR, and NOT gates; the CIRCUIT-SAT problem is to determine if there is an assignment of 0's and 1's to a circuit's inputs so that the circuit outputs 1.

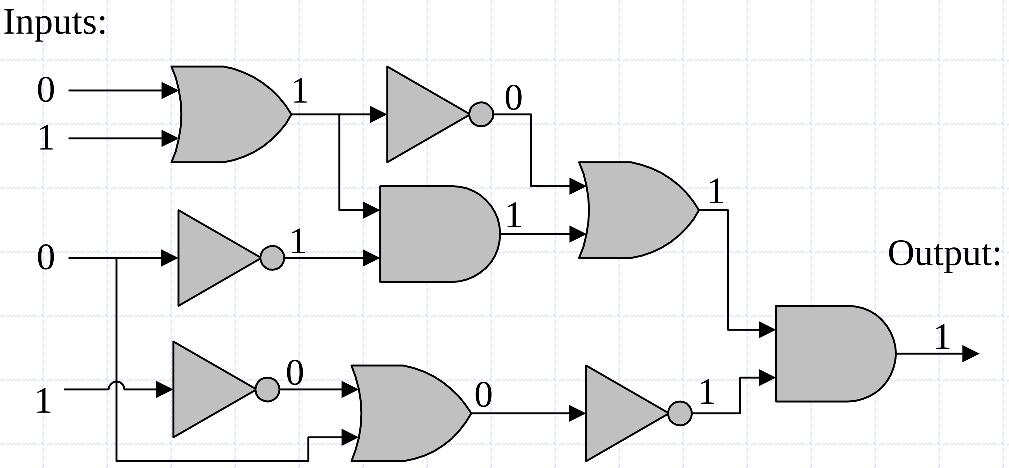




CIRCUIT-SAT is in NP

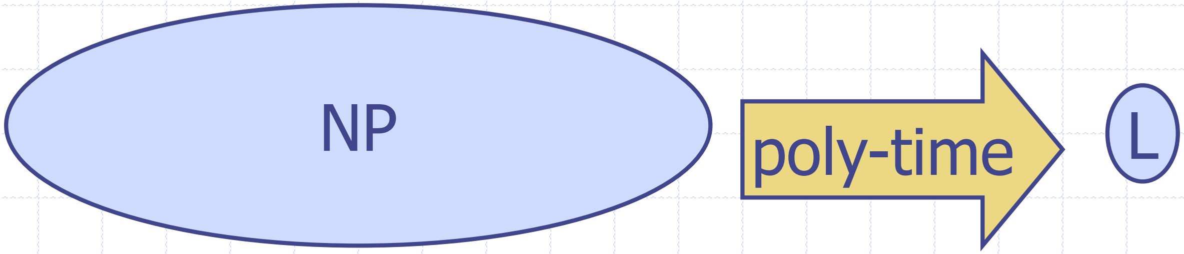
- Non-deterministically choose a set of inputs and the outcome of every gate, then test each gate's I/O.





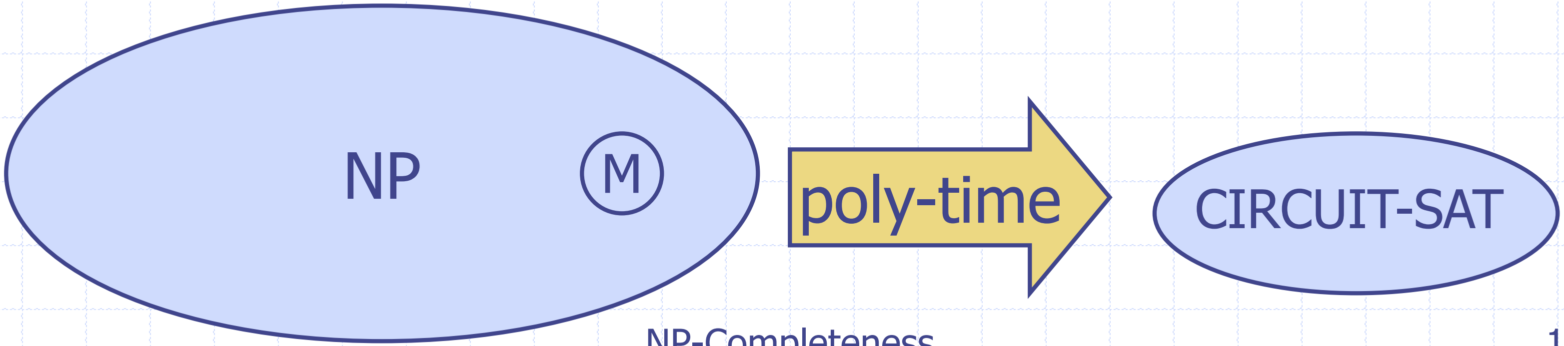
NP-Completeness

- A problem (language) L is **NP-hard** if every problem in NP can be reduced to L in polynomial time.
- That is, for each language M in NP, we can take an input x for M, **transform** it in polynomial time to an input x′ for L such that x is in M if and only if x′ is in L.
- L is **NP-complete** if it's in NP and is NP-hard.



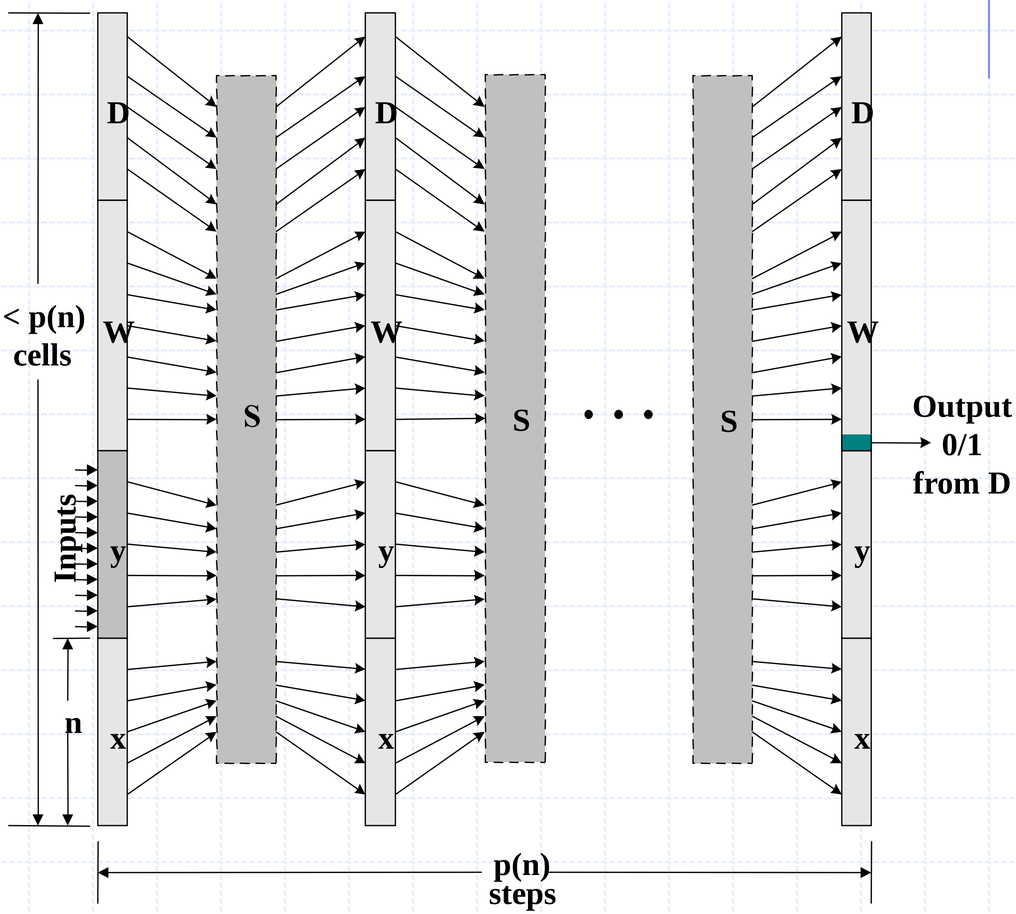
Cook-Levin Theorem

- CIRCUIT-SAT is NP-complete.
  - We already showed it is in NP.
- To prove it is NP-hard, we have to show that every language in NP can be reduced to it.
  - Let M be in NP, and let x be an input for M.
  - Let y be a certificate that allows us to verify membership in M in polynomial time, p(n), by some algorithm D.
  - Let S be a circuit of size at most $O(p(n)^2)$ that simulates a computer (details omitted…)

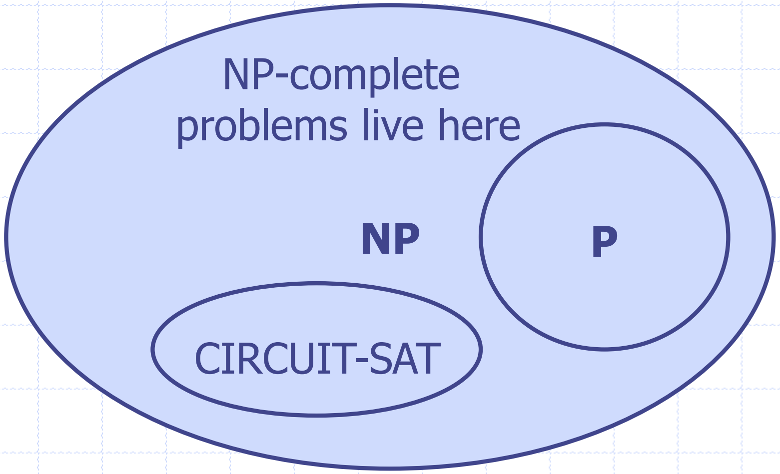


Cook-Levin Proof

- We can build a circuit that simulates the verification of x's membership in M using y.
  - Let W be the working storage for D (including registers, such as program counter); let D be given in RAM "machine code."
  - Simulate p(n) steps of D by replicating circuit S for each step of D. Only input: y.
  - Circuit is satisfiable if and only if x is accepted by D with some certificate y
  - Total size is still polynomial: $O(p(n)^3)$.



Some Thoughts about P and NP



- Belief: P is a proper subset of NP.
- Implication: the NP-complete problems are the hardest in NP.
- Why: Because if we could solve an NP-complete problem in polynomial time, we could solve every problem in NP in polynomial time.
- That is, if an NP-complete problem is solvable in polynomial time, then P=NP.
- Since so many people have attempted without success to find polynomial-time solutions to NP-complete problems, showing your problem is NP-complete is equivalent to showing that a lot of smart people have worked on your problem and found no polynomial-time algorithm.