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DUALLY CHORDAL GRAPHS*

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Abstract. Recently in several papers, graphs with maximum neighborhood orderings were characterized and turned out to be algorithmically useful. This paper gives a unified framework for characterizations of those graphs in terms of neighborhood and clique hypergraphs which have the Helly property and whose line graph is chordal. These graphs are dual (in the sense of hypergraphs) to chordal graphs. By using the hypergraph approach in a systematical way new results are obtained, some of the old results are generalized, and some of the proofs are simplified.

Key words. graphs, hypergraphs, tree structure, hypertrees, duality, chordal graphs, clique hypergraphs, neighborhood hypergraphs, disk hypergraphs, Helly property, chordality of line graphs, maximum neighborhood orderings, linear time recognition, doubly chordal graphs, strongly chordal graphs, bipartite incidence graphs

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1. Introduction. The class of chordal graphs is a by now classical and well-understood graph class which is algorithmically useful and has several interesting characterizations. In the theory of relational database schemes there are close relationships between desirable properties of database schemes, acyclicity of corresponding hypergraphs, and chordality of graphs which corresponds to tree and Helly properties of hypergraphs [2], [5], [25]. Chordal graphs arise also in solving large sparse systems of linear equations [28], [36] and in facility location theory [13].

Recently a new class of graphs was introduced and characterized in [20], [6], [21], [39] which is defined by the existence of a maximum neighborhood ordering. These graphs appeared first in [20] and [16] under the name *HT–graphs* but only a few results have been published in [21]. [34] also introduces maximum neighborhoods but only in connection with chordal graphs (chordal graphs with maximum neighborhood ordering were called there *doubly chordal graphs*).

It is our intention here to attempt to provide a unified framework for characterizations of those graph classes in terms of neighborhood and clique hypergraphs. These graphs are dual (in the sense of hypergraphs) to chordal graphs (this is why we call them *dually chordal*) but have very different properties—thus they are in general not perfect and not closed under taking induced subgraphs. By using the hypergraph approach in a systematical way new results are obtained, a part of the previous results are generalized, and some of the proofs are simplified. The present paper improves the results of the unpublished manuscripts [20] and [6].

Graphs with maximum neighborhood orderings (alias dually chordal graphs) are a generalization of strongly chordal graphs (a well-known subclass of chordal graphs

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for which not only a maximum neighborhood but a linear ordering of neighborhoods of neighbors is required—this leads to the fact that strongly chordal graphs are exactly the hereditary dually chordal graphs, i.e., graphs for which each induced subgraph is a dually chordal graph). Notice also that doubly chordal graphs are precisely those graphs which are chordal and dually chordal.

Maximum neighborhood orderings are also algorithmically useful, especially for domination-like problems and problems which are based on distances. Many problems remaining NP–complete on chordal graphs have efficient algorithms on strongly chordal graphs. In some cases this is due to the existence of maximum neighbors (and not to chordality). Therefore many problems efficiently solvable for strongly chordal and doubly chordal graphs remain polynomial-time solvable for dually chordal graphs, too. In the companion papers [18], [19], [9], [10] the algorithmic use of the maximum neighborhood orderings is treated systematically. Dually chordal graphs seem to represent an important supplement of the world of classical graph classes.

One of our theorems shows that a graph G has a maximum neighborhood ordering if and only if the neighborhood hypergraph of G is a *hypertree*, i.e., it has the Helly property and its line graph is chordal. Due to the self-duality of neighborhood hypergraphs this is also equivalent to the α–acyclicity of the hypergraph which implies a linear time recognition of the graph class. This contrasts with the fact that the best known recognition algorithms for strongly chordal graphs have complexity $O(|E|log|V|)$ [35] and $O(|V|^2)$ [38].

There are several interesting generalizations of this class. Theorem 4 shows that a graph G has a maximum neighborhood ordering if and only if the clique hypergraph (or the disk hypergraph) of G has the Helly property and its line graph is chordal. It is known from [4], [17] that G is a disk–Helly graph (i.e., a graph whose disk hypergraph has the Helly property) if and only if G is a dismantlable clique–Helly graph, and in [3] it is shown that G is an absolute reflexive retract if and only if G is a dismantlable clique–Helly graph. Thus dually chordal graphs are properly contained in the classes of disk–Helly and clique–Helly graphs.

The paper is organized as follows. In section 2 we give standard hypergraph notions and properties. Section 3 is devoted to graphs with maximum neighborhood ordering. There we define some types of hypergraphs associated with graphs and present characterizations of dually chordal graphs, doubly chordal graphs, and strongly chordal graphs via hypergraph properties. The results of this section are from [20]. Section 4 deals with bipartite graphs with maximum neighborhood ordering. There we also describe relationships between graphs and bipartite graphs with different types of maximum neighborhood orderings. A part of the results of this section are from [6] and [22]. In section 5 some results confirming the duality between chordal graphs and dually chordal graphs are established. We conclude with two diagrams which present relationships between classes of graphs, hypergraphs, and some bipartite graphs.

2. Standard hypergraph notions and properties. We mainly use the hypergraph terminology of Berge [7]. A finite hypergraph $\mathcal{E}$ is a family of nonempty subsets (the *edges* of $\mathcal{E}$) from some finite underlying set V (the *vertices* of $\mathcal{E}$). The *subhypergraph* induced by a set $A \subseteq V$ is the hypergraph $\mathcal{E}_A$ defined on A by the edge set $\mathcal{E}_A = \{e \cap A : e \in \mathcal{E}\}$. The *dual hypergraph* $\mathcal{E}^*$ has $\mathcal{E}$ as its vertex set and $\{e \in \mathcal{E} : v \in e\}$ $(v \in V)$ as its edges. The 2*-section graph* $2SEC(\mathcal{E})$ of the hypergraph $\mathcal{E}$ has vertex set V, and two distinct vertices are adjacent if and only if they are contained in a common edge of $\mathcal{E}$. The *line graph* $L(\mathcal{E}) = (\mathcal{E}, E)$ of $\mathcal{E}$ is the intersection graph of $\mathcal{E}$; i.e., $ee' \in E$ if and only if $e \cap e' \neq \emptyset$. A hypergraph $\mathcal{E}$ is *reduced* if no edge $e \in \mathcal{E}$ is contained in another edge of $\mathcal{E}$.

A hypergraph $\mathcal{E}$ is *conformal* if every clique C in $2SEC(\mathcal{E})$ is contained in an edge $e \in \mathcal{E}$. A *Helly hypergraph* is one whose edges satisfy the Helly property; i.e., any subfamily $\mathcal{E}' \subseteq \mathcal{E}$ of pairwise intersecting edges has a nonempty intersection.

First we give a list of well-known properties of hypergraphs (for these and other properties cf. [7]).

(i) Taking the dual of a hypergraph twice is isomorphic to the hypergraph itself; i.e., $(\mathcal{E}^*)^* \sim \mathcal{E}$.

(ii) $L(\mathcal{E}) \sim 2SEC(\mathcal{E}^*)$.

(iii) $\mathcal{E}$ is conformal if and only if $\mathcal{E}^*$ has the Helly property.

A hypergraph $\mathcal{E}$ is a *hypertree* (called *arboreal hypergraph* in [7]) if there is a tree T with vertex set V such that every edge $e \in \mathcal{E}$ induces a subtree in T (T is then called the *underlying vertex tree* of $\mathcal{E}$). A hypergraph $\mathcal{E}$ is a *dual hypertree* if there is a tree T with vertex set $\mathcal{E}$ such that for all vertices $v \in V$ $T_v = \{e \in \mathcal{E} : v \in e\}$ induces a subtree of T (T is then called the *underlying hyperedge tree* of $\mathcal{E}$).

Observe that $\mathcal{E}$ is a hypertree if and only if $\mathcal{E}^*$ is a dual hypertree.

A sequence $C = (e_1, e_2, \ldots, e_k, e_1)$ of edges is a *hypercycle* if $e_i \cap e_{i+1(mod\ k)} \neq \emptyset$ for $1 \leq i \leq k$. The *length* of C is k. A *chord* of the hypercycle C is an edge e with $e_i \cap e_{i+1(mod\ k)} \subseteq e$ for at least three indices i, $1 \leq i \leq k$. A hypergraph $\mathcal{E}$ is *α–acyclic* if it is conformal and contains no chordless hypercycles of length at least 3. Note that the notion of α–acyclicity was introduced in [5] in a different way but the notion given above is equivalent to that given in [5] (cf. [29]).

In a similar way, a graph G is *chordal* if it does not contain any induced (chordless) cycles of length at least 4.

THEOREM 1.

(i) *(See* [23], [27].*) $\mathcal{E}$ is a hypertree if and only if $\mathcal{E}$ is a Helly hypergraph and its line graph $L(\mathcal{E})$ is chordal.*

(ii) *(See* [5], [25], [29].*) $\mathcal{E}$ is a dual hypertree if and only if $\mathcal{E}$ is α–acyclic.*

Due to the dualities between hypertrees and dual hypertrees, the conformality and the Helly property, and the line graph of a hypergraph and the 2-section graph of the dual hypergraph, Theorem 1 can be expressed also in other variants by switching between a property and its dual.

A particular instance of hypertrees are totally balanced hypergraphs. A hypergraph is *totally balanced* if every cycle of length greater than two has an edge containing at least three vertices of the cycle.

THEOREM 2 (see [32]). *A hypergraph $\mathcal{E}$ is totally balanced if and only if every subhypergraph of $\mathcal{E}$ is a hypertree.*

There is a close connection between totally balanced hypergraphs, strongly chordal graphs and chordal bipartite graphs [1], [26], [11]; see [8] for a systematic treatment of these relations. Motivated by these results, we will establish similar connections between hypertrees, dually chordal graphs, and some classes of bipartite graphs.

Hypergraphs can be represented in a natural way by incidence matrices. Let $\mathcal{E} = \{e_1, \ldots, e_m\}$ be a hypergraph and $V = \{v_1, \ldots, v_n\}$ be its vertex set. The *incidence matrix* $\mathcal{IM}(\mathcal{E})$ of the hypergraph $\mathcal{E}$ is a matrix whose (i, j) entry is 1 if $v_i \in e_j$ and 0 otherwise. The (bipartite vertex–edge) *incidence graph* $\mathcal{IG}(\mathcal{E}) = (V, \mathcal{E}, E)$ of the hypergraph $\mathcal{E}$ is a bipartite graph with vertex set $V \cup \mathcal{E}$, where two vertices $v \in V$ and $e \in \mathcal{E}$ are adjacent if and only if $v \in e$. Note that the transposed matrix $\mathcal{IM}(\mathcal{E})^T$ is the incidence matrix of the dual hypergraph $\mathcal{E}^*$, while $\mathcal{IG}(\mathcal{E}) \sim \mathcal{IG}(\mathcal{E}^*)$ if the sides of the bipartite graph are not marked.

Following [33] a matrix M is in *doubly lexical order* if rows and columns as 0-1-vectors are in increasing order. Two rows $r_1 < r_2$ and columns $c_1 < c_2$ form a Γ if the crossing points of these rows and columns define the submatrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. An ordered 0-1 matrix M is *supported* Γ if for every pair $r_1 < r_2$ of rows and pair $c_1 < c_2$ of columns which form a Γ there is a row $r_3 > r_2$ with $M(r_3, c_1) = M(r_3, c_2) = 1$ (r_3 *supports* Γ).

A *subtree matrix* is the incidence matrix of a collection of subtrees of a tree T. A *totally balanced matrix* is the incidence matrix of a totally balanced hypergraph.

THEOREM 3. *Let M be a 0-1 matrix.*

(i) *(See* [33]*.) M is a subtree matrix if and only if it has a supported $\Gamma-$ordering.*
(ii) *(See* [1], [31], [33]*.) M is a totally balanced matrix if and only if it has a $\Gamma-$free ordering.*

Due to the duality shown later, part (i) of this theorem provides a matrix characterization of chordal graphs as well as dually chordal graphs by transposing the incidence matrix.

3. Maximum neighborhood orderings in graphs. Let $G = (V, E)$ be a finite undirected simple (i.e., without loops and multiple edges) and connected graph. For two vertices $x, y \in V$ the *distance* $d_G(x, y)$ is the length (i.e., number of edges) of a shortest path connecting x and y. Let $I(x, y) = \{v \in V : d_G(x, v) + d_G(v, y) = d_G(x, y)\}$ be the *interval* between vertices x and y. By $N_G(v) = \{u : uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ we denote the *open neighborhood* and the *closed neighborhood* of v, respectively. If no confusion can arise we will omit the index G. Let $\mathcal{N}^0(G) = \{N(v) : v \in V\}$ and $\mathcal{N}(G) = \{N[v] : v \in V\}$ be the *open neighborhood hypergraph* and the *closed neighborhood hypergraph* of G, respectively. Let also $\mathcal{C}(G) = \{C : C$ is a maximal clique in $G\}$ be the *clique hypergraph* of G.

It is easy to see that the following holds:

(i) $2SEC(\mathcal{C}(G))$ is isomorphic to G (and thus $\mathcal{C}(G)$ is conformal).
(ii) $(\mathcal{N}(G))^*$ is isomorphic to $\mathcal{N}(G)$ (where it is assumed that the hypergraph $\mathcal{N}(G) = \{N[v] : v \in V\}$ is a multiset) and the same holds for $\mathcal{N}^0(G)$.

Concerning clique hypergraphs of chordal graphs, from Theorem 1 we have the following well-known equivalence:

(iii) A graph G is chordal if and only if its clique hypergraph $\mathcal{C}(G)$ is α–acyclic if and only if $\mathcal{C}(G)$ is a dual hypertree.

Let v be a vertex of G. The *disk* centered at v with radius k is the set of all vertices having distance at most k to v: $N^k[v] = \{u : u \in V$ and $d(u, v) \leq k\}$. Denote by $\mathcal{D}(G) = \{N^k[v] : v \in V,\ k$ a positive integer$\}$ the *disk hypergraph* of G.

First we present some results establishing a connection between the closed neighborhood, the clique, and the disk hypergraphs of a given graph G.

Let a maximal induced cycle of G be an induced cycle of G with a maximum number of edges. Denote by $l(G)$ the number of edges of a maximal induced cycle of G.

LEMMA 1. *Let G be an arbitrary graph.*

(i) $l(L(\mathcal{D}(G))) = l(L(\mathcal{N}(G)))$*. In particular, $L(\mathcal{D}(G))$ is chordal if and only if $L(\mathcal{N}(G))$ is so.*
(ii) $l(L(\mathcal{N}(G))) \leq l(L(\mathcal{C}(G)))$*. In particular, if $L(\mathcal{C}(G))$ is chordal, then $L(\mathcal{N}(G))$ is so.*
(iii) *If $\mathcal{N}(G)$ is conformal, then $(\mathcal{C}(G))^*$ is so.*

Proof. (i) Among all maximal induced cycles of the graph $L(\mathcal{D}(G))$ choose a cycle

$$C = (N^{r_1}[v_1], \dots, N^{r_k}[v_k], N^{r_1}[v_1])$$

with a minimal sum $s = r_1 + \dots r_k$. We claim that C is formed by unit disks only; i.e., $r_1 = r_2 = \dots = r_k = 1$. Assume to the contrary that $r_1 \geq 2$.

Pick arbitrary vertices $a \in N^{r_1}[v_1] \cap N^{r_2}[v_2]$ and $b \in N^{r_k}[v_k] \cap N^{r_1}[v_1]$. Now consider two neighbors $v_1' \in I(a, v_1)$ and $v_1'' \in I(b, v_1)$ of the vertex v_1. If

$$N^{r_1-1}[v_1'] \cap N^{r_k}[v_k] = \emptyset, N^{r_1-1}[v_1''] \cap N^{r_2}[v_2] = \emptyset$$

holds then the disks

$$(N^{r_1-1}[v_1'], N^{r_2}[v_2], \dots, N^{r_k}[v_k], N^{r_1-1}[v_1''])$$

form an induced cycle with $k+1$ edges, contradicting the maximality of C.

So assume for example that $N^{r_1-1}[v_1'] \cap N^{r_k}[v_k] \neq \emptyset$. Then replacing the disk $N^{r_1}[v_1]$ by $N^{r_1-1}[v_1']$ in the cycle C we obtain an induced cycle with k edges and total radius sum $s-1$. This again contradicts the choice of C. Thus C consists of unit disks; i.e., C is an induced cycle of the graph $L(\mathcal{N}(G))$.

(ii) Consider vertices $v_1, \dots, v_k$ whose neighborhoods generate a maximal induced cycle C in the graph $L(\mathcal{N}(G))$. Let

$$B_2 = N[v_1] \cap N[v_2], \dots, B_k = N[v_{k-1}] \cap N[v_k], B_1 = N[v_k] \cap N[v_1].$$

In each set B_i pick a vertex b_i such that the sum $s = d(b_1, b_2) + \dots + d(b_{k-1}, b_k) + d(b_k, b_1)$ is minimal. Now define a cycle C' of the graph $L(\mathcal{C}(G))$ using the following rules: if the vertices $b_i, b_{i+1(mod\ k)}$ are adjacent, then add a clique K_i to C' which contains the vertices $b_i, b_{i+1(mod\ k)}$ and v_i; otherwise add two cliques K_i' and K_i'' (in this order) to C' which contain the edges $v_i b_i$ and $v_i b_{i+1(mod\ k)}$, respectively.

The cycle C' has at least k edges. Assume that C' is not induced; i.e., two nonconsecutive cliques K' and K'' of C' have a nonempty intersection. By the definition of C' any clique of C' contains a center of some neighborhood from C. Since C is an induced cycle the cliques K' and K'' contain centers of two consecutive neighborhoods of C. Let us assume that $v_1 \in K'$ and $v_2 \in K''$. Up to symmetry we have one of the following possibilities: $K' = K_1$ and $K'' = K_2''$ or $K' = K_1'$ and $K'' = K_2'$ or $K' = K_1'$ and $K'' = K_2''$.

In all of these cases the inequality $d(b_1, b_2) + d(b_2, b_3) \geq 3$ holds. Let $b_2^* \in K' \cap K'' \subset B_2$. Since $d(b_1, b_2^*) + d(b_2^*, b_3) = 2$ this leads to a contradiction with the choice of the vertices $b_1, \dots, b_k$. Hence C' is an induced cycle of $L(\mathcal{C}(G))$ and its length is at least $k = l(L(\mathcal{N}(G)))$.

(iii) By the duality properties of hypergraphs it is sufficient to show that $\mathcal{C}(G)$ is a Helly hypergraph. Let $\mathcal{F} = \{C_1, \dots, C_m\}$ be a family of pairwise intersecting cliques. For each vertex $v \in \bigcup_{i=1}^m C_i$ consider the closed neighborhood $N[v]$. Evidently, any two such neighborhoods intersect. Therefore the vertices of the set $\bigcup_{i=1}^m C_i$ induce in $2SEC(\mathcal{N}(G))$ a clique. By the conformality of $\mathcal{N}(G)$ there exists a vertex w such that $N[w]$ contains the union $\bigcup_{i=1}^m C_i$. Due to the maximality of the cliques $C_1, \dots, C_m$ the vertex w belongs to all of them. ∎

3.1. Characterization of dually chordal graphs. Let $G = (V, E)$ be a graph. A vertex $v \in V$ is *simplicial* in G if $N[v]$ is a clique in G. Let $G_i = G(\{v_i, v_{i+1}, \dots, v_n\})$ be the subgraph induced by $\{v_i, v_{i+1}, \dots, v_n\}$ and $N_i[v]$ be the closed neighborhood of v in G_i. A linear ordering $(v_1, \ldots, v_n)$ of V is a *perfect elimination ordering of* G if for all $i \in \{1, \ldots, n\}$, $N_i[v_i]$ is a clique; i.e., v_i is simplicial in G_i.

It is known that a graph G is chordal if and only if G has a perfect elimination ordering. Moreover, every noncomplete chordal graph has two nonadjacent simplicial vertices (see [28]).

A vertex $u \in N[v]$ is a *maximum neighbor of* v if for all $w \in N[v]$, $N[w] \subseteq N[u]$ holds (note that $u = v$ is not excluded). A linear ordering $(v_1, v_2, \ldots, v_n)$ of V is a *maximum neighborhood ordering* of G if for all $i \in \{1, \ldots, n\}$, there is a maximum neighbor $u_i \in N_i[v_i]$; i.e.,

$$\text{for all } w \in N_i[v_i], \quad N_i[w] \subseteq N_i[u_i] \text{ holds.}$$

Note that graphs with maximum neighborhood orderings are in general not perfect. Indeed, let $G = (V, E)$ be any graph and $x \notin V$ be a new vertex. Then for $G' = (V \cup \{x\}, E \cup \{vx : v \in V\})$ the ordering $(v_1, \ldots, v_n, x)$ is a maximum neighborhood ordering. Thus, e.g., the C_5 with an additional dominating vertex (the wheel W_5) has a maximum neighborhood ordering and is not perfect.

THEOREM 4. *For a graph G the following conditions are equivalent:*

(i) *G has a maximum neighborhood ordering*;

(ii) *there is a spanning tree T of G such that any maximal clique of G induces a subtree in T*;

(iii) *there is a spanning tree T of G such that any disk of G induces a subtree in T*;

(iv) *$\mathcal{N}(G)$ is a hypertree (is a dual hypertree).*

Proof. (i) $\Longrightarrow$ (ii). We proceed by induction on the number of vertices of the graph G. Let x be the first vertex in a maximum neighborhood ordering of G. Let y be a maximum neighbor of x; i.e., $N^2[x] = N[y]$. If $x = y$, then x is adjacent to all other vertices of G and the desired tree T could be a star with center x. Thus (ii) is fulfilled. Assume now that $x \neq y$. By induction hypothesis there exists a spanning tree of the graph $G - x = G(V \setminus \{x\})$ which satisfies condition (ii). Among all such spanning trees choose a tree T in which y is adjacent with a maximum number of vertices from $N(x)$. We claim that y is adjacent with all vertices from $N(x) \setminus \{y\}$.

Assume the contrary and pick a vertex $z \in N(x)$ which is nonadjacent to y in T. In T consider a path $y - \cdots - v - z$ connecting vertices y and z. Denote by T_v with $v \in T_v$ and T_z with $z \in T_z$ the connected components of T obtained by deleting an edge (v, z). Adding to these subtrees a new edge (y, z) we transform the tree T into a new tree T'. Since y and z are adjacent vertices of $G - x$ the tree T' is a spanning tree of $G - x$. Now we show that T' fulfills the condition (ii), too. Let C be a maximal clique of $G - x$. If $z \notin C$, then C is completely contained in one of the subtrees T_v or T_z; i.e., C induces in both trees T and T' one and the same subtree. So, suppose that $z \in C$. Since $N[z] \subseteq N[y] = N^2[x]$ we have $y \in C$. Let u_1, u_2 be arbitrary vertices from C. If both vertices u_1 and u_2 belong to one and the same subtree T_v or T_z, then these vertices are connected in T and T' by one and the same path, and we are done. Now, let $u_1 \in T_v$ and $u_2 \in T_z$. In T_v the vertices u_1 and y are connected by a path l_1, consisting of vertices from C. In a similar way, the vertices u_2 and z are joined in T_z by a path $l_2 \subseteq C$. Gluing together the paths l_1 and l_2 and the edge yz we obtain a path which connects the vertices u_1 and u_2 in T'. Hence any clique C of $G - x$ induces a subtree in T'; i.e., T' also satisfies condition (ii). This, however, contradicts the choice of the spanning tree T. The contradiction shows that y is adjacent in T to all vertices of $N(x) \setminus \{y\}$.

Consider a spanning tree T^* of G obtained from T by adding a leaf x adjacent to y. Evidently T^* fulfills condition (ii) of the theorem; i.e., T^* is the required tree.

(ii) $\Longrightarrow$ (iii). Let T be a spanning tree of G such that any clique of G induces a subtree in T. We claim that any disk $N^r[z]$ of G induces a subtree in T, too. In order to prove this, it is sufficient to show that the vertex z and any vertex $v \in N^r[z]$ may be connected in T by a path consisting of vertices from $N^r[z]$. Let $v = v_1 - v_2 - \cdots - v_k - v_{k+1} = z$ be a shortest path of G between v and z. By C_i we denote a maximal clique of G containing the edge $v_i v_{i+1}$, $i \in \{1, \ldots, k\}$. From the choice of T it follows that the vertices v_i and v_{i+1} are connected in T by a path $l_i \subseteq C_i$. The vertices of the set $L = \bigcup_{i=1}^{k} l_i$ induce a subtree $T(L)$ of the tree T. Therefore the vertices v and z may be connected in $T(L)$ (and in T, too) by a path l. Since $d(z,w) \leq d(z,v_i) \leq r$ for any vertex $w \in C_i$ any clique C_i belongs to the disk $N^r[z]$. So our claim follows from the following evident inclusions:

$$l \subseteq L \subseteq \bigcup_{i=1}^{k} C_i \subseteq N^r[z].$$

(iii) $\Longrightarrow$ (iv) is evident.

(iv) $\Longrightarrow$ (i). Suppose T is a tree with the same vertex set as G such that $N_G[v]$ induces a subtree T_v of T for all vertices v in G. Consider T as a tree rooted at a chosen vertex r. Every $N_G[v]$ has a unique vertex v^* such that

$$d_T(r, v^*) < d_T(r, u) \text{ for all vertices } u \in N_G[v] \setminus \{v^*\},$$

which can be considered as the *root* of the subtree T_v of T. Sort the vertices of G into $v_1, v_2, \ldots, v_n$ such that

$$d(r, v_1^*) \geq d(r, v_2^*) \geq \cdots \geq d(r, v_n^*).$$

We claim that this ordering is a maximum neighborhood ordering of G. Note that $v_i^* \in N_i[v_i]$. For each $v_j \in N_i[v_i]$ and $v_k \in N_i[v_j]$, v_j is in both T_{v_i} and T_{v_k}. So v_i^* and v_k^* are both ancestors of v_j. Also, $d_T(r, v_k^*) \leq d_T(r, v_i^*)$. Thus v_i^* is in the path from v_j to v_k^* in T. Since v_j and v_k^* are both in $N_G[v_k]$; i.e., in the subtree T_{v_k} of T, v_i^* is also in T_{v_k}; i.e., $v_i^* \in N_G[v_k]$ and so $v_k \in N_i[v_i^*]$. Thus v_i^* is a maximum neighbor of v_i for $1 \leq i \leq n$. This proves that $v_1, v_2, \ldots, v_n$ is a maximum neighborhood ordering of G. □

This result was also presented in [21].

In [40] a linear time algorithm for recognizing α–acyclicity of a hypergraph is given. Since dual hypertrees are exactly the α–acyclic hypergraphs by Theorem 4 we have the following.

COROLLARY 1. *It can be recognized in linear time $O(|V| + |E|)$ whether a graph G has a maximum neighborhood ordering.*

In [18], [9] we show that for a given dually chordal graph a maximum neighborhood ordering can be generated in linear time, too.

From Theorem 4 it also follows that G has a maximum neighborhood ordering if and only if $\mathcal{C}(G)$ is a hypertree. Recall that the graph G is chordal if and only if $(\mathcal{C}(G))^*$ is a hypertree. Thus graphs with maximum neighborhood ordering are dual to chordal graphs in this sense. Therefore we call them *dually chordal graphs*. The further results will confirm this term and will show the deepness of this duality. Note that unlike for chordal graphs where the number of maximal cliques is linearly bounded, this is not the case for dually chordal graphs.

Furthermore from Theorem 4 it follows that G has a maximum neighborhood ordering if and only if $\mathcal{D}(G)$ is a hypertree. Using this fact in [9] we present efficient algorithms for r–domination and r–packing problems on dually chordal graphs.

The *kth power* $G^k, k \geq 1$, of G has the same vertices as G, and two distinct vertices are joined by an edge in G^k if and only if their distance in G is at most k.

COROLLARY 2. *Any power of a dually chordal graph is dually chordal.*

Proof. Let G be a dually chordal graph, and let G^k be some power of this graph. A unit disk of G^k with center in v coincides with the disk $N^k[v]$ of G. Therefore $\mathcal{N}(G^k)$ is the family of all disks of radius k of the graph G. Since G is dually chordal $\mathcal{N}(G^k)$ has the Helly property and $L(\mathcal{N}(G^k))$ is chordal as an induced subgraph of the chordal graph $L(\mathcal{D}(G))$. By Theorem 4 it follows that G^k is dually chordal. □

3.2. Doubly chordal, power–chordal, and strongly chordal graphs. A vertex v of a graph G is *simple* [26] if the set $\{N[u] : u \in N[v]\}$ is totally ordered by inclusion. A linear ordering $(v_1, \ldots, v_n)$ of V is a *simple elimination ordering of* G if for all $i \in \{1, \ldots, n\}$ v_i is simple in G_i. A graph is *strongly chordal* if it admits a simple elimination ordering. A *k-sun* [11], [14], [26] is a graph with $2k$ vertices for some $k \geq 3$ whose vertex set can be partitioned into two sets $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$ such that U induces a complete graph, W forms an independent set, and u_i is adjacent to w_j if and only if $i = j$ or $i = j + 1 (\text{mod } k)$.

COROLLARY 3. *For a graph G the following conditions are equivalent:*

(i) *G is a strongly chordal graph*;

(ii) *G is a sun-free chordal graph*;

(iii) *G is a hereditary dually chordal graph*; *i.e., any induced subgraph of G is dually chordal.*

Proof. The equivalence of (i) and (ii) is contained in [11], [14], [26]. Since every induced subgraph of a strongly chordal graph is strongly chordal we deduce that (i) $\Longrightarrow$ (iii). Furthermore, any simple vertex v of G evidently has a maximum neighbor. Finally (iii) $\Longrightarrow$ (ii) because induced cycles of length at least four and suns do not contain a vertex which has a maximum neighbor. □

By Lemma 1(iii) conformality of $\mathcal{N}(G)$ implies conformality of $(\mathcal{C}(G))^*$. Moreover in [16], [17] it has been shown that for chordal graphs $\mathcal{N}(G)$ is a Helly hypergraph if and only if $\mathcal{C}(G)$ is so. By Lemma 1(ii) we also know that $L(\mathcal{N}(G))$ is chordal if $L(\mathcal{C}(G))$ is chordal. The following result shows that for chordal graphs the converse is also true.

LEMMA 2. *For a chordal graph G the following conditions are equivalent:*

(i) $G^2 \sim L(\mathcal{N}(G))$ *is chordal*;

(ii) $L(\mathcal{C}(G))$ *is chordal.*

Proof. (ii) $\Longrightarrow$ (i) follows from Lemma 1(ii). Conversely, assume that there is an induced cycle $\Gamma = (C_1, \ldots C_m, C_1), m \geq 4$, of the graph $L(\mathcal{C}(G))$. Let $C = \bigcup_{i=1}^m C_i$. $G^2(C)$ as an induced subgraph of the chordal graph G^2 contains a simplicial vertex x. Suppose that $x \in C_1$. This means $C_2, C_m \subseteq N^2[x]$. Because of the simpliciality of x in G^2 for arbitrary vertices $u \in C_2$ and $v \in C_m$ we have $d(u, v) \leq 2$. Let $C_2 = \{x_1, \ldots x_s\}$ and $C_m = \{y_1, \ldots y_t\}$. We claim that any vertex of C_2 has in G a neighbor in C_m and vice versa. Assume to the contrary that this is not the case for x_1; i.e.,

$$d(x_1, y_1) = d(x_1, y_2) = \cdots = d(x_1, y_t) = 2.$$

Since G is chordal there exists a common neighbor of the vertices x_1 and $y_1, \ldots, y_t$. However, this contradicts the fact that C_m is a maximal clique of G. Thus our claim is true.

In the clique C_2 choose a vertex x_i which is adjacent to a maximum number of vertices from C_m. Suppose that x_i is adjacent to $y_1, \ldots, y_{l-1}$. Note that $l \leq t$, otherwise $C_2 \cap C_m \neq \emptyset$. By our claim we conclude that y_l is adjacent to some vertex $x_j \in C_2$. A unique chord of the cycle $(x_i, y_k, y_l, x_j, x_i)$, $k \in \{1, \ldots, l-1\}$ may be only $x_j y_k$. Therefore x_j is adjacent with $y_1, \ldots, y_{l-1}, y_l$, contradicting the choice of x_i. So, our initial assumption that Γ is an induced cycle of $L(\mathcal{C}(G))$ leads to a contradiction. □

A graph is *power–chordal* if all of its powers are chordal. For the next theorem we need the following lemma.

LEMMA 3. *Let G be a noncomplete graph. If both graphs G and G^2 are chordal, then there exist two nonadjacent vertices of G which are simplicial in G and G^2.*

Proof. The assertion is evident when G^2 is complete. Assume that G^2 is noncomplete and let the assertion be true for all smaller graphs. Since G^2 is chordal there are two nonadjacent simplicial vertices in G^2. If both vertices are also simplicial in G, we are done. So, suppose that the simplicial vertex x of G^2 has in G two nonadjacent neighbors u and v. Consider a minimal $(u-v)$-separator F of the graph G. From [15] it follows that F is a complete subgraph of G. Evidently $x \in F$. Let $G(A)$ and $G(B)$ be connected components of $G(V \setminus F)$ containing u and v, respectively.

By the induction hypothesis either the subgraph $G_1 = G(A \cup F)$ contains a pair of two nonadjacent vertices which are simplicial in G_1 and G_1^2 or G_1 is a complete graph. In the first case at least one of the obtained vertices is in A (since F induces a complete subgraph). In the second case any vertex from A is simplicial in $G_1 = G_1^2$.

Summarizing we conclude that the set A contains a vertex y which is simplicial in G_1 and G_1^2. It is evident that y is simplicial in G. Now we show that y is simplicial in G^2, too. It is enough to consider only the case when y is adjacent in G with a vertex of F. For any vertex $w \notin A \cup F$ we have $d(w,x) \leq 2$ if $d(w,y) \leq 2$. Since y is simplicial in G_1^2 a similar implication also holds for any vertex $u \in A \cup F$: if $d(u,y) \leq 2$, then $d(u,x) \leq 2$. Hence for arbitrary vertices v and w such that $d(y,w) \leq 2$ and $d(y,v) \leq 2$ we have analogous inequalities $d(x,w) \leq 2$ and $d(x,v) \leq 2$. Now, recall that x is simplicial in G^2. This implies that $d(v,w) \leq 2$ and y is simplicial in G^2.

In a similar way we obtain the existence of a vertex $z \in B$ which is simplicial in G and G^2. It remains to notice that y and z are nonadjacent. □

THEOREM 5. *For a graph G the following conditions are equivalent:*

(i) *G is power–chordal*;
(ii) *G and G^2 are chordal*;
(iii) *there exists a common perfect elimination ordering of G and G^2 (i.e., an ordering$(v_1, \ldots, v_n)$ of V such that v_i is simplicial in both graphs G_i and G_i^2, $i \in \{1, \ldots, n\}$).*

Proof. In [24] it is shown that if G^k is chordal, then so is G^{k+2}. Consequently, powers of chordal graphs are chordal provided that G^2 is chordal; i.e., (i) $\Longleftrightarrow$ (ii). The implication (iii) $\Longrightarrow$ (ii) is evident. To prove that (ii) $\Longrightarrow$ (iii) we proceed by induction on the number of vertices. By Lemma 3 there is a simplicial vertex v of G and G^2. It is easy to see that $(G-v)^2 = G^2 - v$; i.e., both graphs $G - v$ and $(G-v)^2$ are chordal. Applying to these graphs the induction hypothesis we obtain the required common perfect elimination ordering. □

A vertex v of a graph G is *doubly simplicial* [34] if v is simplicial and has a maximum neighbor. A linear ordering $(v_1, \ldots, v_n)$ of the vertices of G is *doubly perfect* if for all $i \in \{1, \ldots, n\}$ v_i is a doubly simplicial vertex of G_i. A graph G is *doubly chordal* [34] if it admits a doubly perfect ordering. The following result justifies the term "doubly chordal graphs."

COROLLARY 4 (See [20], [34]). *For a graph G the following conditions are equiv-*

alent:

(i) *G is doubly chordal*;

(ii) *G is chordal and dually chordal*;

(iii) *both hypergraphs $\mathcal{C}(G)$ and $(\mathcal{C}(G))^*$ are hypertrees.*

Proof. From the previous results it is sufficient to show that (ii) $\Longrightarrow$ (i). Since G and G^2 are chordal, Theorem 5 ensures the existence of a vertex v which is simplicial in G and G^2. For any two vertices $x, y \in N^2[v]$ the inequality $d(x, y) \leq 2$ is fulfilled. Hence $N[x] \cap N[y] \neq \emptyset$. Since $\mathcal{N}(G)$ is a hypertree the family of pairwise intersecting disks $\{N[x] : x \in N^2[v]\}$ has a nonempty intersection. Let w be a vertex from this intersection. Then w is a maximum neighbor of v. As we already mentioned $(G - v)^2 = G^2 - v$. It remains to show that $\mathcal{N}(G - v)$ has the Helly property. But this is obvious, because any neighborhood containing v contains the vertex w, too. ∎

From these results we conclude that powers of doubly chordal graphs are doubly chordal. For strongly chordal graphs a similar result was established in [33]: Powers of strongly chordal graphs are strongly chordal.

4. Maximum neighborhood orderings in bipartite graphs. Let $G = (V, E)$ be an arbitrary graph, and let v be a vertex of G. Following [3] the sets

$$HD_{odd}(v) = \{u \in V : d(u, v) \leq k \text{ and } d(u, v) \text{ is odd}\},$$

$$HD_{even}(v) = \{u \in V : d(u, v) \leq k \text{ and } d(u, v) \text{ is even}\}$$

are called the *half-disks* centered at v with radius k. By $\mathcal{HD}(G)$ we denote the family of all half-disks of G and call it the *half–disk hypergraph* of the graph G.

4.1. Bipartite graphs with maximum X–neighborhood ordering. For bipartite graphs $B = (X, Y, E)$ there are also standard hypergraph constructions: $\mathcal{N}^X(B) = \{N(y) : y \in Y\}$ denotes the *X–sided neighborhood hypergraph* of B (analogously define $\mathcal{N}^Y(B)$). Note that $(\mathcal{N}^X(B))^*$ is isomorphic to $\mathcal{N}^Y(B)$ and the same for X and Y exchanged. In addition, $\mathcal{N}^0(B) = \mathcal{N}^X(B) \cup \mathcal{N}^Y(B)$.

The half-disks of a bipartite graph B are defined as follows: for $z \in X$ let $HD^X_B(z, k) = \{x : x \in X \text{ and } d(z, x) \leq k \text{ and } d(z, x) \text{ even}\}$ and for $z \in Y$ let $HD^X_B(z, k) = \{x : x \in X \text{ and } d(z, x) \leq k \text{ and } d(z, x) \text{ odd}\}$ (the half-disks in X). Analogously define the half-disks in Y. Again if no confusion can arise we will omit the index B. The half–disk hypergraph $\mathcal{HD}(B)$ of the bipartite graph B splits into two components: $\mathcal{HD}^X(B) = \{HD^X(y, 2k + 1) : y \in Y \text{ and } k \text{ a positive integer}\} \cup \{HD^X(x, 2k) : x \in X \text{ and } k \text{ a positive integer}\}$, called the *$X$–sided half–disk hypergraph* (consisting of subsets of X), and $\mathcal{HD}^Y(B)$ (defined analogously) called the *Y–sided half–disk hypergraph* (consisting of subsets of Y); i.e., $\mathcal{HD}(B) = \mathcal{HD}^X(B) \cup \mathcal{HD}^Y(B)$.

A bipartite graph $B = (X, Y, E)$ is called *X–conformal* [2] if for any set $S \subseteq Y$ with the property that all vertices of S have pairwise distance 2 there is a vertex $x \in X$ with $S \subseteq N(x)$. B is *X–chordal* [2] if for every cycle C in B of length at least 8 there is a vertex $x \in X$ which is adjacent to at least two vertices in C whose distance in C is at least 4 (a *bridge vertex*). Analogously define Y–chordality and Y–conformality. In [2] it is also shown that the following connection holds.

LEMMA 4. *Let $B = (X, Y, E)$ be a bipartite graph. Then B is X–chordal and X–conformal if and only if $\mathcal{N}^Y(B)$ is a dual hypertree if and only if $\mathcal{N}^X(B)$ is a hypertree.*

A vertex $y \in N(x)$ of $B = (X, Y, E)$ is a *maximum neighbor of* x if for all $y' \in N(x)$ $N(y') \subseteq N(y)$ holds. Let $B_i^Y = B(X \cup \{y_i, y_{i+1}, \ldots, y_n\})$ and $N_i(x)$ be the neighborhood of $x \in X$ in B_i^Y. A linear ordering $(y_1, \ldots, y_n)$ of Y is a *maximum* X*–neighborhood ordering* of B if for all $i \in \{1, \ldots, n\}$ there is a maximum neighbor $x_i \in N(y_i)$ of y_i; i.e.,

$$\text{for all } x \in N(y_i) \quad N_i(x) \subseteq N_i(x_i) \text{ holds.}$$

Analogously define a *maximum* Y*–neighborhood ordering.*

THEOREM 6. *Let* $B = (X, Y, E)$ *be a bipartite graph. Then the following conditions are equivalent:*

(i) *B has a maximum X–neighborhood ordering*;

(ii) *B is X–chordal and X–conformal*;

(iii) *$\mathcal{N}^X(B)$ is a hypertree*;

(iv) *the X–sided half-disk hypergraph $\mathcal{HD}^X(B)$ is a hypertree.*

Proof. The equivalence (ii) $\Longleftrightarrow$ (iii) follows from Lemma 4. The direction (iv) $\Longrightarrow$ (iii) is obvious.

(i) $\Longrightarrow$ (ii). Let $(y_1, \ldots, y_n)$ be a maximum X–neighborhood ordering of Y. Consider a chordless cycle $C = (x_{i_1}, y_{i_1}, \ldots, x_{i_k}, y_{i_k})$, $k \geq 4$. Assume that y_{i_1} is the leftmost Y–vertex of C in $(y_1, \ldots, y_n)$ which appears in this ordering in the jth position: $y_{i_1} = y_j$. Since $y_{i_k} \in N_j(x_{i_1}) \setminus N_j(x_{i_2})$ and $y_{i_2} \in N_j(x_{i_2}) \setminus N_j(x_{i_1})$ the sets $N_j(x_{i_1})$ and $N_j(x_{i_2})$ are incomparable with respect to set inclusion. Thus neither x_{i_1} nor x_{i_2} are maximum neighbors of y_{i_1}. Let x be a maximum neighbor of $y_{i_1} = y_j$. Then $y_{i_1}, y_{i_2}, y_{i_k} \in N_j(x)$ and x is a bridge vertex. (Note that x is even a neighbor of three Y–vertices of C.) Thus B is X-chordal.

Now let $S \subseteq Y$ be a subset of vertices of pairwise distance 2. Let $y \in S$ be the leftmost element of S in $(y_1, \ldots, y_n)$ and assume that $y = y_j$. For all $y' \in S$ there are common neighbors $x' \in X$ of y and y'. If x is a maximum neighbor of y_j, then $S \subseteq N_j(x)$. Thus B is X–conformal.

(ii) $\Longrightarrow$ (i). Assume that B is X–chordal and X–conformal. By Lemma 4 the graph $G' = 2SEC(\mathcal{N}^Y(B))$ is chordal. Let $(y_1, \ldots, y_n)$ be a perfect elimination ordering of G'. Thus $N_{G'}[y_1]$ is a clique; i.e., for all $u, v \in N_{G'}[y_1]$, $u \neq v$, there is a common neighbor in X and so the distance between u and v is 2. Since B is X–conformal there is an $x \in X$ with $N_{G'}[y_1] \subseteq N_B(x)$. Necessarily x is a neighbor of y_1 in B and is also a maximum neighbor of y_1 in B since for all $x' \in X$ with $x' \in N_B[y_1]$ $N_B(x') \subseteq N_{G'}[y_1]$.

The same argument can be applied repeatedly to the graph B_i^Y since $G' \setminus \{y_1\}$ is again chordal. Thus the perfect elimination ordering $(y_1, \ldots, y_n)$ of G' is a maximum X–neighborhood ordering of B and vice versa.

(iii) $\Longrightarrow$ (iv). Suppose that T_N is a tree with vertex set X such that for all $y_i \in Y$, $i \in \{1, \ldots, n\}$, $N(y_i)$ induces a subtree in T_N, and let $(y_1, \ldots, y_n)$ be a maximum X–neighborhood ordering of Y. We have to show that then also each half–disk of $\mathcal{HD}^X(B)$ induces a subtree in T_N, too.

The proof is done along the maximum X–neighborhood ordering $(y_1, \ldots, y_n)$ of Y. Let Y_i denotes the subset $\{y_i, \ldots, y_n\}$, and let B_i be the bipartite graph B restricted to Y_i. For Y_n the assertion is obviously true since the only X–sided half-disks in this case are the one-vertex sets $\{x\}, x \in X$, and the neighborhood $N(y_n)$. Obviously, these sets induce subtrees of T_N. Assume now that the half-disks of $\mathcal{HD}^X(B_{i+1})$ induce subtrees in T_N, $i \geq 1$. We will show that then also the half-disks of $\mathcal{HD}^X(B_i)$ induce subtrees in T_N. Without loss of generality let $i = 1$. Let x be a maximum neighbor of y_1. In order to show that the half-disks of $\mathcal{HD}^X(B)$ induce subtrees of T_N we describe their structure. Consider for example the half–disk centered at z with radius $k \geq 2$. We distinguish two cases.

Case 1. $z \in N(y_1)$. First suppose that the degree of z is 1; i.e., $N(z) = \{y_1\}$.

Then $HD_B^X(z,k) = HD_{B_2}^X(x,k-2) \cup N(y_1)$ as can be easily seen: for every vertex $w \notin N(y_1)$ we have $d(x,w) = d(z,w) - 2$ and thus $w \in HD_{B_2}^X(x,k-2)$. Otherwise, if the degree of z is larger than 1, then $HD_B^X(z,k) = HD_{B_2}^X(z,k) \cup N(y_1)$; indeed, for every vertex $w \notin N(y_1)$ there is a path of length $d(z,w)$ which avoids the vertex y_1.

Case 2. $z \notin N(y_1)$.

If $d(z,y_1) > k$, then $HD_B^X(z,k) = HD_{B_2}^X(z,k)$. Otherwise, we obtain that $HD_B^X(z,k) = HD_{B_2}^X(z,k) \cup N(y_1)$. Furthermore in the latter case the vertex x belongs to $HD_{B_2}^X(z,k)$; as $HD_{B_2}^X(z,k)$ contains a neighbor x_j of y_1 and since x is a maximum neighbor of y_1 the half–disk also contains x itself.

Thus in all cases either $HD_B^X(z,k)$ is the same as before or is a union of two subtrees of T_N which both contain x. Thus it is again a subtree of T_N. □

From the proof of the implication (ii) $\Longrightarrow$ (i) it follows also that $(y_1,\dots,y_n)$ is a maximum X–neighborhood ordering of B if and only if $(y_1,\dots,y_n)$ is a perfect elimination ordering of $2\mathrm{SEC}(\mathcal{N}^Y(B))$.

4.2. Graphs with *b*–extremal ordering. Now let G be again an arbitrary graph. Lemma 1 gives a connection between the closed neighborhood and the disk hypergraphs of a given graph G. The next lemma establishes a similar connection between the open neighborhood hypergraph and the half–disk hypergraph of a graph G.

LEMMA 5. *For any graph G $l(L(\mathcal{HD}(G))) = l(L(\mathcal{N}^0(G)))$ holds. In particular, $L(\mathcal{HD}(G))$ is chordal if and only if $L(\mathcal{N}^0(G))$ is so.*

In a graph G a vertex v is *dominated* by another vertex $u \neq v$ if $N(v) \subseteq N(u)$. A vertex v is *b–extremal* if it is dominated by another vertex and there exists a vertex w such that $N(N(v)) = N(w)$. The ordering $(v_1,\dots,v_n)$ of V is a *b–extremal ordering* of G if for all $i \in \{1,\dots,n\}$ v_i is b–extremal in G_i. It is quite evident that a graph G admitting a b–extremal ordering must be bipartite. Indeed, consider the following iterative coloring of G. Let the vertices v_n and v_{n-1} be colored. Then for any i $(i < n-1)$ if the vertex v_i is dominated by v_j, then v_i gets the same color as v_j.

THEOREM 7. *For a graph G the following conditions are equivalent:*

(i) *$\mathcal{N}^0(G)$ is a hypertree*;
(ii) *$\mathcal{HD}(G)$ is a hypertree*;
(iii) *G is bipartite, and G has a maximum X–neighborhood ordering and a maximum Y–neighborhood ordering*;
(iv) *G has a b–extremal ordering.*

Proof. The equivalence of (i), (ii), and (iii) is an immediate consequence of Theorem 6 and the fact that if $\mathcal{N}^0(G)$ is a hypertree, then G is bipartite (which has a straightforward proof).

(i) $\Longrightarrow$ (iv). Let $\mathcal{N}^0(G)$ be a hypertree. Then G is bipartite, say, $G = (X,Y,E)$. Consider the chordal graphs $G^Y = 2SEC(\mathcal{N}^Y(G))$ and $G^X = 2SEC(\mathcal{N}^X(G))$. Let $x \in X$ be a simplicial vertex of G^X. Additionally suppose that x is an opposite vertex in G for some v; i.e., $x \notin I(v,x')$ for any vertex x' of G. Since x is simplicial in G^X the distance between every two vertices from $N(N(x))$ is 2. By the Helly property there is a vertex $y \in Y$ such that $N(y) = N(N(x))$. Moreover, since x is an opposite vertex for v and G is bipartite, necessarily $N(x) \subseteq I(x,v)$. Consider the family of half-disks consisting of open neighborhoods centered at vertices of $N(x)$ and a half-disk centered at v with radius $d(x, v) - 1$. By the Helly property for half-disks there is a common vertex $z \neq x$ of these open neighborhoods. So $N(x) \subseteq N(z)$. Hence any vertex x, which is opposite in G and simplicial in G^X or G^Y, is a b-extremal vertex of the graph G.

Now we prove that such a vertex always exists. Let diam(G), diam(G^X), and diam(G^Y) be the diameters of the graphs G, G^X, and G^Y, respectively. First assume that diam(G) is even; i.e., diam(G)=$2k$. Then max(diam(G^X), diam(G^Y))=k. Let diam(G^X)=k and let x' and x'' be a diametral pair in G^X. Then $d(x', x'') = 2k =$ diam(G); i.e., any diametral pair of G^X is a diametral pair of G, too. It is known [41] that any chordal graph contains a diametral pair of simplicial vertices. Now assume that diam(G) is odd; i.e., diam(G)=$2k+1$. Then max(diam(G^X), diam(G^Y))=k. Let diam(G^X)=k, and let x' and x'' be simplicial vertices which constitute a diametral pair of G^X. Then either x' and x'' are mutually opposite vertices in G or one of them is an end of a diameter in G.

(iv) $\Longrightarrow$ (iii). If $G = (V, E)$ has a b–extremal ordering $(v_1, \ldots, v_n)$, then by arguments above G is bipartite: $G = (X, Y, E)$. Assume that $v_1 \in Y$. Let $G' = G - v_1 = (X, Y - v_1, E')$ with a maximum Y–neighborhood ordering $(x_1, \ldots, x_r)$ and a maximum X–neighborhood ordering $(y_1, \ldots, y_s)$. Then $(v_1, y_1, \ldots, y_s)$ is also a maximum X–neighborhood ordering of G: it is obvious that v_1 has a maximum neighbor in X. Furthermore, as we will show, $(x_1, \ldots, x_r)$ is still a maximum Y–neighborhood ordering of G. Assume by way of contradiction that for x_1 this is not so. Let z be a maximum neighbor of x_1 in G', and assume that $N(z)$ and $N(v_1)$ are incomparable with respect to set inclusion. Since v_1 is b–extremal there is a vertex $u \in Y \setminus \{v_1\}$ such that $N(v_1) \subseteq N(u)$, contrary to the maximality of $N(z)$ in G'. □

Recall [30] that a graph G is *chordal bipartite* if G is bipartite and any induced cycle of G has length 4.

COROLLARY 5. *For a graph the following conditions are equivalent:*

(i) *Every induced subgraph of G admits a b–extremal ordering*;

(ii) *G is a chordal bipartite graph.*

We conclude this section by establishing some relationships between dually chordal graphs and their bipartite relatives. For this we recall two standard transformations of graphs. The first transformation associates with a graph $G = (V, E)$ the bipartite graph $B(G)$, called the *bigraph* of G. The vertex set of $B(G)$ consists of two disjoint copies V' and V'' of V, with $v' \in V'$ and $w'' \in V''$ adjacent in $B(G)$ if and only if v and w either coincide or are adjacent in G. Equivalently, $B(G)$ is the (vertex–closed-neighborhood) incidence graph of G; i.e., $B(G) = \mathcal{IG}(\mathcal{N}(G))$. In a similar way we define the bipartite graph $B_C(G) = \mathcal{IG}(\mathcal{C}(G))$.

From Theorems 4, 6, and 7 we obtain the following result.

COROLLARY 6. *Let G be a graph. Then G has a maximum neighborhood ordering if and only if $B(G)$ has a maximum X–neighborhood ordering (maximum Y–neighborhood ordering) if and only if $B(G)$ has a b–extremal ordering.*

Let $B = (X, Y, E)$ be a bipartite graph. Then the graph $\text{split}_X(B) = (X \cup Y, E_X)$ is obtained from B by completing X to a clique. Assume that X is a maximal clique in $\text{split}_X(B)$, i.e., for no $y \in Y$ $X \subseteq N(y)$. Note that the set of maximal cliques in $\text{split}_X(B)$ is

$$\mathcal{C}(\text{split}_X(B)) = \{\{y\} \cup N(y) : y \in Y\} \cup \{X\}.$$

LEMMA 6. *Let $B = (X, Y, E)$ be a bipartite graph.*

(i) *$\mathcal{N}^X(B)$ has the Helly property if and only if $\mathcal{C}(\text{split}_X(B))$ has the Helly property (analogously for Y instead of X).*

(ii) *$L(\mathcal{N}^X(B))$ is chordal if and only if $L(\mathcal{C}(\text{split}_X(B)))$ is chordal.*

Thus $\mathcal{N}^X(B)$ is a hypertree if and only if $\mathcal{C}(\text{split}_X(B))$ is a hypertree.

The assertion (i) follows from the definition of $\mathcal{N}^X(B)$ and $\mathcal{C}(\text{split}_X(B))$. To show (ii) let $L = L(\mathcal{N}^X(B)) = (\{N(y) : y \in Y\}, E')$ and $(N(y_1), \dots, N(y_k))$ be a perfect elimination ordering of L. Then $N(y_1)$ is a simplicial vertex in L; i.e., all $N(y)$ intersecting $N(y_1)$ are pairwise intersecting (the elements in the intersection are elements of X). Then for $R = L(\mathcal{C}(\text{split}_X(B)))$ $(N[y_1], \dots, N[y_k], X)$ is a perfect elimination ordering of R and vice versa.

COROLLARY 7. *Let $B = (X, Y, E)$ be a bipartite graph. Then B is X–chordal and X–conformal if and only if $\text{split}_X(B)$ is doubly chordal.*

The proof of this result is a sequence of equivalences: B is X–chordal and X–conformal if and only if $\mathcal{N}^X(B)$ is a hypertree if and only if $\mathcal{C}(\text{split}_X(B))$ is a hypertree if and only if $\text{split}_X(B)$ is doubly chordal.

In section 2 we gave the notion of the incidence graph $\mathcal{IG}(\mathcal{E})$ of a hypergraph $\mathcal{E}$. In the particular case of one-sided neighborhood hypergraphs $\mathcal{N}^V(\mathcal{IG}(\mathcal{E}))) = \mathcal{E}$ and $\mathcal{N}^{\mathcal{E}}(\mathcal{IG}(\mathcal{E}))) = \mathcal{E}^*$ hold.

COROLLARY 8. *Let $\mathcal{E}$ be a hypergraph. Then $\mathcal{E}$ is a hypertree if and only if $\mathcal{IG}(\mathcal{E})$ has a maximum X–neighborhood ordering if and only if $\text{split}_V(\mathcal{IG}(\mathcal{E}))$ has a maximum neighborhood ordering.*

5. The duality between chordal and dually chordal graphs. In this section we take advantage of the previous results to explain the duality between chordal and dually chordal graphs.

THEOREM 8. *Let $G = (V, E)$ be a graph.*

(i) *G is chordal if and only if $B_C(G)$ has a maximum y–neighborhood ordering.*

(ii) *G is dually chordal if and only if $B_C(G)$ has a maximum X–neighborhood ordering.*

(iii) *G is doubly chordal if and only if $B_C(G)$ has a X–neighborhood ordering and a maximum Y–neighborhood ordering if and only if $B_C(G)$ has a b–extremal ordering.*

It is well known [12] that chordal graphs are exactly the intersection graphs of subtrees of a tree. The next result shows that a dual property characterizes the class of dually chordal graphs.

THEOREM 9. *Let $G = (V, E)$ be a graph.*

(i) *(See [12]) G is chordal if and only if it is the line graph of some hypertree if and only if it is the 2–section graph of some α–acyclic hypergraph.*

(ii) *G is dually chordal if and only if it is the line graph of some α–acyclic hypergraph if and only if it is the 2–section graph of some hypertree if and only if it is the 2–section graph of paths of a tree.*

(iii) *G is doubly chordal if and only if it is the line graph of some α–acyclic hypertree if and only if it is the 2–section graph of some α–acyclic hypertree.*

Proof. To show (ii) let G be a dually chordal graph. By Theorem 4 $\mathcal{E} = \mathcal{C}(G)$ is a hypertree. Recall also that $G = 2SEC(\mathcal{C}(G)) = 2SEC(\mathcal{E})$. Let T be a representing tree of $\mathcal{E}$. We obtain the hypergraph $\mathcal{E}'$ of paths of the tree T from $\mathcal{E}$ by replacing every subtree T_C $(C \in \mathcal{C}(G))$ by a collection of all paths connecting in T the leaves of T_C. Obviously, $2SEC(\mathcal{E})$=$2SEC(\mathcal{E}')$.

Now assume that G is the 2–section graph of some hypertree $\mathcal{E}$ with representing tree T. Consider a neighborhood $N[v]$ in G. Since $N[v]$ is a union of subtrees con-

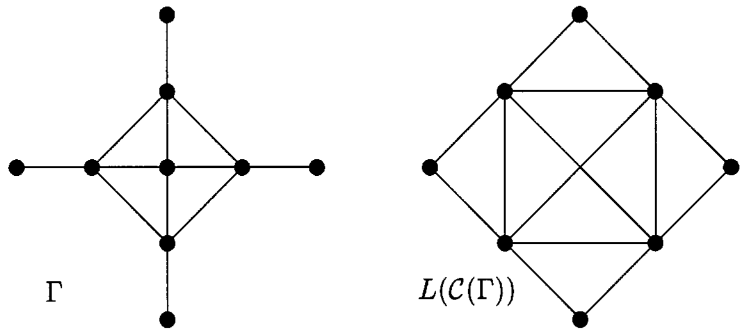

FIG. 1.

taining v, $N[v]$ is a subtree of T; i.e., $\mathcal{N}(G)$ is a hypertree. By Theorem 4 G is dually chordal. ∎

It is well known that there is a simple method to obtain an underlying hyperedge tree for the clique hypergraph $\mathcal{C}(G)$ of a chordal graph G (the so-called *clique–tree* [37]): Weight the edges of the intersection graph $L(\mathcal{C}(G))$ by the size of the intersection and find a maximum spanning tree on this graph.

For a dually chordal graph G there is a dual variant of this method (see [18]): Weight every edge of the graph $G = 2SEC(\mathcal{C}(G))$ by the number of maximal cliques of G containing this edge and find a maximum spanning tree on this weighted graph. Then G is dually chordal if and only if every maximum spanning tree on G is an underlying vertex tree for $\mathcal{C}(G)$.

As it was shown in [33] the matrix M^TM (M^T the transpose of M) is totally balanced provided that M is so. Unfortunately a similar property does not hold for subtree matrices; see Figure 1. The graph Γ is dually chordal. So the incidence matrix $M = \mathcal{IM}(\mathcal{C}(\Gamma))$ is a subtree matrix. The matrix M^TM is the neighborhood matrix $M = \mathcal{IM}(\mathcal{N}(L(\mathcal{C}(\Gamma))))$ of the clique graph $L(\mathcal{C}(\Gamma))$ of Γ. Since $L(\mathcal{C}(\Gamma))$ is not dually chordal M^TM is not a subtree matrix. Nevertheless the following is true.

COROLLARY 9. *If M is a subtree matrix then so is MM^T.*

Proof. Let $\mathcal{E}_M$ be a hypertree whose incidence matrix is M. By Theorem 9 the graph $G = 2SEC(\mathcal{E}_M)$ is dually chordal. Note that the matrix MM^T is the neighborhood matrix $\mathcal{IM}(\mathcal{N}(G))$. Since $\mathcal{N}(G)$ is a hypertree (Theorem 4) MM^T is a subtree matrix. ∎

The graph Γ of Figure 1 shows that the clique graph of a dually chordal graph is not necessarily dually chordal. The results below characterize the clique graphs of chordal, dually chordal, and doubly chordal graphs.

Subsequently we use the following notations: A graph G is *clique–Helly* if $\mathcal{C}(G)$ has the Helly property. G is *Helly chordal* if G is chordal and clique–Helly. G is *clique–chordal* if $L(\mathcal{C}(G))$ is chordal.

COROLLARY 10. *G is a Helly chordal graph if and only if G is the clique graph of some dually chordal graph G'; i.e., $G \sim L(\mathcal{C}(G'))$.*

Proof. By Theorem 4 the clique hypergraph $\mathcal{C}(G')$ has the Helly property. By Theorem 9 $L(\mathcal{C}(G'))$ is chordal. On the other hand, as follows from [4, Theorem 3.2], cliques of the graph $L(\mathcal{C}(G'))$ have the Helly property. Conversely, assume that G is a Helly chordal graph. By Theorem 9 G is the line graph of some conformal hypertree $\mathcal{E}$. It is easy to see that any conformal and reduced hypergraph is the

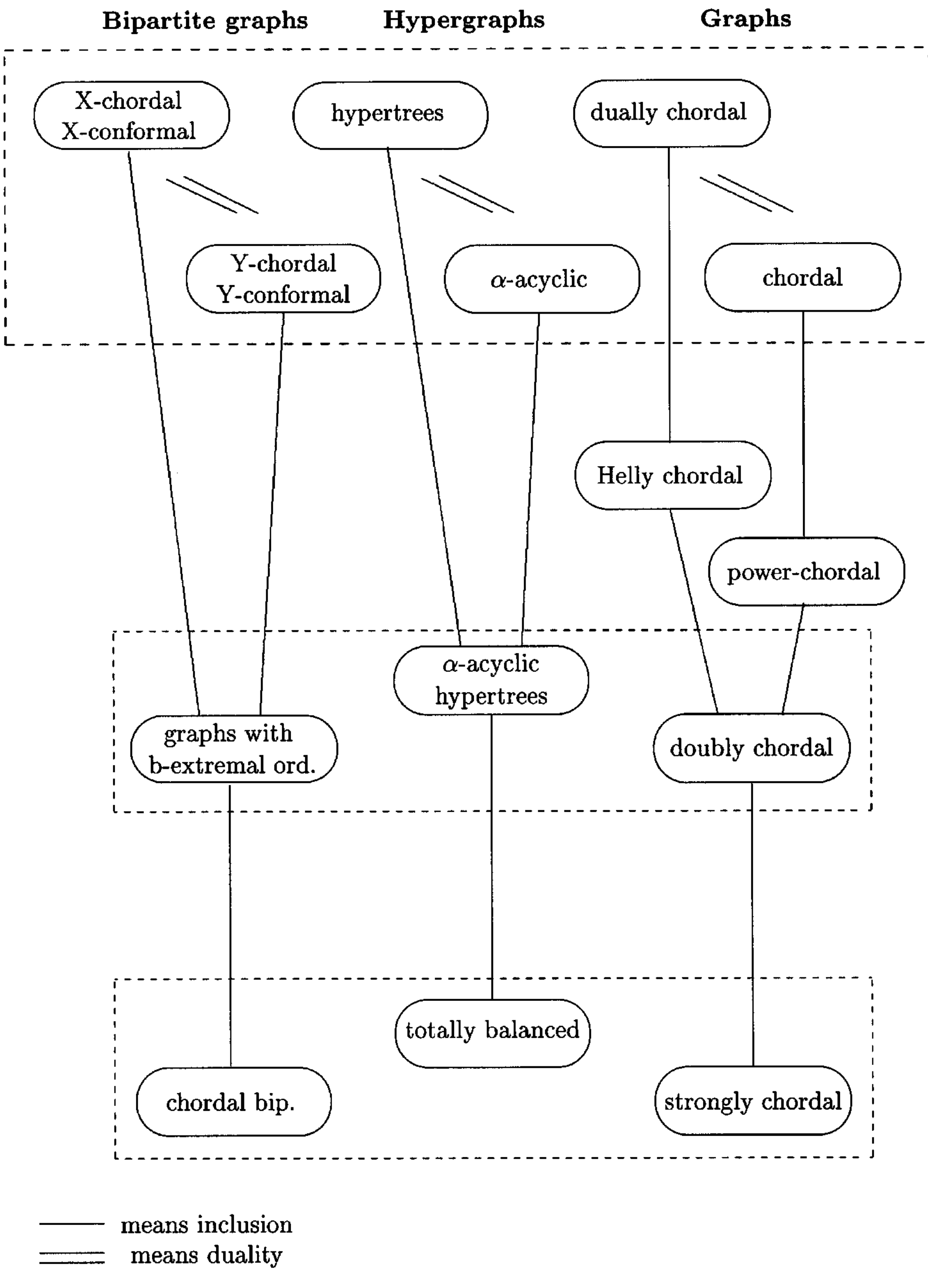

Fig. 2.

clique hypergraph of its 2–section graph. Then any conformal and reduced hypertree is the clique hypergraph of some dually chordal graph. So it is sufficient to transform $\mathcal{E}$ into such a hypergraph $\mathcal{E}'$ without changing its line graph. We obtain the hypergraph $\mathcal{E}'$ from $\mathcal{E}$ by adding to each edge e_i of $\mathcal{E}$ one new vertex u_i incident to e_i only. ◻

Corollary 11 (see [39]). *G is a dually chordal graph if and only if G is the clique graph of some chordal graph if and only if G is the clique graph of some intersection*

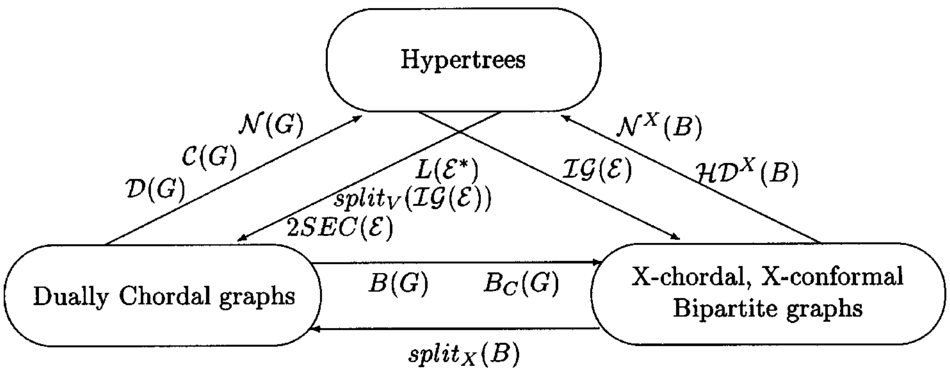

FIG. 3.

graph of paths in a tree.

The proof follows from Theorem 9 by using similar arguments as in the proof of Corollary 10.

Combining Corollaries 10 and 11 and Theorem 9 we obtain the following.

COROLLARY 12. *G is a doubly chordal graph if and only if G is the clique graph of some doubly chordal graph.*

Our duality results are established using the clique hypergraph $\mathcal{C}(G)$ of a graph G. The following four properties of this hypergraph play a crucial role:

- Conformality of $\mathcal{C}(G)$;
- Chordality of $G = 2SEC(\mathcal{C}(G))$;
- Helly property of $\mathcal{C}(G)$;
- Chordality of $L(\mathcal{C}(G))$.

The conformality of $\mathcal{C}(G)$ is fulfilled for all graphs. Chordal graphs are a well–investigated class; see, for instance, [28]. Clique–Helly graphs are characterized in [4], [16], [17].

In different combinations the four conditions above characterize the graph classes considered in this paper.

dually chordal = **clique–Helly** $\bigcap$ **clique–chordal**
doubly chordal = **clique–Helly** $\bigcap$ **clique–chordal** $\bigcap$ **chordal**
Helly chordal = **clique–Helly** $\bigcap$ **chordal**
Power–chordal = **clique–chordal** $\bigcap$ **chordal**

We conclude with the hint to two diagrams (Figures 2 and 3) which show the relations between graph classes and hypergraphs associated with these graphs.

6. Concluding remarks. We have shown the close relationship of graphs with maximum neighborhood ordering and hypergraph properties as the Helly property and tree-like representations of maximal cliques and neighborhoods. Thus in the sense of hypergraph duality these graphs are dual to chordal graphs but have different properties, especially they are in general not perfect. On the other hand maximum neighborhood orderings turn out to be very useful for domination-like problems (see [21], [34], [18], [6], [19]). In the papers [9], [10] the algorithmic use of maximum neighborhood orderings is treated systematically.

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