Collective Tree Spanners of Graphs with Bounded Parameters

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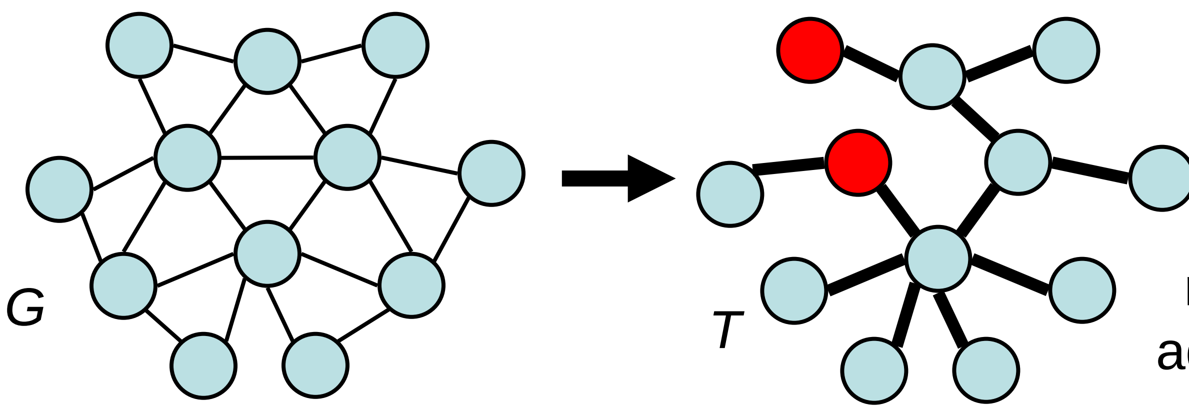
Well-known **Tree *t* -Spanner Problem**

Given unweighted undirected graph *G=(V,E)* and integers *t,r.*
Does *G* admit a spanning tree *T =(V,E')* such that

$\forall u,v \in V,\quad dist_T(v,u) \le t \times dist_G(v,u)$ (a *multiplicative* tree *t-spanner* of *G*)

or

$\forall u,v \in V,\quad dist_T(u,v) - dist_G(u,v) \le r$ (an *additive* tree *r-spanner* of *G*)?



multiplicative tree 4-, additive tree 3-spanner of *G*

Some known results for the tree spanner problem

(mostly multiplicative case)

- general graphs [CC'95]
  - $t \geq 4$ is NP-complete. (t=3 is still open, $t \leq 2$ is P)
- approximation algorithm for general graphs [EP'04]
  - $O(\log n)$ approximation algorithm
- chordal graphs [BDLL'02]
  - $t \geq 4$ is NP-complete. (t=3 is still open.)
- planar graphs [FK'01]
  - $t \geq 4$ is NP-complete. (t=3 is polynomial time solvable.)
- easy to construct for some special families of graphs.

Well-known **Sparse *t*-Spanner Problem**

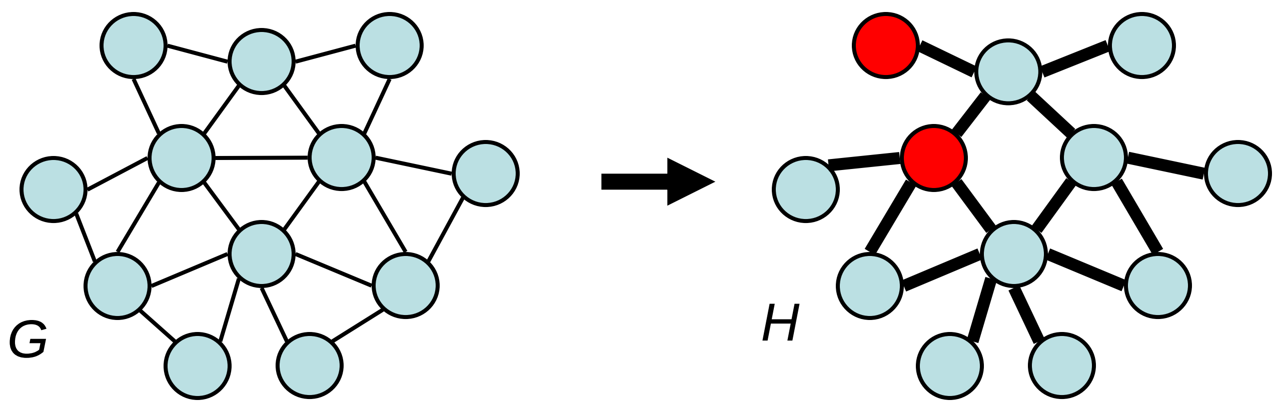
Given unweighted undirected graph $G=(V,E)$ and integers t,m,r.

Does G admit a spanning graph $H=(V,E')$ with $|E'| \leq m$ s.t.

$$\forall u,v \in V,\ \ dist_H(v,u) \leq t \times dist_G(v,u)$$ (a *multiplicative t-spanner* of *G*)

or

$$\forall u,v \in V,\ \ dist_H(u,v) - dist_G(u,v) \leq r$$ (an *additive r-spanner* of *G*)?



multiplicative 2- and additive 1-spanner of *G*

Some known results for sparse spanner problems

- general graphs
  - *t, m≥1* is NP-complete [PS'89]
  - multiplicative (2k-1)-spanner with $n^{1+1/k}$ edges [TZ'01, BS'03]
- *n*-vertex chordal graphs (multiplicative case) [PS'89]

  (*G* is chordal if it has no chordless cycles of length >3)
  - multiplicative 3-spanner with *O(n* log*n)* edges
  - multiplicative 5-spanner with *2n-2* edges
- *n*-vertex *c*-chordal graphs (additive case) [CDY'03, DYL'04]

  (*G* is *c*-chordal if it has no chordless cycles of length >c)
  - additive (c+1)-spanner with *2n-2* edges
  - additive $(2\lfloor c/2 \rfloor)$-spanner with *n log n* edges

  ➔ For chordal graphs: additive *4*-spanner with *2n-2* edges, additive *2*-spanner with *n log n* edges

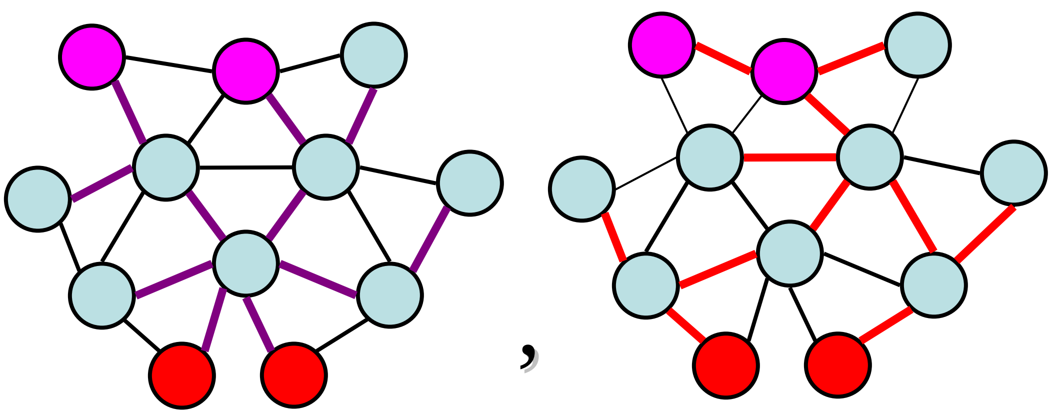
New Collective Additive Tree *r*-Spanners Problem

Given unweighted undirected graph *G=(V,E)* and integers μ, *r.*
Does *G* admit a system of μ collective additive tree *r-spanners* $\{T_1, T_2, \ldots, T_\mu\}$ such that

$$\forall u,v \in V \; and \; \exists 0 \le i \le \mu, \; dist_{T_i}(v,u) - dist_G(v,u) \le r$$

(a system of μ collective *additive tree r-spanners* of *G*)?

surplus



2 collective additive tree 2-spanners

collective multiplicative
tree t-spanners
can be defined similarly

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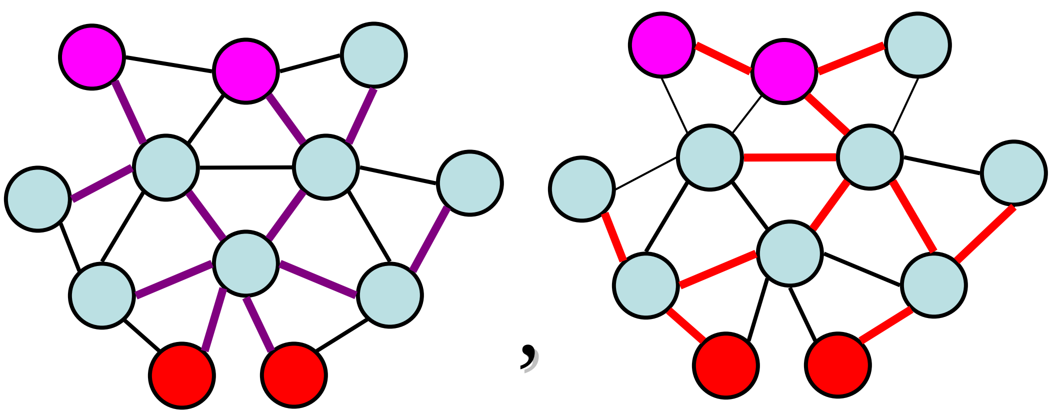
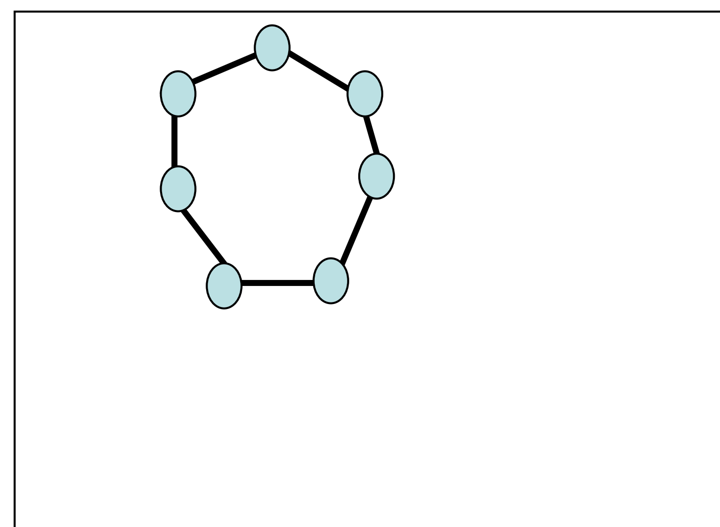


2 **collective additive tree *2*-spanners**



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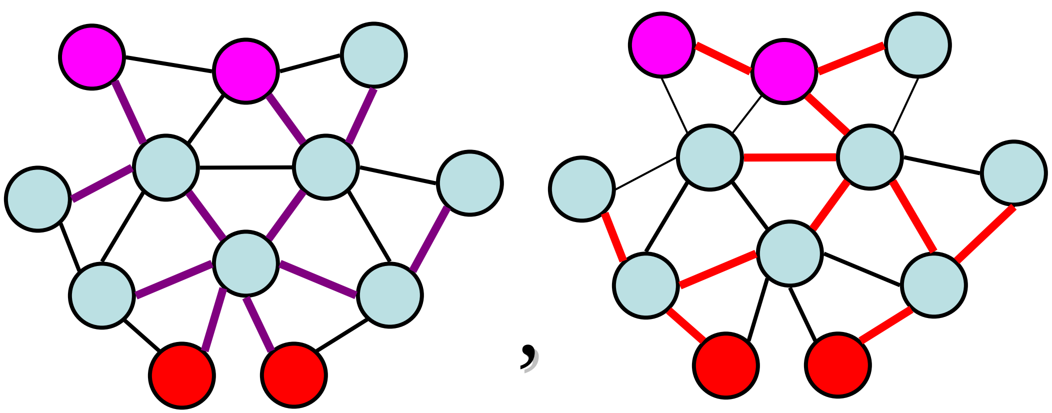
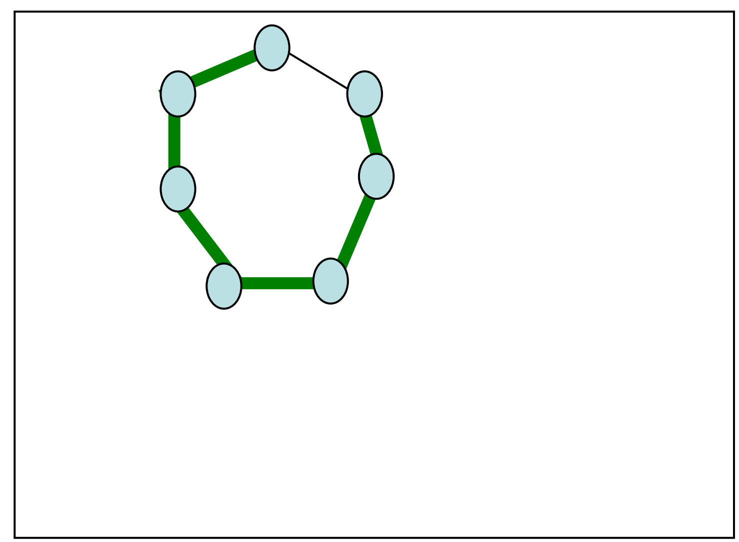


2 collective additive tree *2*-spanners



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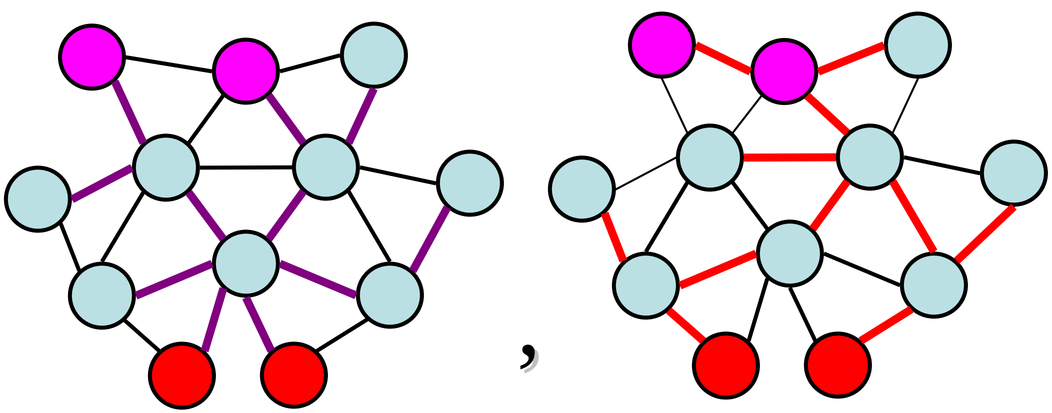
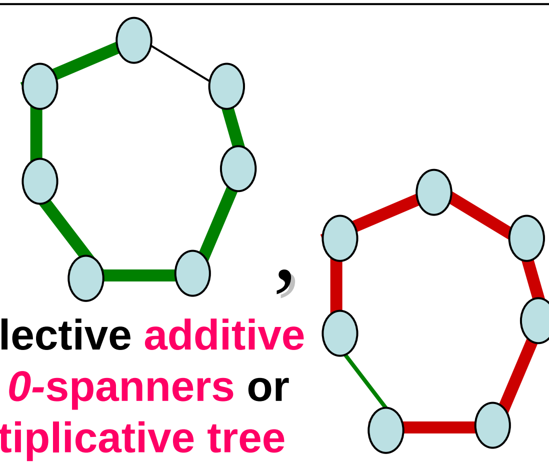


2 **collective additive tree *2*-spanners**



2 **collective additive tree *0*-spanners or multiplicative tree 1-spanners**

Applications of Collective Tree Spanners

- message routing in networks

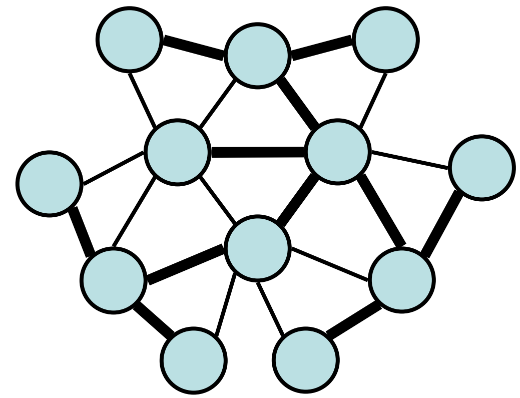
Efficient routing schemes are known for trees but not for general graphs. For any two nodes, we can route the message between them in one of the trees which approximates the distance between them.

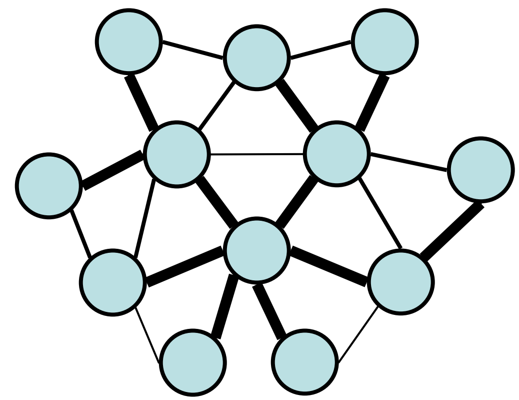
- *(μ log²n/ log log n)-bit labels,*

- *O(μ) initiation, O(1) decision*

- solution for sparse *t*-spanner problem

If a graph admits a system of μ collective additive tree *r*-spanners, then the graph admits a sparse additive *r*-spanner with at most μ*(n-1)* edges, where *n* is the number of nodes.





2 collective tree 2-spanners for *G*

Previous results on the collective tree spanners problem

(Dragan, Yan, Lomonosov [SWAT'04])

(Corneil, Dragan, Köhler, Yan [WG'05])

- chordal graphs, chordal bipartite graphs
  - log *n* collective additive tree *2*-spanners in polynomial time
  - $\Omega(n^{1/2})$ or $\Omega(n)$ trees necessary to get +1
  - no constant number of trees guaranties +2 (+3)
- circular-arc graphs
  - 2 collective additive tree *2*-spanners in polynomial time
- *c*-chordal graphs
  - log *n* collective additive tree $2\lfloor c/2 \rfloor$-spanners in polynomial time
- interval graphs
  - log *n* collective additive tree 1-spanners in polynomial time
  - no constant number of trees guaranties +1

Previous results on the collective tree spanners problem

(Dragan, Yan, Corneil [WG'04])

- AT-free graphs
  - include: interval, permutation, trapezoid, co-comparability
  - 2 collective additive tree *2*-spanners in linear time
  - an additive tree *3*-spanner in linear time (before)
- graphs with a dominating shortest path
  - an additive tree *4*-spanner in polynomial time (before)
  - 2 collective additive tree *3*-spanners in polynomial time
  - 5 collective additive tree *2*-spanners in polynomial time
- graphs with asteroidal number an(G)=k
  - k(k-1)/2 collective additive tree 4-spanners in polynomial time
  - k(k-1) collective additive tree 3-spanners in polynomial time

Previous results on the collective tree spanners problem

(Gupta, Kumar,Rastogi [SICOMP'05])

- the only paper (before) on collective multiplicative tree spanners in weighted planar graphs
- any weighted planar graph admits a system of *O(log n)* collective multiplicative tree 3-spanners
- they are called there the tree-covers.
- it follows from (Corneil, Dragan, Köhler, Yan [WG'05]) that
  - no constant number of trees guaranties +c (for any constant c)

New results on collective additive tree spanners of weighted graphs with bounded parameters

Graph class	μ	r
planar	$O(\sqrt{n})$	*0*
with genus g	$O(\sqrt{gn})$	*0*
W/o an h-vertex minor	$O(\sqrt{h^3 n})$	*0*
tw(G) ≤ k-1	$k\log_2 n$	*0*
cw(G) ≤ k	$k\log_{3/2} n$	2w
c-chordal	*next*	*slide*

$\Omega(\sqrt{n}\log\log n/\log^2 n)$ to get +0

No constant number of trees guaranties $+r$ for any constant r even for outer-planar graphs

$\Omega(n)$ to get +1

- w is the length of a longest edge in G

New results on collective additive tree spanners of weighted c-chordal graphs

Graph class	μ	r
c-chordal *(c>4)*	$\log_2 n$	$2\left\lfloor \frac{c}{2} \right\rfloor w$
	$4\log_2 n$	$2(\left\lfloor \frac{c}{3} \right\rfloor + 1)w$
	$5\log_2 n$	$2\left\lfloor \frac{c+2}{3} \right\rfloor w$
4-chordal	$6\log_2 n$	$2w$
weakly chordal	$4\log_2 n$	$2w$

No constant number of trees guaranties $+r$ for any constant r even for weakly chordal graphs

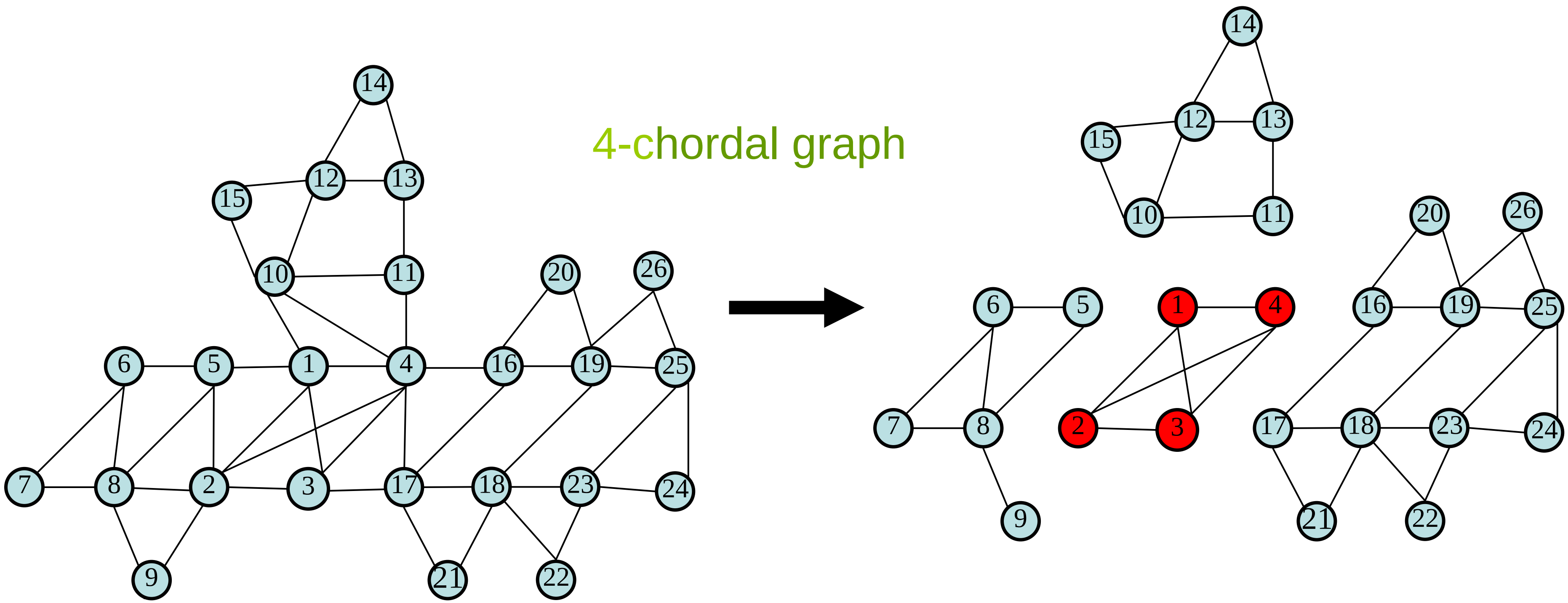
(α, γ, r)-Decomposable Graphs

- A graph *G=(V, E)* is (α, γ, r)-decomposable if there exists a vertex-separator ***S*** in ***G*** such that

Balanced separator: each conn. comp. of *G-S* has $\leq \alpha n$ vertices;

Bounded *r*-dominating set: *S* has an *r*-dominating set *D* in *G* with $|D| \leq \gamma$;

Hereditary family: any induced subgraph of *G* is (α, γ, r)-decomposable.



Main Results of the Paper

Theorem: Any (α, γ, r)-decomposable graph admits a system of at most $\gamma \log_{1/\alpha} n$ collective additive tree $2r$-*spanners.*

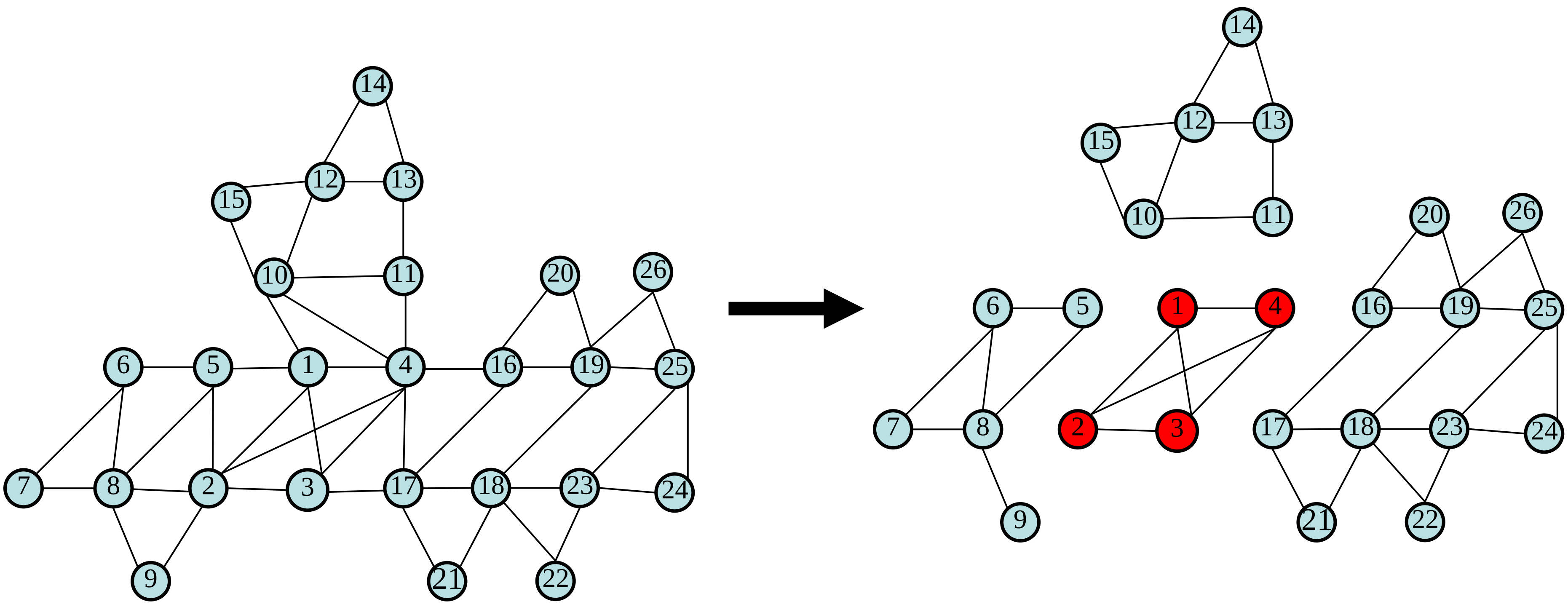
+

Graph class	decomposition
planar	$(2/3, \sqrt{6n}, 0)$
with genus *g*	$(2/3, O(\sqrt{gn}), 0)$
W/o an *h*-vertex minor	$(2/3, O(\sqrt{h^3 n}), 0)$
tw(G) ≤ k-1	$(1/2, k, 0)$
cw(G) ≤ k	$(2/3, k, w)$
c-chordal	$(1/2, 1, \lfloor c/2 \rfloor w)$, $(1/2, 5, \lfloor (c+2)/3 \rfloor w)$, $(1/2, 4, (\lfloor c/3 \rfloor + 1) w)$

Polynomial time constructions

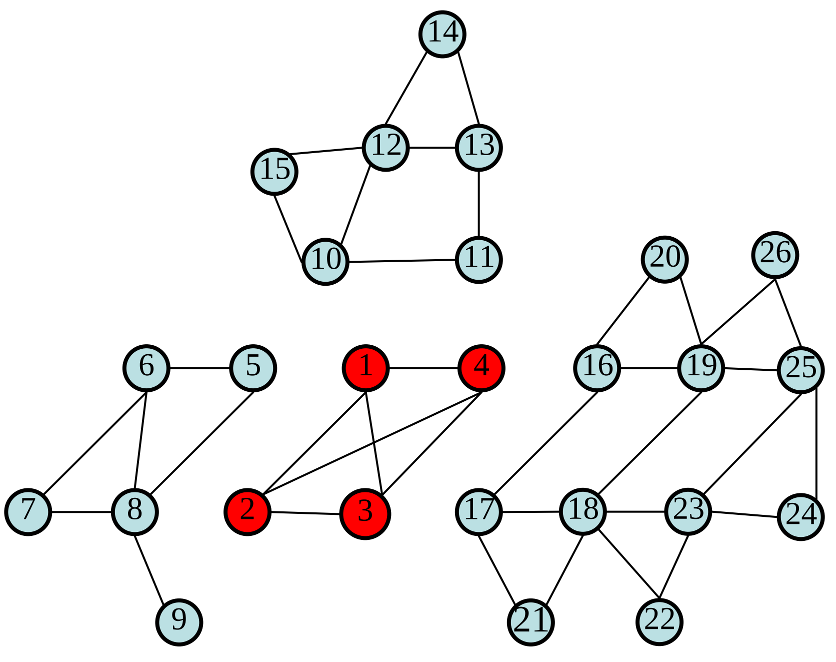
Constructing a Rooted Balanced Decomposition Tree for an (α, γ, r)-Decomposable Graph

- Find a good *balanced separator S* of *G*.



Constructing a Rooted Balanced Decomposition Tree for an (α, γ, r)-Decomposable Graph

- Use *S* as the *root* of the *rooted balanced decomposition tree*.



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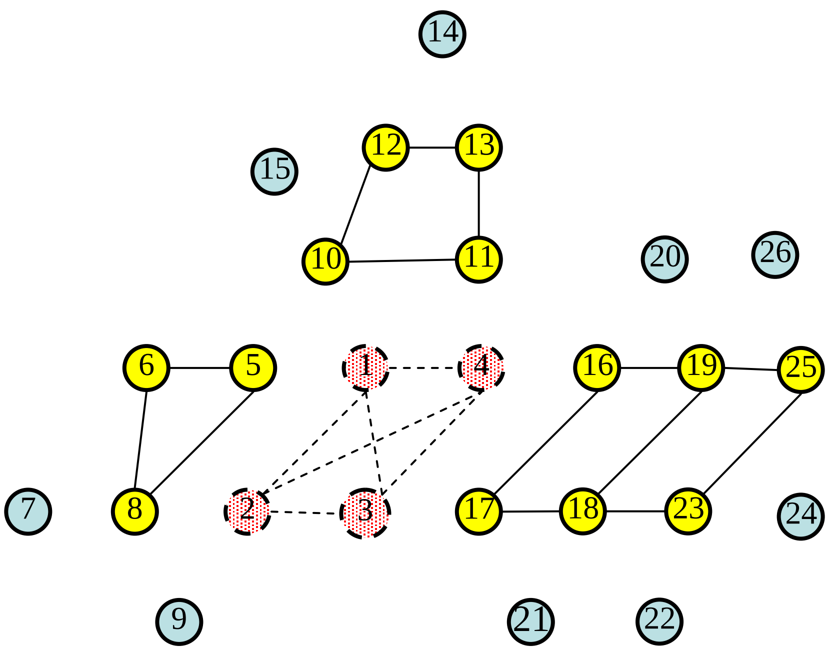
Constructing a Rooted Balanced Decomposition Tree for an (α, γ, r)-Decomposable Graph

- For each connected component of *G-S,* find its good *balanced separator*.



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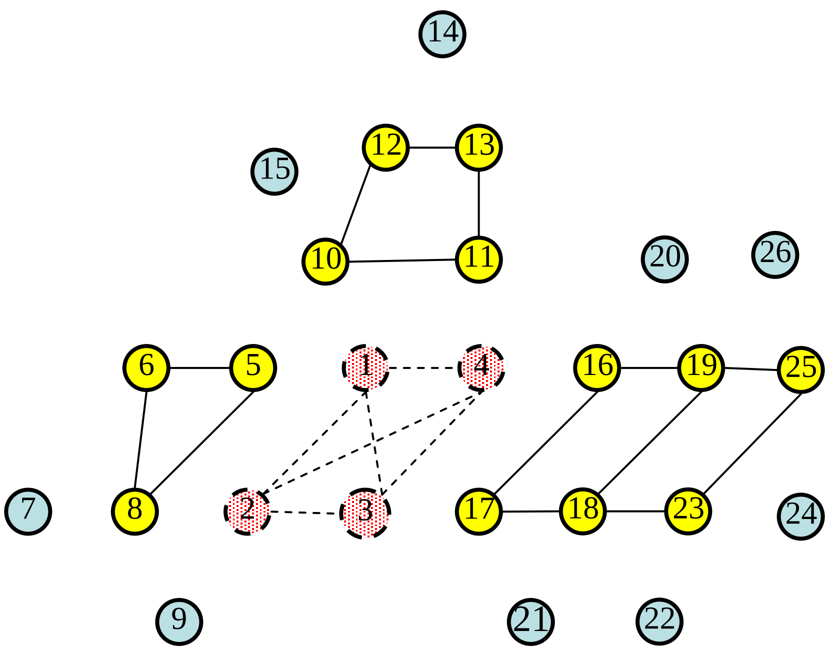
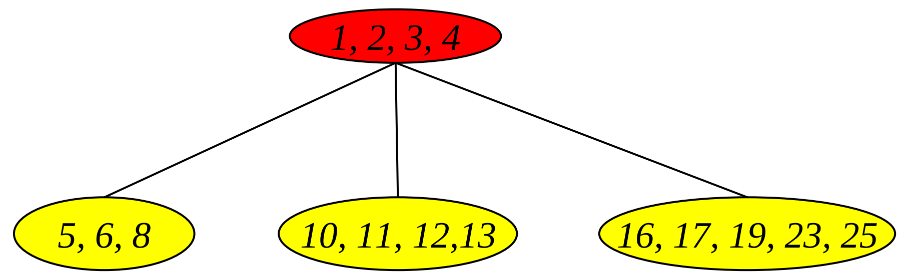
Constructing a Rooted Balanced Decomposition Tree for an (α, γ, r)-Decomposable Graph

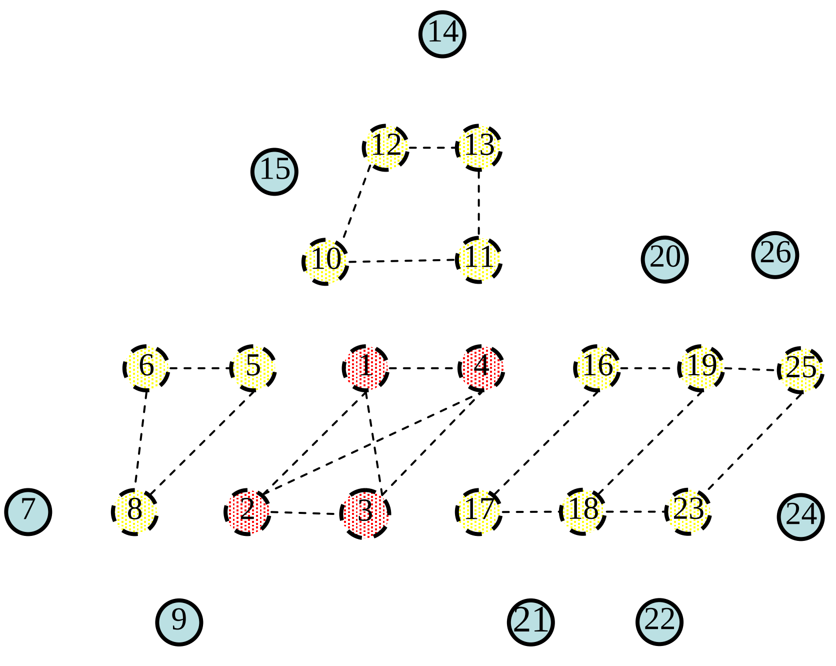
- Use the separators as nodes of the *rooted balanced decomposition tree* and let *S* be their father.

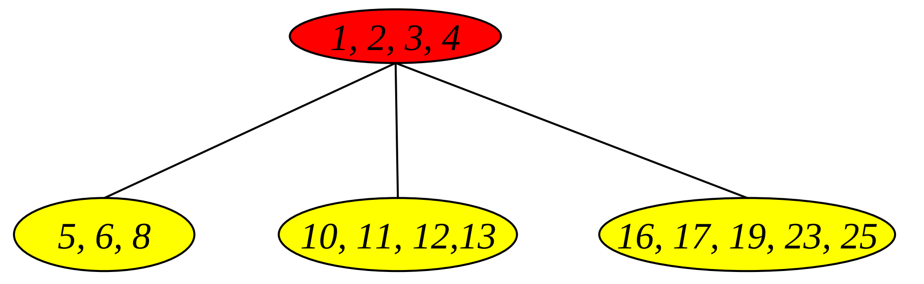




Constructing a Rooted Balanced Decomposition Tree for an (α, γ, r)-Decomposable Graph

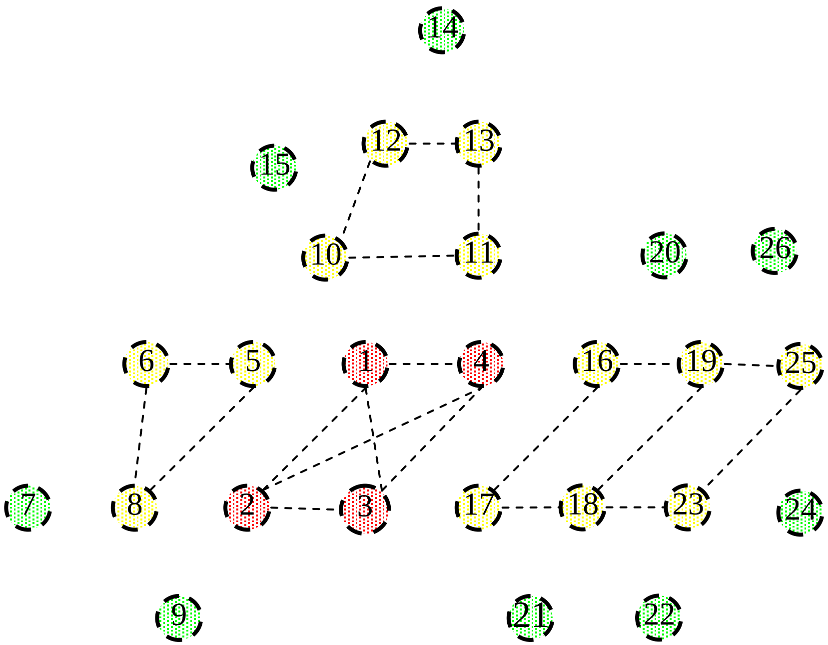
- Recursively repeat previous procedure until each connected component has an *r*-dominating set of size at most γ.

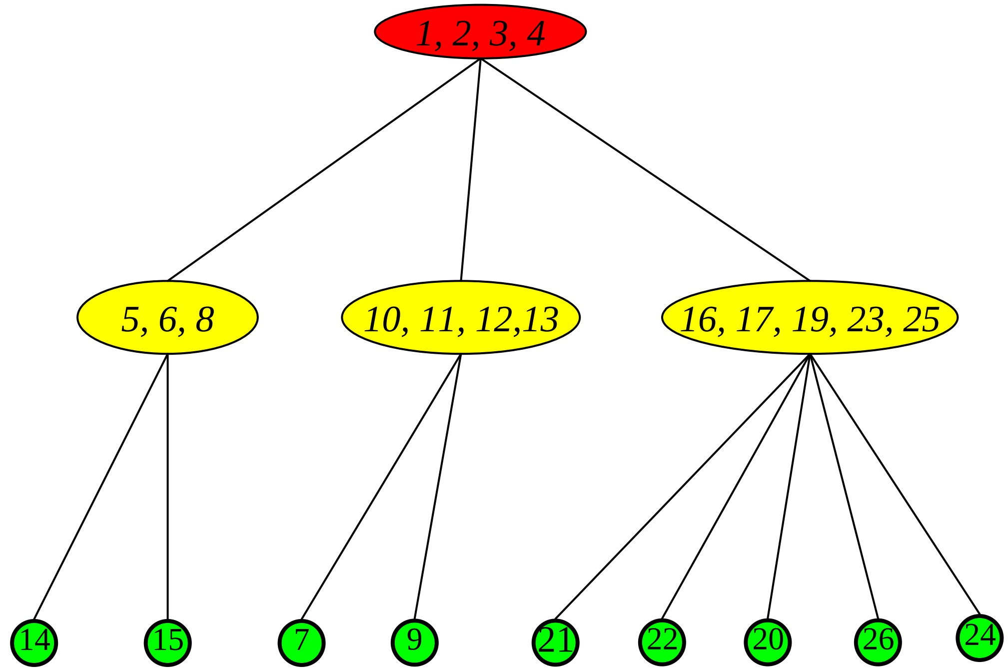




Constructing a Rooted Balanced Decomposition Tree for an (α, γ, r)-Decomposable Graph

- Get the *rooted balanced decomposition tree*.





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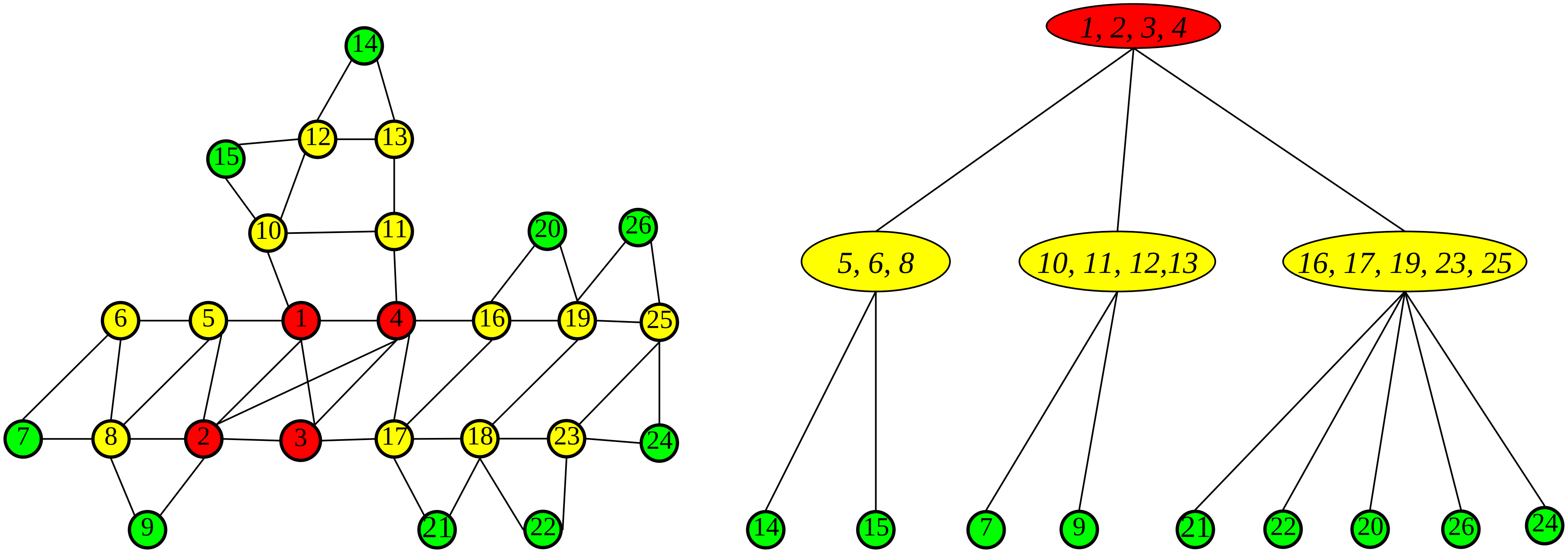
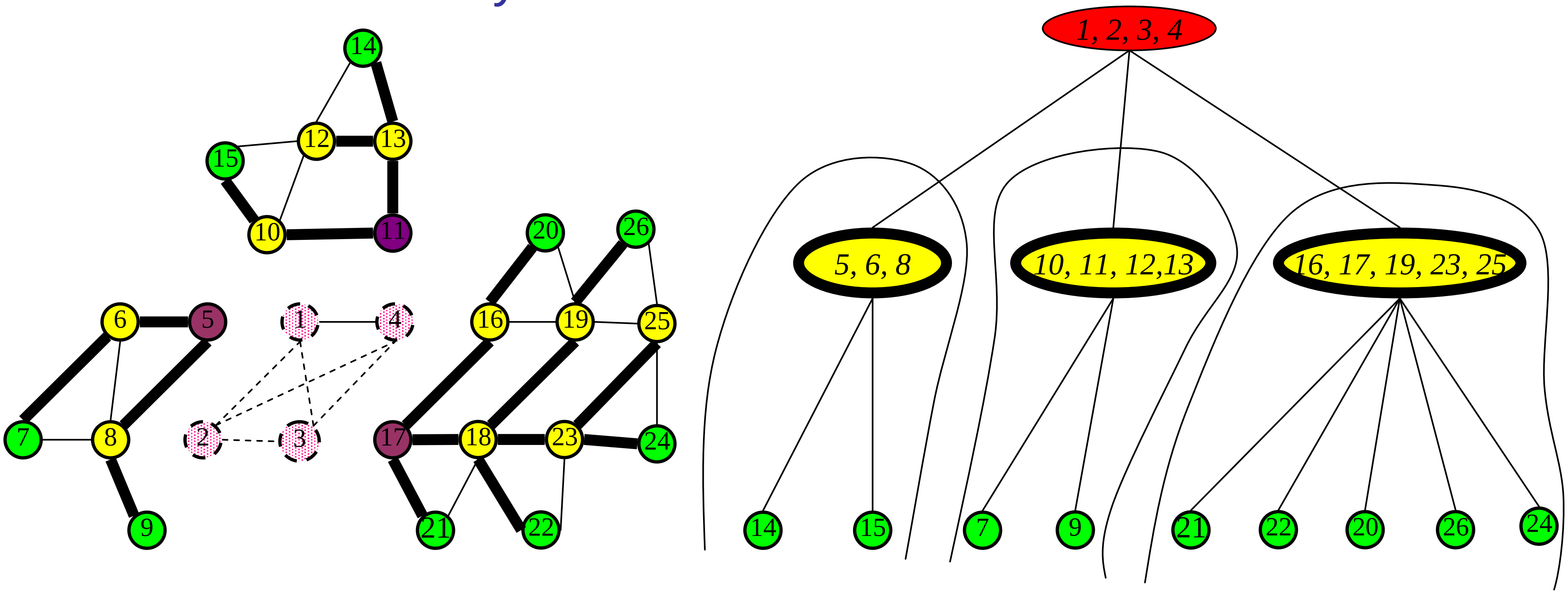
Rooted Balanced Decomposition Tree



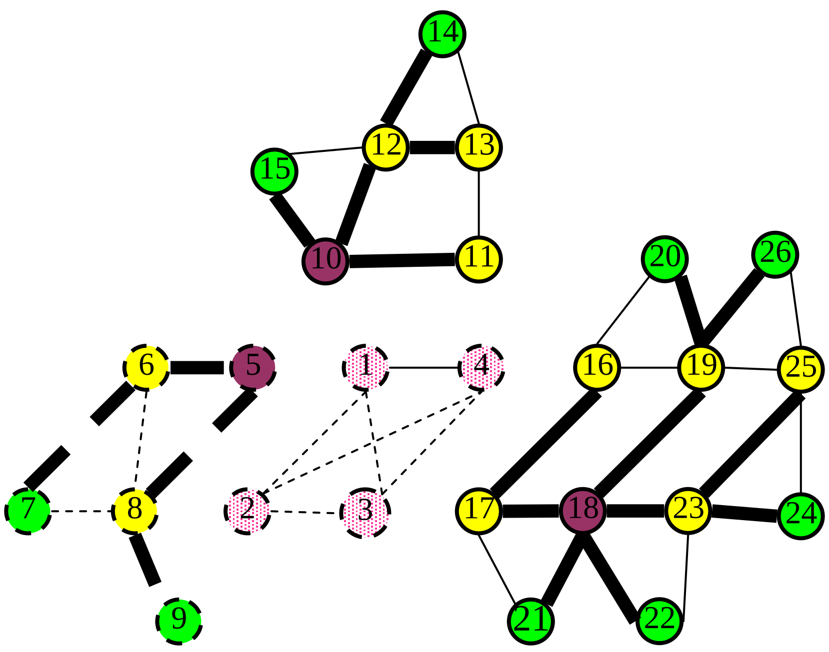
Constructing Local Spanning Trees

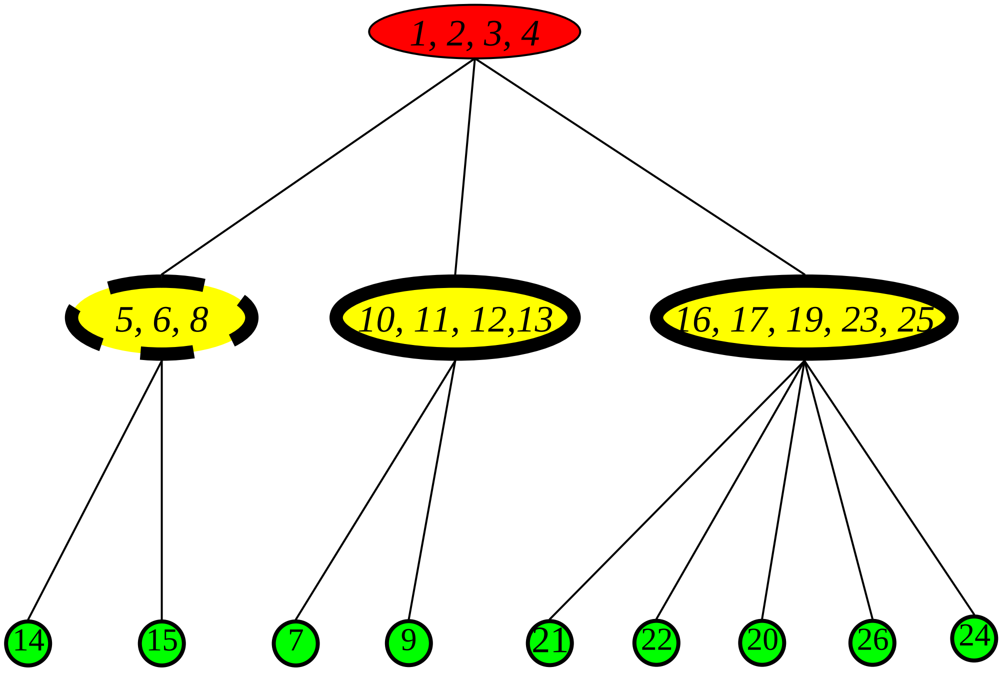
- for each layer of the decomposition tree, construct *local spanning trees (shortest path trees in the subgraph)*.
- we use second layer for illustration.



Constructing Local Spanning Trees

• each time, pick a different vertex from the *r*-dominating set to grow a shortest path tree in the subgraph.





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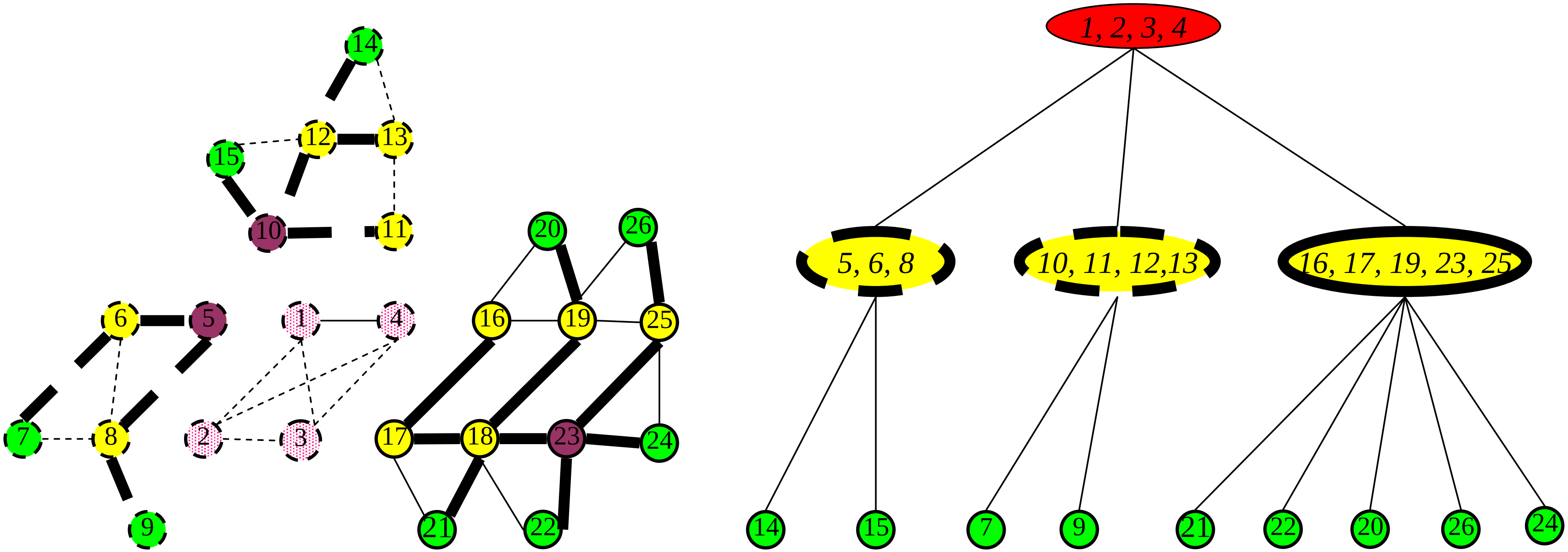
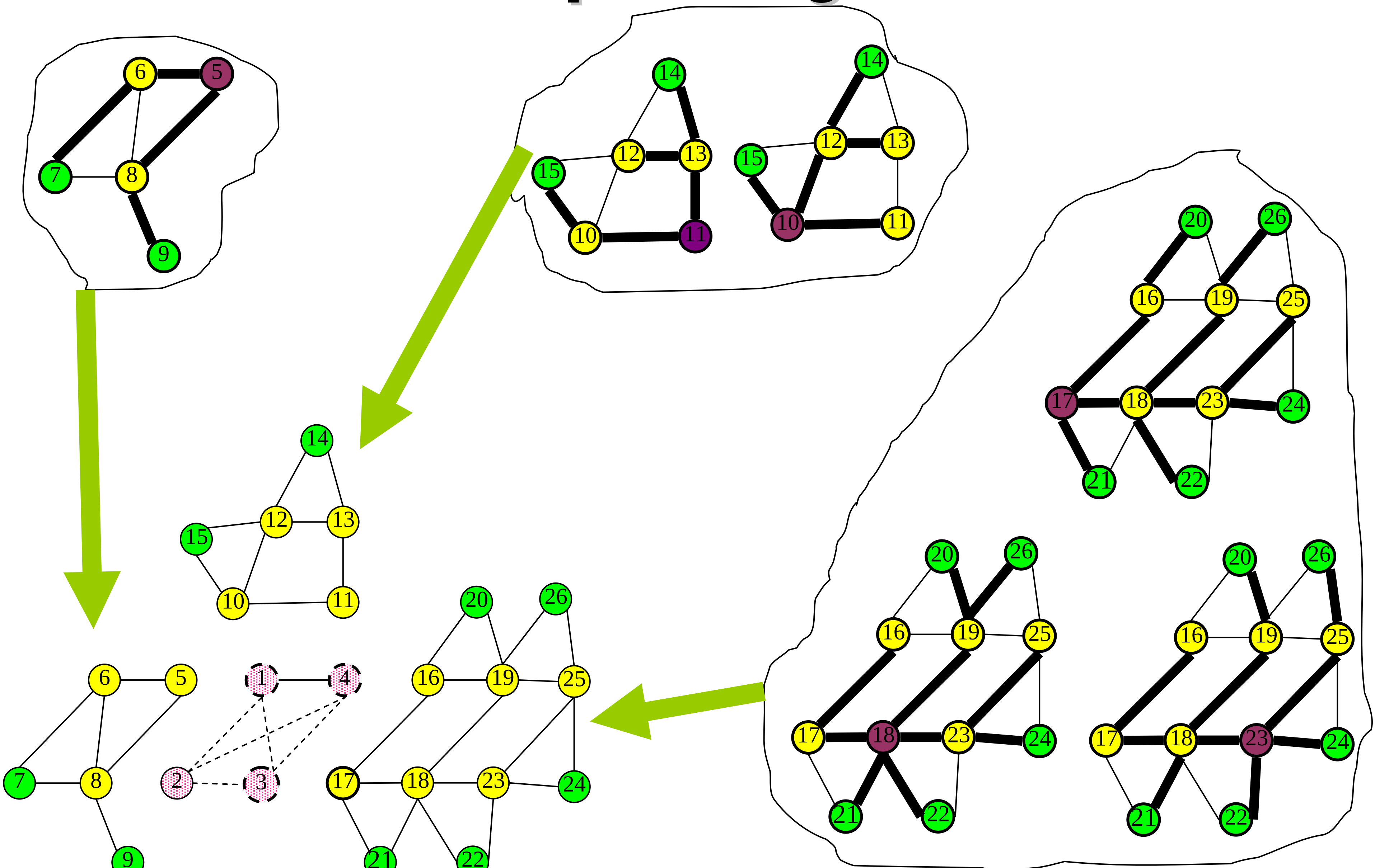
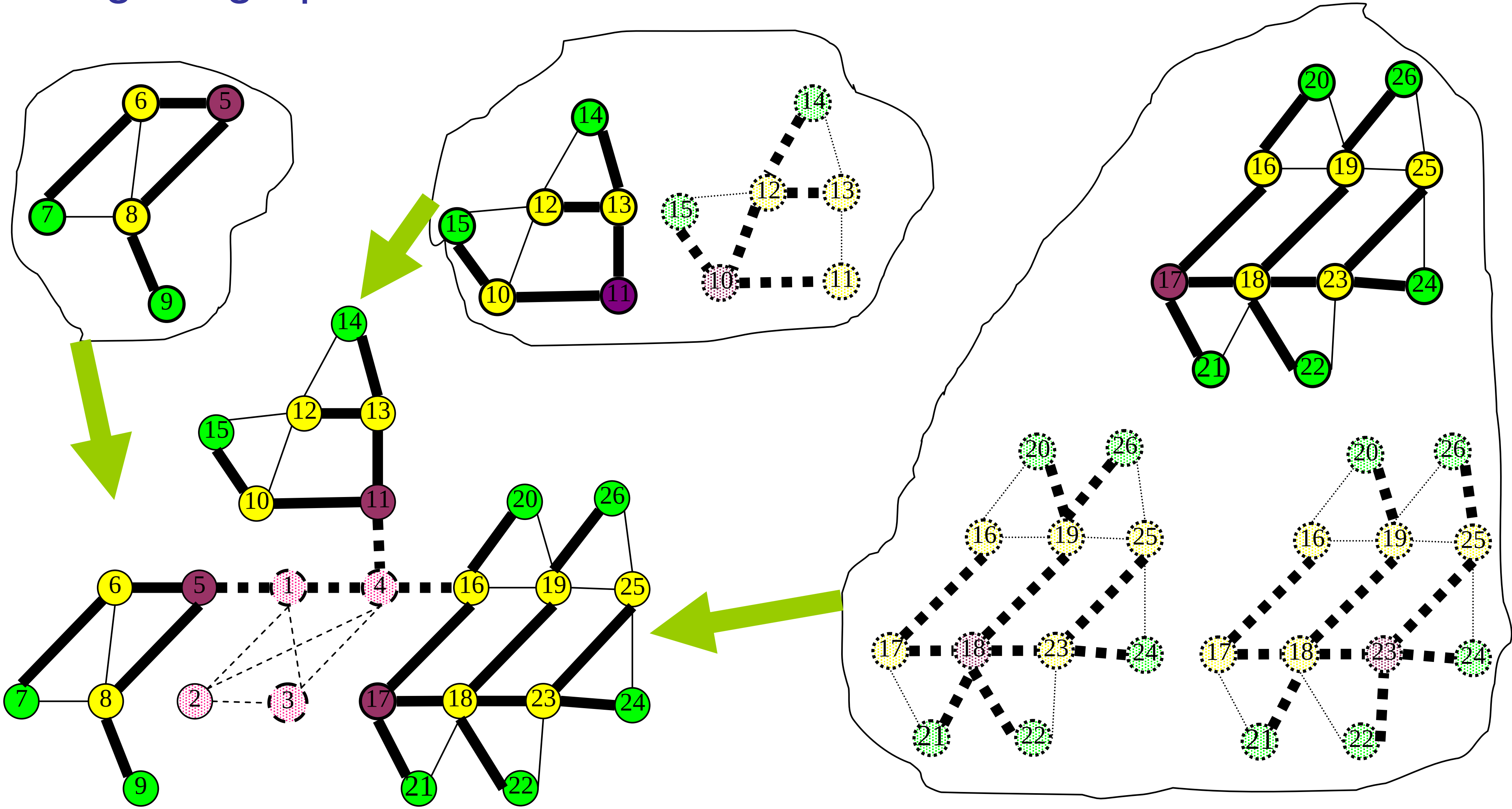


Local Spanning Trees



Spanning Trees Construction

- Connect *local spanning trees* to form spanning trees for the original graph.



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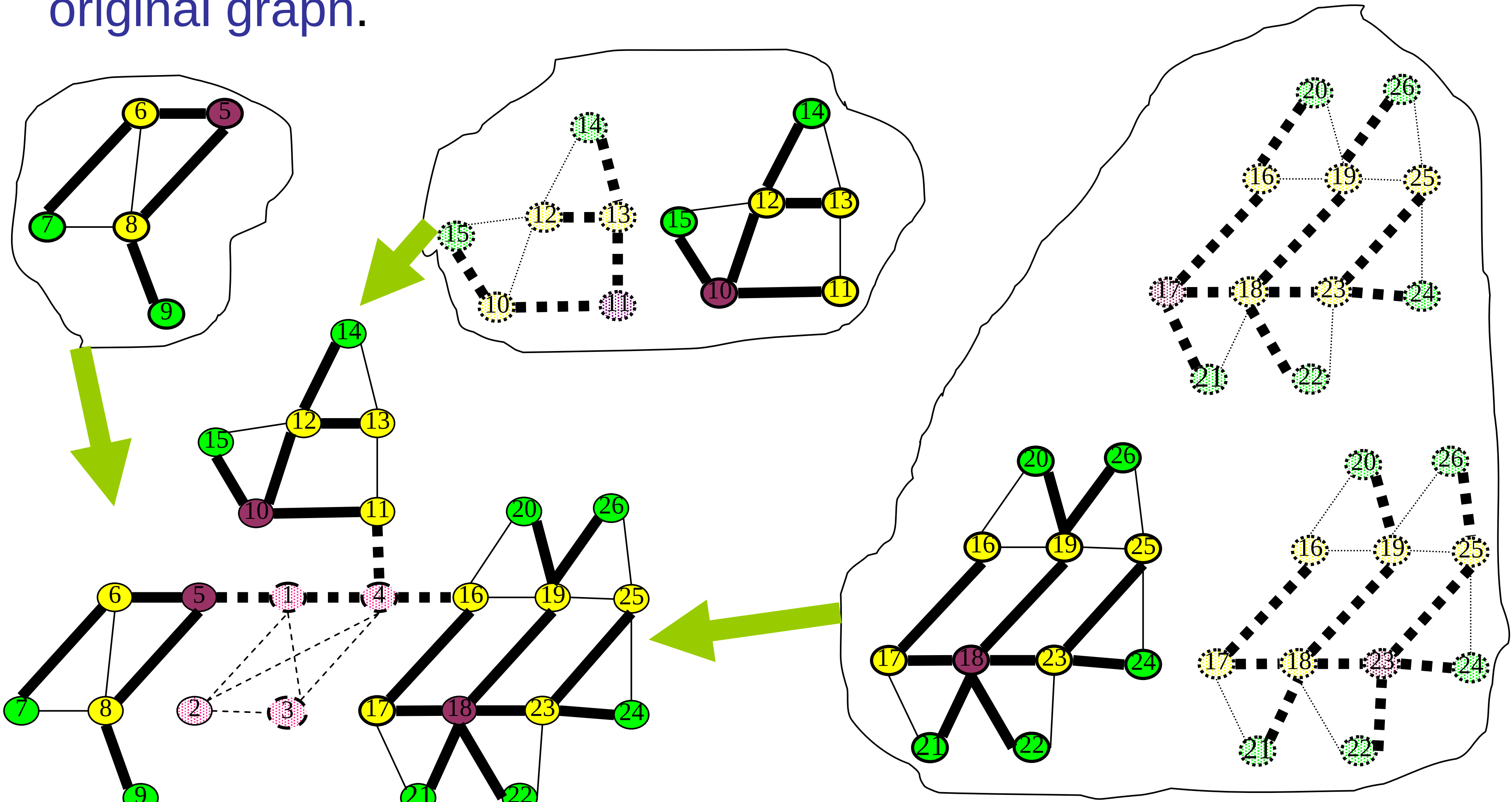

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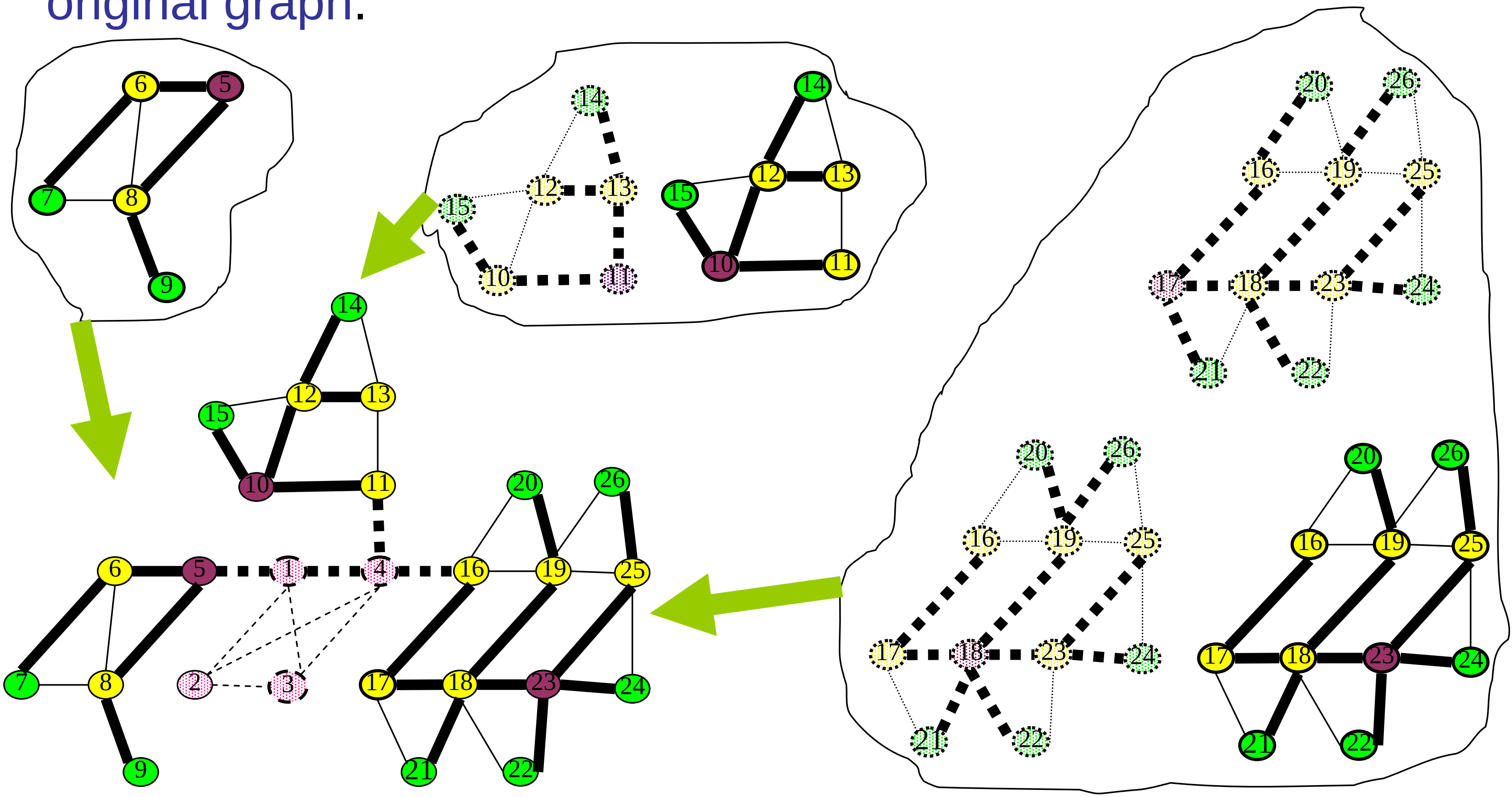
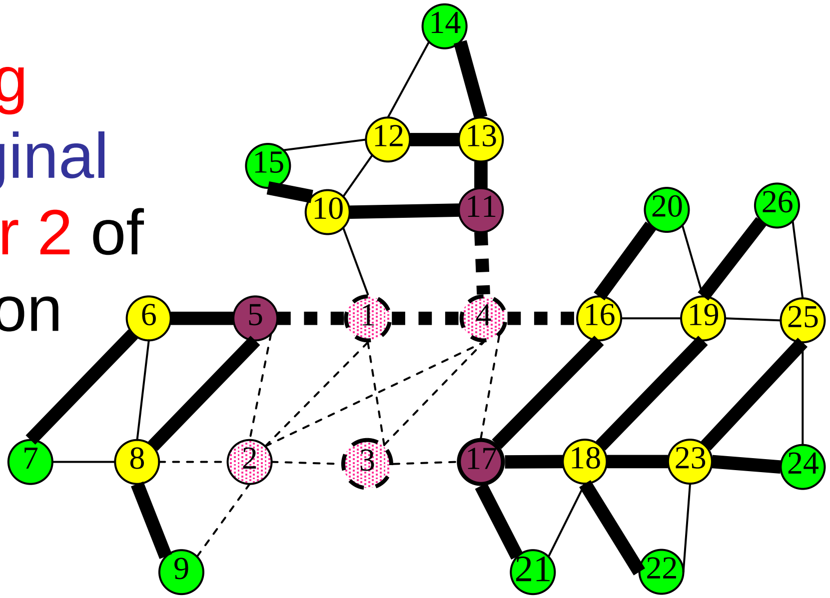
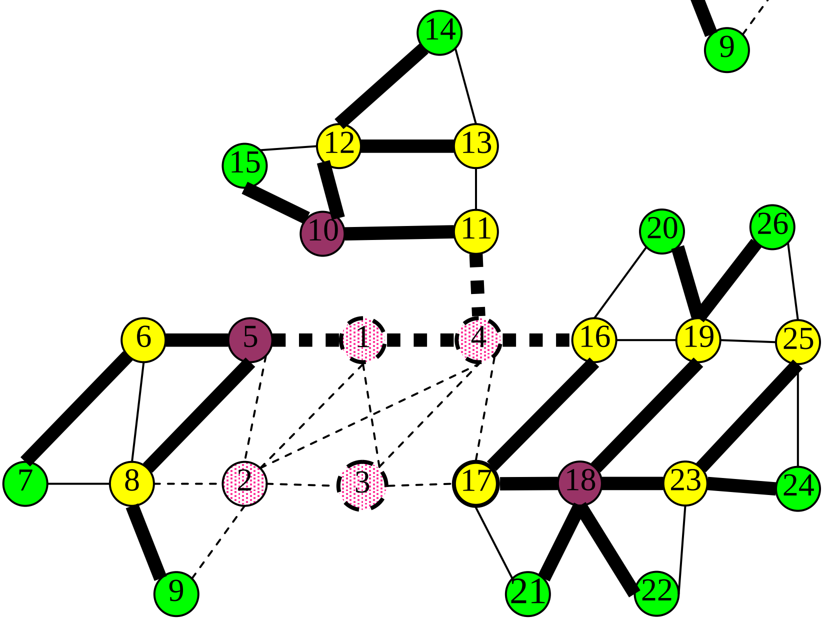
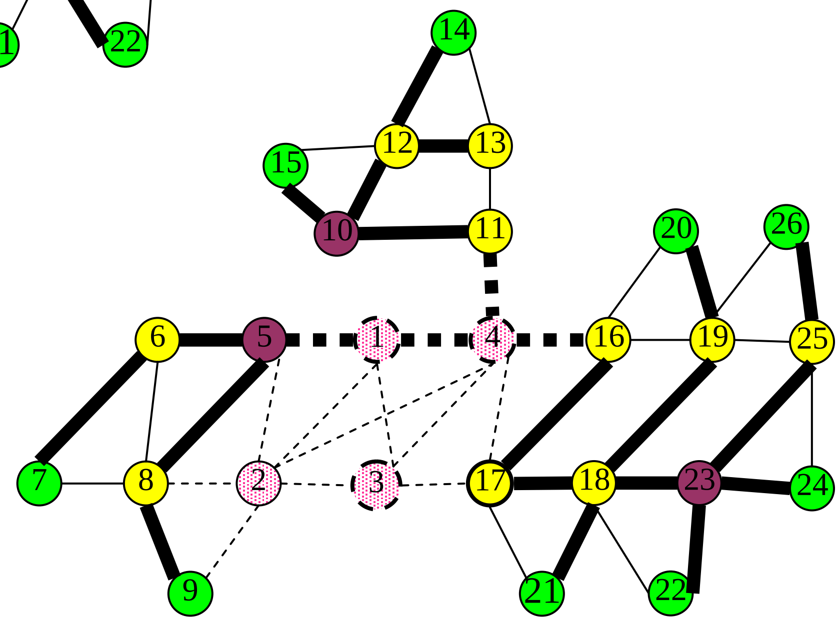


Spanning Trees Construction

- Three spanning trees for the original graph w.r.t. layer 2 of the decomposition tree.

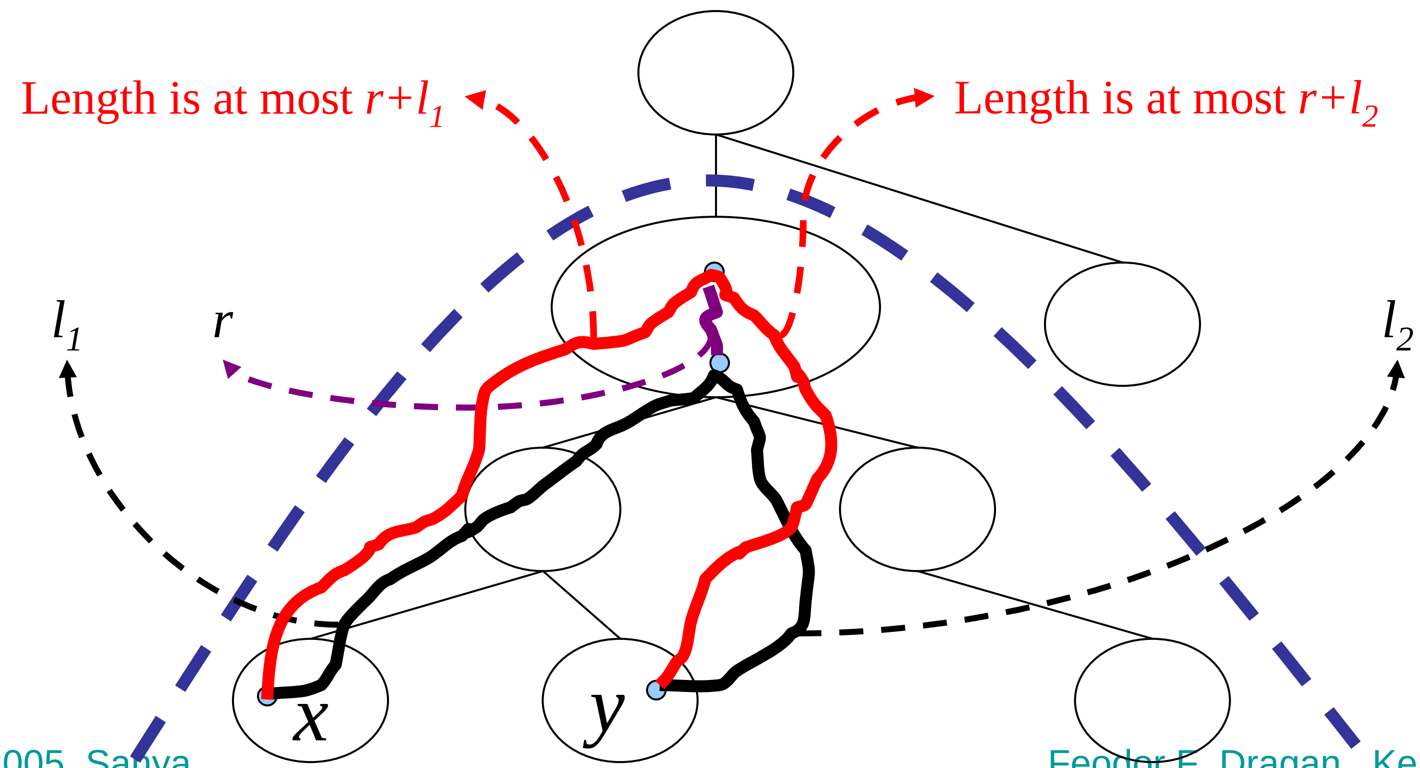






Analysis

Theorem: Any (α, γ, r)-decomposable graph admits a system of at most $\gamma \log_{1/\alpha} n$ collective additive tree $2r$-*spanners.*



Further Results

➔ Any (α, γ, r)-decomposable graph G admits an additive $2r$-spanner with at most $\gamma n \log_{1/\alpha} n$ edges which can be constructed in polynomial time.

➔ Any (α, γ, r)-decomposable graph G admits a routing scheme of deviation $2r$ and with labels of size $O(\gamma \log_{1/\alpha} n \log^2 n / \log \log n)$ bits per vertex. Once computed by the sender in $\gamma \log_{1/\alpha} n$ time, headers never change, and the routing decision is made in constant time per vertex.

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Open questions and future plans

- Given a graph $G=(V, E)$ and two integers μ and ***r*,** what is the complexity of finding a system of μ collective additive (multiplicative) tree ***r***-spanner for G? (Clearly, for most μ and ***r*,** it is an NP-complete problem.)

- Find better trade-offs between μ and ***r*** for planar graphs, genus ***g*** graphs and graphs w/o an ***h***-minor.

- We may consider some other graph classes. What's the optimal μ for each ***r***?

- More applications of collective tree spanner…

Thank You