Tree Spanners on Chordal Graphs: Complexity, Algorithms, Open Problems

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Abstract. A *tree t-spanner* T in a graph G is a spanning tree of G such that the distance in T between every pair of vertices is at most t times their distance in G. The TREE t-SPANNER problem asks whether a graph admits a tree t-spanner, given t. We substantially strengthen the hardness result of Cai and Corneil [*SIAM J. Discrete Math.* 8 (1995) 359–387] by showing that, for any $t \geq 4$, TREE t-SPANNER is **NP**-complete even on chordal graphs of diameter at most $t+1$ (if t is even), respectively, at most $t+2$ (if t is odd). Then we point out that every chordal graph of diameter at most $t-1$ (respectively, $t-2$) admits a tree t-spanner whenever $t \geq 2$ is even (respectively, $t \geq 3$ is odd), and such a tree spanner can be constructed in linear time.

The complexity status of TREE 3-SPANNER still remains open for chordal graphs, even on the subclass of undirected path graphs that are strongly chordal as well. For other important subclasses of chordal graphs, such as very strongly chordal graphs (containing all interval graphs), 1-split graphs (containing all split graphs) and chordal graphs of diameter at most 2, we are able to decide TREE 3-SPANNER efficiently.

1 Introduction and Results

All graphs considered are connected. For two vertices in a graph G, $d_G(x,y)$ denotes the distance between x and y; that is, the number of edges in a shortest path in G joining x and y. The value $\text{diam}(G) := \max d_G(x,y)$ is the *diameter* of the graph G.

Let $t \geq 2$ be a fixed integer. A spanning tree T of a graph G is a *tree t-spanner* of G if, for every pair of vertices x, y of G, $d_T(x,y) \leq t \cdot d_G(x,y)$. TREE t-SPANNER is the following problem: Given a graph G, does G admit a tree t-spanner?

There are many applications of tree spanners in different areas; especially in distributed systems and communication networks. In [1], for example, it was shown that tree spanners can be used as models for broadcast operations; see also [21]. Moreover, tree spanners also appear in biology [2], and in [25], tree

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spanners were used in approximating the bandwidth of graphs. We refer to [7,6, 22,24] for more background information on tree spanners.

In [6] Cai and Corneil gave a linear time algorithm solving Tree 2-Spanner and proved that Tree t-Spanner is **NP**-complete for any $t \geq 4$. A graph is *chordal* if it does not contain any chordless cycle of length at least four. For a popular subclass of chordal graph, the strongly chordal graphs, Brandstädt *et al.* [3] proved that, for every $t \geq 4$, Tree t-Spanner is solvable in linear time. Indeed, they show that every strongly chordal graph admits a tree 4-spanner. In contrast, one of our results is

Theorem 1 *For any $t \geq 4$,* Tree t-Spanner *is* ***NP****-complete on chordal graphs of diameter at most $t+1$ (if t is even), respectively, of at most $t+2$ (if t is odd).*

Comparing with a recent result due to Papoutsakis [20], it is interesting to note that the union of two tree t-spanners, $t \geq 4$, may contain chordless cycles of any length. This perhaps indicates the difficulty in proving Theorem 1. Indeed, our reduction from 3SAT to Tree t-Spanner given in Section 2 is quite involved.

Moreover, to the best of our knowledge, Theorem 1 is the first hardness result for Tree t-Spanner on a restricted, well-understood graph class. Notice that in [12] it is shown that Tree t-Spanner, $t \geq 4$, is **NP**-complete on planar graphs *if* the integer t is part of the input.

In view of the diameter constraints in Theorem 1, we note that Tree t-Spanner remains open on chordal graphs of diameter t (t is even) and of diameter $t-1$, t or $t+1$ (if t is odd). For "smaller" diameter we have

Theorem 2 *For any even integer t, every chordal graph of diameter at most $t-1$ admits a tree t-spanner, and such a tree spanner can be constructed in linear time. For any odd integer t, every chordal graph of diameter at most $t-2$ admits a tree t-spanner, and such a tree spanner can be constructed in linear time.*

We were able also to show that chordal graphs of diameter at most $t-1$ (t is odd) admit tree t-spanners if and only if chordal graphs of diameter 2 admit tree 3-spanners. This result is used to show that every chordal graph of diameter at most $t-1$ (t is odd), if it is planar or a k-tree, for $k \leq 3$, has a tree t-spanner and such a tree spanner can be constructed in polynomial time. Note that, for any fixed t, there is a 2-tree without a tree t-spanner [16]. So, even those kind of results are of interest. Unfortunately, the reduction above (from arbitrary odd t to $t = 3$) is of no direct use for general chordal graphs because not every chordal graph of diameter at most 2 admits a tree 3-spanner. One of our theorems characterizes those chordal graphs of diameter at most 2 that admit such spanners.

We now discuss Tree t-Spanner on important subclasses of chordal graphs. It is well-known that chordal graphs are exactly the intersection graphs of subtrees in a tree [13]. Thus, intersection graphs of paths in a tree, called *path graphs*, form a natural subclass of chordal graphs. Tree t-Spanner remains unresolved even on this natural subclass of chordal graphs.

The complexity status of Tree 3-Spanner remains a long standing open problem. However, it can be solved efficiently for many particular graph classes, such as *cographs* and *complements of bipartite graphs* [5], *directed path graphs* [17] (hence for all interval graphs [16,18,23]), *split graphs* [5,16,25], *permutation graphs* and *regular bipartite graphs* [18], *convex bipartite graphs* [25], and recently for *planar graphs* [12]. In [5,19,20], some properties of graphs admitting a tree 3-spanner are discussed.

On chordal graphs, however, TREE 3-SPANNER remains even open on path graphs which are strongly chordal as well. For some important subclasses of chordal graphs we can decide TREE 3-SPANNER efficiently. Graphs considered in the theorem below are defined in Section 5.

Theorem 3 *All* very strongly chordal *graphs and all* 1-split *graphs admit a tree* 3*-spanner, and such a tree* 3*-spanner can be constructed in linear time.*

Theorem 4 *For a given chordal graph* $G = (V, E)$ *of diameter at most 2,* TREE 3-SPANNER *can be decided in* $O(|V||E|)$ *time. Moreover, a tree 3-spanner of* G*, if it exists, can be constructed within the same time bound.*

Theorem 3 improves previous results on tree 3-spanners in interval graphs [16,18, 23] and on split graphs [5,16,25]. The complexity status of TREE t-SPANNER on chordal graphs considered in this paper is summarized in Figure 1 and Table 1.

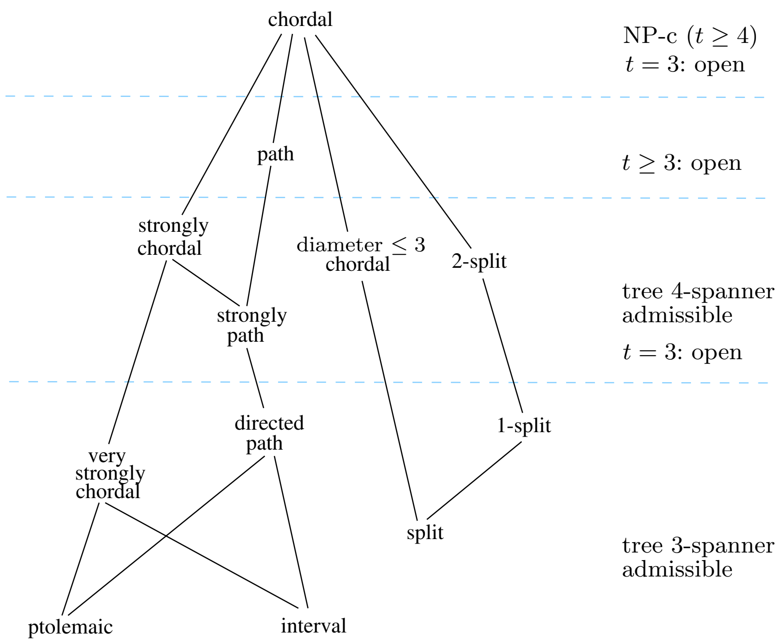

Fig. 1. The complexity status of TREE t-SPANNER on chordal graphs and important subclasses

Table 1. The complexity status of TREE t-SPANNER on chordal graphs under diameter constraints

Diameter at most	Complexity
$t+2$, $t \geq 5$ odd	**NP**-complete
$t+1$, $t \geq 4$ even	**NP**-complete
$t+1$, $t \geq 3$ odd	?
t, $t \geq 3$	?
$t-1$, $t \geq 5$ odd	?
$t-1$, $t=3$	polynomial time
$t-1$, $t \geq 2$ even	linear time
$t-2$, $t \geq 3$ odd	linear time

Notations and definitions not given here may be found in any standard textbook on graphs and algorithms. We write xy for the edge joining vertices x, y; x and y are also called *endvertices* of xy. For a set C of vertices, $N(C)$ denotes the set of all vertices outside C adjacent to a vertex in C; $N(x)$ stands for $N(\{x\})$ and $deg(x)$ stands for $|N(x)|$. We set $d(v, C) := \min\{d(v, x) : x \in C\}$. The *eccentricity* of a vertex v in G is the maximum distance from v to other vertices in G. The *radius* $r(G)$ of G is the minimum of all eccentricities and the *diameter* diam(G) of G is the maximum of all eccentricities. A *cutset* of a graph is a set of vertices whose deletion disconnects the graph. A graph is *non-separable* if it has no one-element cutset, and *triconnected* if it has no cutset with ≤ 2 vertices. *Blocks* in a graph are maximal non-separable subgraphs of that graph.

Clearly, a graph contains a tree t-spanner if and only if each of its blocks contains a tree t-spanner. Note also that dividing a graph into blocks can be done in linear time. Thus, in this paper, we consider *non-separable graphs* only. Graphs having a tree t-spanner are called *tree t-spanner admissible.*

Finally, we will use of the following fact in checking whether a spanning tree is a tree t-spanner.

Proposition 1 ([6]) *A spanning tree T of a graph G is a tree t-spanner if and only if, for every edge xy of G, $d_T(x, y) \leq t$.*

2 NP-Completeness, $t \geq 4$

In this section we will show that, for any fixed $t \geq 4$, TREE t-SPANNER is **NP**-complete on chordal graphs. The proof is a reduction from 3SAT, for which the following family of chordal graphs will play an important role.

First, $S_1[x, y]$ stands for a triangle with two labeled vertices x and y. Next, for a fixed integer $k \geq 1$, $S_{k+1}[x, y]$ is obtained from $S_k[x, y]$ by taking to every edge $e \neq xy$ in $S_k[x, y]$ that belongs to exactly one triangle a new vertex v_e and joining v_e to exactly the two endvertices of e. We write also S_k for $S_k[x, y]$ for some suitable labeled vertices x, y. See Figure 2.

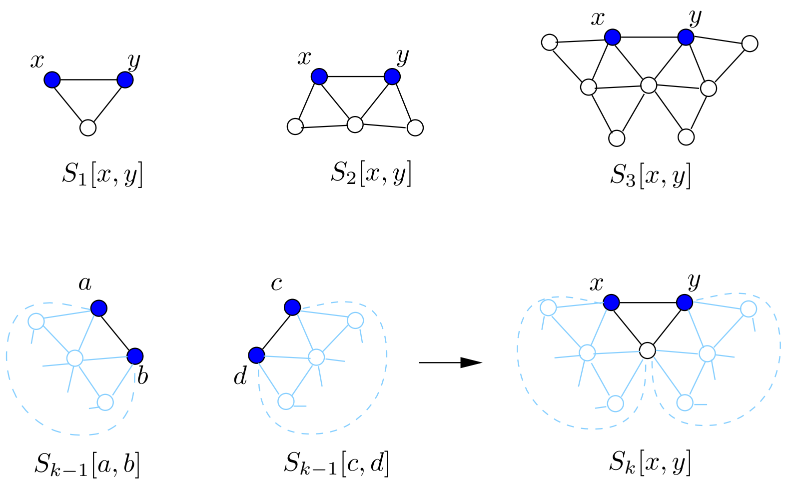

Fig. 2. The graph $S_k[x, y]$ obtained from $S_{k-1}[a, b]$ and $S_{k-1}[c, d]$ by identifying $b = d$ and joining $x = a$ with $y = c$

Equivalently, $S_{k+1}[x, y]$ is obtained from two disjoint $S_k[a, b]$ and $S_k[c, d]$ by identifying the two vertices c, d to a vertex z and joining the vertices $x := a$ and $y := b$ by an edge. With this notation, z is the common neighbor of x and y in $S_{k+1}[x, y]$, and we call $S_k[x, z]$ and $S_k[y, z]$ the two *corresponding* S_k in $S_{k+1}[x, y]$.

We denote by $S_k[x, y)$ the graph $S_k[x, y] - y$, that is, the graph obtained from $S_k[x, y]$ by deleting the vertex y. The following observations collect basic facts on S_k used in the reduction later.

Observation 1

(1) *For every* $v \in S_k[x, y]$, $d_{S_k[x,y]}(v, \{x, y\}) \le \lceil \frac{k}{2} \rceil$,
(2) $S_k[x, y]$ *has a tree* $(k + 1)$*-spanner containing the edge* xy,
(3) $S_k[x, y)$ *has a tree* k*-spanner* T *such that, for each neighbor* y' *of* y *in* $S_k[x, y)$, $d_T(x, y') \le k$.

Proofs of these and all other results are omitted in this extended abstract. They will be given in the journal version.

Observation 2 *Let* H *be an arbitrary graph and let* e *be an arbitrary edge of* H. *Let* K *be an* $S_k[x, y]$ *disjoint from* H. *Let* G *be the graph obtained from* H *and* K *by identifying the edges* e *and* xy*; see Figure 3. Suppose that* T *is a tree* t*-spanner in* G, $t > k$, *such that the* xy*-path in* T *belongs to* H. *Then*

(1) $d_T(x, y) \le t - k$, *and*
(2) *there exists an edge* $uv \in K$ *with* $d_T(u, v) \ge d_T(x, y) + k$.

Part (1) of Observation 2 indicates a way to force an edge xy to be a tree edge, or to force a path of the tree to belong to certain part of the graph: Choosing

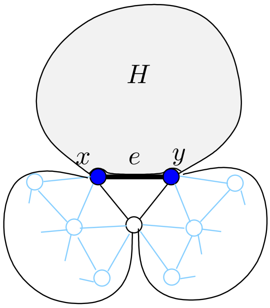

Fig. 3. The graph obtained from H and $S_k[x, y]$ by identifying the edge $e = xy$

$k = t - 1$ shows that xy must be an edge of T, or else the xy-path in T must belong to the part $S_{t-1}[x, y]$.

We now describe the reduction. Let F be a 3SAT formula with m clauses $C_j = (c_{j1}, c_{j2}, c_{j3})$ over n variables v_i. Set $\ell := \lfloor \frac{t}{2} \rfloor - 2$ and $\lambda := \lceil \frac{\ell}{2} \rceil$. Since $t \geq 4$, $\ell \geq 0$ and $\lambda \geq 0$.

For each variable v_i create the graph $G(v_i)$ as follows.

- Set $q_i^0 := v_i$, $q_i^{\ell+1} := \overline{v_i}$. We will use $u \in \{q_i^0, q_i^{\ell+1}\}$ as a vertex in our graph as well as a literal in the given 3SAT formula F.
- Take a clique Q_i on $\ell + 2$ vertices $q_i^0, \ldots, q_i^\ell, q_i^{\ell+1}$.
- For each edge $xy \in \{q_i^k q_i^{k+1} : 0 \leq k \leq \ell\}$ create an $S_{t-1}[x, y]$. No two of the S_{t-1} have a vertex in common unless those in $\{x, y\}$.
- Take a chordless path on vertices $s_i^0, s_i^1, \ldots, s_i^\lambda$ and edges $s_i^k s_i^{k+1}$, $0 \leq k < \lambda$.
- Connect each s_i^k, $0 \leq k \leq \lambda$, to exactly q_i^0 and $q_i^{\ell+1}$.
- For each edges $xy \in \{s_i^k s_i^{k+1} : 0 \leq k < \lambda\}$ create an $S_{t-2}[x, y]$.
- For each edges $xy \in \{s_i^0 q_i^0, s_i^0 q_i^{\ell+1}, s_i^\lambda q_i^0, s_i^\lambda q_i^{\ell+1}\}$ create an $S_{t-(\ell+2)}[x, y]$.

Note that the clique Q_i is a cutset of $G(v_i)$ and the components of $G(v_i) - Q_i$ are chordal. Thus, $G(v_i)$ is a chordal graph. See also Figure 4.

For each clause C_j create the graph $G(C_j)$ as follows. If t is even, $G(C_j)$ is simply a single vertex a_j. If t is odd, $G(C_j)$ is the graph $S_{t-1}[a_j^1, a_j^2]$. In any case, $G(C_j)$ is a chordal graph.

Finally, the graph $G = G(F)$ is obtained from all $G(v_i)$ and $G(C_j)$ by identifying all vertices s_i^0 to a single vertex s, and adding the following additional edges:

- connect every vertex in Q_i with every vertex in $Q_{i'}$, $i \neq i'$. Thus, the cliques Q_i, $1 \leq i \leq n$, form together a clique Q in G,
- for each literal $u_i \in \{q_i^0, q_i^{\ell+1}\}$, if $u_i \in C_j$ then connect u_i with a_j, respectively, with a_j^1 and a_j^2, according to the parity of t.

The description of the graph $G = G(F)$ is complete. Clearly, G can be constructed in polynomial time.

Lemma 1 *G is chordal, and* $\operatorname{diam}(G) \leq \begin{cases} t+1 \text{ } \textit{if } t \textit{ is even,} \\ t+2 \text{ } \textit{if } t \textit{ is odd.} \end{cases}$

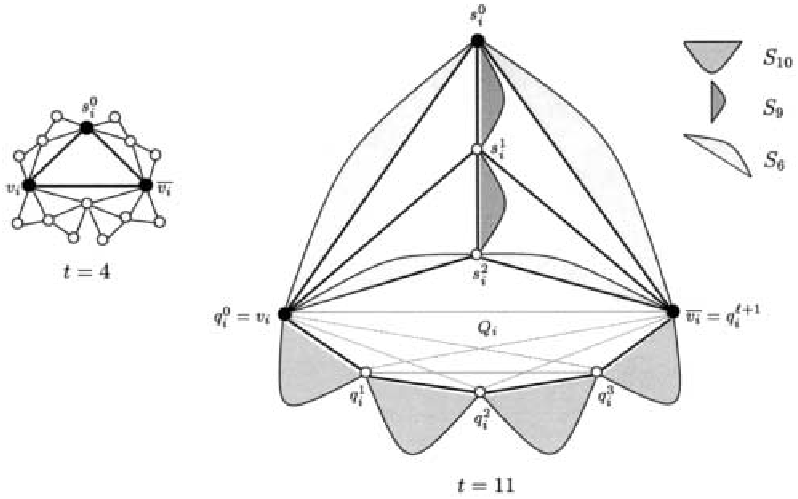

Fig. 4. The graph $G(v_i)$

Lemma 2 *Suppose G admits a tree t-spanner. Then F is satisfiable.*

Lemma 3 *Suppose F is satisfiable. Then G admits a tree t-spanner.*

Theorem 1 follows from Lemmas 1-3. We remark that the chordal graph G constructed above always admits a tree $(t+1)$-spanner.

3 Tree Spanners in Chordal Graphs of "Smaller" Diameter

It is known [8,9] that for a chordal graph G, $\text{diam}(G) \geq 2r(G) - 2$ holds. This already yields the following.

Theorem 5 *Let $t \geq 2$ be an even integer. Every chordal graph of diameter at most $t - 1$ admits a tree t-spanner, and such a tree spanner can be constructed in linear time.*

We remark that there are chordal graphs of diameter t without tree t-spanner. Thus, Theorem 5 is best possible under diameter constraints.

Corollary 1 *Every chordal graph of diameter at most 3 has a tree 4-spanner, and such a tree spanner can be constructed in linear time.*

It remains an interesting open question whether existence of a tree 3-spanner in a given chordal graph of diameter at most 3 can be tested in polynomial time.

Lemma 4 *Every chordal graph G admits a tree $(2r(G))$-spanner, and such a tree spanner can be constructed in linear time.*

Let now t be an odd integer ($t \geq 3$). From Lemma 4 and the fact that $2r(G) \geq \mathrm{diam}(G) \geq 2r(G) - 2$ holds for any chordal graph G, we immediately deduce.

Theorem 6 *Every chordal graph of diameter at most $t-2$ admits a tree t-spanner, and such a tree spanner can be constructed in linear time.*

It would be nice to show also that, if $t \geq 3$ is an odd integer, then every chordal graph of diameter at most $t-1$ admits a tree t-spanner. But, although for chordal graphs with $\mathrm{diam}(G) \geq 2r(G) - 1$ this is true, it fails to hold for chordal graphs of diameter $2r(G)-2$. There are even chordal graphs of diameter 2 without tree 3-spanners. In what follows we will show that the existence of a tree $(2r(G)-1)$-spanner in a chordal graph of diameter $2r(G)-2$ "depends" on the existence of a tree 3-spanner in a chordal graph of diameter 2.

First we present some auxiliary results. A subset $S \subseteq V$ is *m–convex* if S contains every vertex on every chordless path between vertices of S. For a subset $S \subseteq V$ and a vertex $v \in V$, let $proj(v, S) = \{u \in S : d_G(v, u) = d_G(v, S)\}$ be the *metric projection* of v to S. For a subset $X \subseteq V$, let $proj(X, S) = \bigcup_{v \in X} proj(v, S)$. A subset $A \subseteq V$ is a *two-set* in G if $d_G(v, u) \leq 2$ holds for every $v, u \in A$.

Lemma 5 *Let G be a (not necessarily chordal) graph. The metric projection $proj(A, S)$ of any two-set A to an m–convex set S of G is a two-set.*

Lemma 6 *In every chordal graph $G = (V, E)$ of diameter $2r(G)-2$ there exists a two-set S such that $d_G(v, S) \leq r(G) - 2$ for every $v \in V$. Moreover such a two-set can be determined within time $O(|V|^3)$.*

Lemma 7 *Every maximal by inclusion two-set of a chordal graph is m-convex.*

Theorem 7 *Chordal graphs of diameter $2r(G) - 2$ admit tree $(2r(G) - 1)$-spanners if and only if chordal graphs of diameter 2 admit tree 3-spanners. Moreover, if a tree 3-spanner of any chordal graph of diameter 2 can be found in polynomial time, then a tree $(2r(G)-1)$-spanner of a chordal graph of diameter $2r(G) - 2$ can be found in polynomial time, too.*

We do not know how to use this theorem for general chordal graphs (since not all chordal graphs of diameter 2 have tree 3-spanners), but this theorem could be very useful for those hereditary subclasses of chordal graphs where each graph of diameter 2 is tree 3-spanner admissible. Then, for every graph of diameter at most $t-1$ from those classes, a tree t-spanner will exist and it could be found in polynomial time if corresponding tree 3-spanner is constructable in polynomial time. For an arbitrary chordal graph G with $\mathrm{diam}(G) = 2r(G)-2$, it can happen that a chordal graph of diameter at most 2, generated by a two-set of G (found as described in Lemma 6 and Theorem 7), does not have a tree 3-spanner, but yet G itself admits a $(2r(G)-1)$-spanner. We are still working on TREE $(2r(G)-1)$-SPANNER problem in chordal graphs of diameter $2r(G) - 2$. It is natural to ask whether a combination of Theorem 7 and Theorem 9 works.

4 Tree 3-Spanners in Chordal Graphs of Diameter 2

In this section, we give an application of Theorem 7 as well as a criterion for the tree 3-spanner admissibility of chordal graphs of diameter at most 2.

A graph G is *non-trivial* if it has at least one edge.

Lemma 8 *Let G be a non-trivial chordal graph of diameter at most 2. If G does not contain a clique K_5 on five vertices, then G has a dominating edge, i.e., an edge $e \in E$ such that $d_G(v, e) \leq 1$ for any $v \in V$.*

Since neither planar graphs nor 3-trees have cliques on 5 vertices and any graph with a dominating edge is trivially tree 3-spanner admissible, we conclude.

Corollary 2 *Let G be a non-trivial graph of diameter at most 2. If G is a planar chordal graph or a k-tree for $k \leq 3$, then G has a dominating edge and hence a tree 3-spanner.*

As we mentioned in introduction, there is no constant t such that planar chordal graphs or k-trees ($k \geq 2$) are tree t-spanner admissible. So, it is interesting to mention the following result.

Theorem 8 *Every chordal graph of diameter at most $t - 1$, if it is planar or a k-tree ($k \leq 3$), has a tree t-spanner and such a tree spanner can be constructed in polynomial time.*

In what follows we will assume that G is an arbitrary chordal graph which admits a tree 3-spanner T. Note that any tree of diameter at most 2 is a star and any tree of diameter 3 has a dominating edge (in this case T is called a *bistar*).

Lemma 9 *For any maximal (by inclusion) clique C of G one of the following conditions holds.*

- a) *C induces a star in T,*
- b) *either C induces a bistar in T or there is a vertex $v \notin C$ such that $C \cup \{v\}$ induces a bistar in T.*

Clearly, T is a star only if G has an universal vertex, and the diameter of T is 3 only if G has a dominating edge. The following theorem handles the case of all chordal graphs of diameter at most 2. Unfortunately, not every such graph has a dominating edge. There are chordal graphs of diameter 2 which do not have any tree 3-spanners, and there are chordal graphs of diameter 2 that have a tree 3-spanner but all those spanners are of diameter 4. Theorem 4 will follow from this theorem.

Theorem 9 *A chordal graph G of diameter at most 2 admits a tree 3-spanner if and only if there is a vertex v in G such that any connected component of the second neighborhood of v has a dominating vertex in $N(v)$.*

5 Tree Spanners in Strongly Chordal Graphs and *k*-Split Graphs

For an integer $k \geq 3$, a *k-sun* consists of a k-clique $\{v_1, \ldots, v_k\}$ and a k-vertex stable set $\{u_1, \ldots, u_k\}$, and edges u_iv_i, u_iv_{i+1}, $1 \leq i < k$, and u_kv_k, u_kv_1. A chordal graph is *strongly chordal* [11] if it does not contain a k-sun as an induced subgraph. In [3], it is proved that every strongly chordal graph admits a tree 4-spanner and such a tree spanner can be constructed in linear time. Not every strongly chordal graph has a tree 3-spanner. Actually, TREE 3-SPANNER remains open on strongly chordal graphs.

A *k-planet* is obtained from a k-path $v_1v_2v_3 \cdots v_k$ and a triangle abc by adding edges av_i, $1 \leq i \leq k-1$ and bv_i, $2 \leq i \leq k$; see Figure 5.

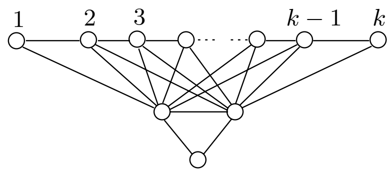

Fig. 5. A k-planet

Definition 1 *A chordal graph is called* very strongly chordal *if it does not contain a k-planet as an induced subgraph.*

As a 3-sun is a 3-planet and every k-sun ($k \geq 4$) contains an induced 4-planet, the class of very strongly chordal graphs is properly contained in the class of strongly chordal graphs. Moreover, the class of very strongly chordal graphs contains all interval graphs and all distance hereditary chordal graphs, called *ptolemaic graphs* [15]. The nice feature of this subclass of strongly chordal graphs is

Theorem 10 *Every very strongly chordal graph admits a tree* 3*-spanner and such a tree spanner can be constructed in linear time.*

Another well-known subclass of strongly chordal graphs consists of the intersection graphs of directed paths in a rooted directed tree, called *directed path graphs*. The class of directed path graphs generalizes interval graphs naturally, and contains all ptolemaic graphs, and is tree 3-spanner admissible [17].

The intersection graphs of paths in a tree are called (undirected) *path graph*. We call shortly a graph *strongly path graph* if it is strongly chordal as well as a path graph. Clearly, every directed path graph is a strongly path graph, but not vice versa. Indeed, there are many strongly path graphs having no tree 3-spanner (while every directed path graph does [17]). Moreover, in contrast to strongly chordal graphs, for every t, there is a path graph having no tree t-spanner [16].

A *split graph* is one whose vertex set can be partitioned into a clique and a stable set. Split graphs are exactly those chordal graphs whose complements are chordal as well. It is known (and easy to see; cf. [5,16,25]) that every split graph admits a tree 3-spanner. We are going to describe a new subclass of chordal graphs containing all split graphs and still are tree 3-spanner admissible.

First, for an arbitrary graph G let $S(G)$ be the set of all simplicial vertices of G. We also use $S(G)$ for the subgraph of G induced by $S(G)$.

Lemma 10 *If $G \setminus S(G)$ has a tree $(t-1)$-spanner then G has a tree t-spanner.*

Definition 2 *For an arbitrary graph G and an integer $k \geq 0$ let $G_k := G_{k-1} \setminus S(G_{k-1})$; $G_0 := G$. A graph G is called k-*split *if G_k is a clique.*

Clearly, 0-split graphs are exactly the cliques, and all split graphs are 1-split but not vice versa. The following fact is probably known.

Proposition 2 *A graph G is chordal if and only if G is k-split for some k.*

Theorem 11 *Every k-split graph admits a tree $(k+2)$-spanner.*

Corollary 3 *All 1-split graphs, hence all split graphs, admit a tree 3-spanner, and a such a tree 3-spanner can be constructed in linear time, given the set of all simplicial vertices.*

Note that the existence of a tree $(k+2)$-spanner in k-split graphs is best possible: there are many k-split graphs without tree $(k+1)$-spanner; for example, the 3-sun is 1-split (even split) and has no tree 2-spanner.

6 Conclusion

In this paper we have proved that, for any $t \geq 4$, Tree t-Spanner is **NP**-complete on chordal graphs of diameter at most $t+1$ (if t is even), respectively, at most $t+2$ (if t is odd), improving the hardness result in [6] on a restricted well-understood graph class. We have shown that every chordal graph G of diameter at most $t-1$ is tree t-spanner admissible if $\mathrm{diam}(G) \neq 2r(G) - 2$.

The complexity of Tree t-Spanner remains unresolved on chordal graphs of diameter t (if t is even) and of diameter t or $t+1$ (if t is odd). Tree t-Spanner remains also open on path graphs and the case $t = 3$ remains even open on path graphs that are strongly chordal graphs as well. However, we have shown that all very strongly chordal graphs, a subclass of strongly chordal graphs that contains all interval graphs and all ptolemaic graphs, are tree 3-spanner admissible, and a tree 3-spanner for a given very strongly chordal graph can be constructed in linear time. This improves known results on tree 3-spanners in interval graphs [16,18,23]. We have also improved known results on tree 3-spanners in split graphs [5,16,25] by showing that all 1-split graphs, a subclass of chordal graphs containing all split graphs, are tree 3-spanner admissible, and a tree 3-spanner for a 1-split graph can be constructed in linear time, given the set of its simplicial vertices. We presented a polynomial time algorithm for the Tree 3-Spanner problem on chordal graphs of diameter at most 2.

Many questions remain still open. Among them:

(1) Can Tree 3-Spanner be decided efficiently on (strongly) chordal graphs?
(2) Can Tree $(2r(G)-1)$-Spanner be decided efficiently on chordal graphs of diameter $2r(G) - 2$?
(3) What is the complexity of Tree t-Spanner for chordal graphs of diameter at most t?.

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