A PTAS for the Sparsest Spanners Problem on Apex-Minor-Free Graphs

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Outline

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   - History and related work
   - Our results
2. **A PTAS for the Sparsest Spanners Problem**
   - Graphs of bounded local treewidth
   - Partial spanners
   - Idea of the algorithm
   - Apex-minor-free graphs

t-spanners

Definition (t-spanner)

Let t be a positive integer. A subgraph S of G, such that $V(S) = V(G)$, is called a *(multiplicative) t-spanner*, if $\mathrm{dist}_S(u, v) \leq t \cdot \mathrm{dist}_G(u, v)$ for every pair of vertices u and v. The parameter t is called the *stretch factor* of S.

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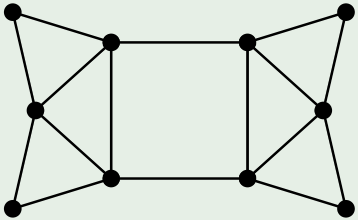
Observation (t-spanner)

Let G be a connected graph, and t be a positive integer. A spanning subgraph S of G is a t-spanner of G if and only if for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \leq t$.

Examples of spanners

3 and 2-spanners



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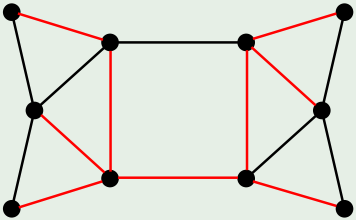

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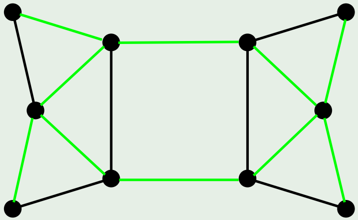

History and related work

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- W. Duckworth, N. C. Wormald, and M. Zito, 2003: A PTAS for the sparsest 2-spanner problem on 4-connected planar triangulations.

Sparsest spanners for planar graphs

Problem (Sparsest t-spanner)

The SPARSEST t-SPANNER *problem asks to find, for a given graph G and an integer t, a t-spanner of G with the minimum number of edges.*

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Theorem (PTAS)

For every $t \geq 1$, the SPARSEST t-SPANNER *problem admits a PTAS with linear running time for the class of apex-minor-free graphs (and, hence, for the planar graphs and for the graphs with bounded genus).*

Graphs of bounded local treewidth

Definition (Graphs of bounded local treewidth)

A graph class $\mathcal{G}$ has bounded local treewidth if there is function $f(r)$ (which depends only on r) such that for any graph G in $\mathcal{G}$, the treewidth of the subgraph of G induced by the set of vertices at distance at most r from any vertex is bounded above by $f(r)$.

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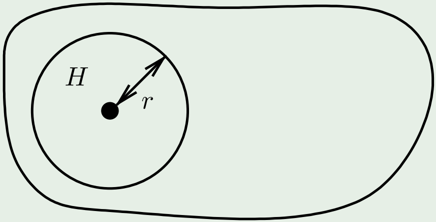

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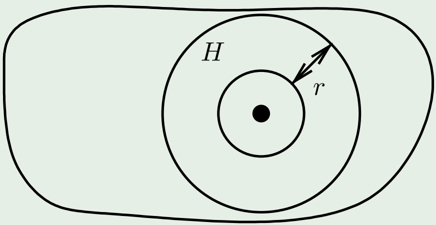

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Partial spanners

Definition (Partial t-spanners)

Let $A \subseteq E(G)$. We call a subgraph S of G, such that for every edge $(x, y) \in A$ $\mathrm{dist}_S(x, y) \le t$, a *partial t-spanner for A*.

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Lemma

Let k and t be positive integers. Let also G be a graph of treewidth at most k, and let $A \subseteq E(G)$. The SPARSEST PARTIAL t-SPANNER *problem can be solved by a linear-time algorithm (the constant which is used in the bound of the running time depends only on k and t) if a corresponding tree decomposition of G is given.*

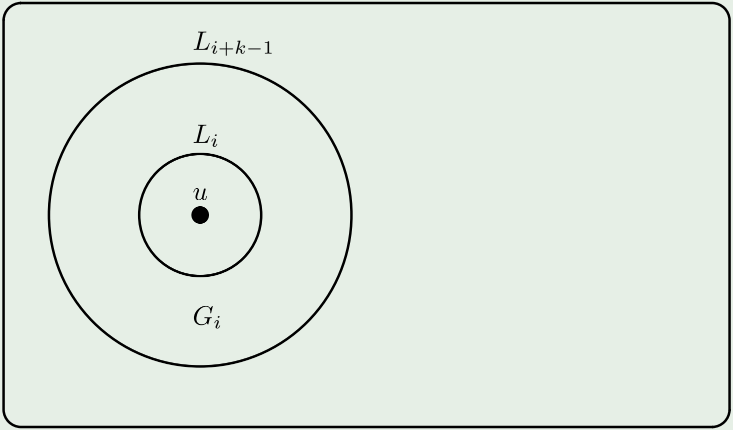
Idea of the algorithm

Definition (BFS decomposition)

Let u be a vertex of a graph G. For $i \geq 0$ we denote by L_i the i-th level of breadth first search, i.e. the set of vertices at distance i from u. We call the partition of the vertex set $V(G)$ $\mathcal{L}(G, u) = \{L_0, L_1, \dots, L_r\}$ *breadth first search (BFS) decomposition* of G.

Idea of the algorithm

Graphs G_i and S_i



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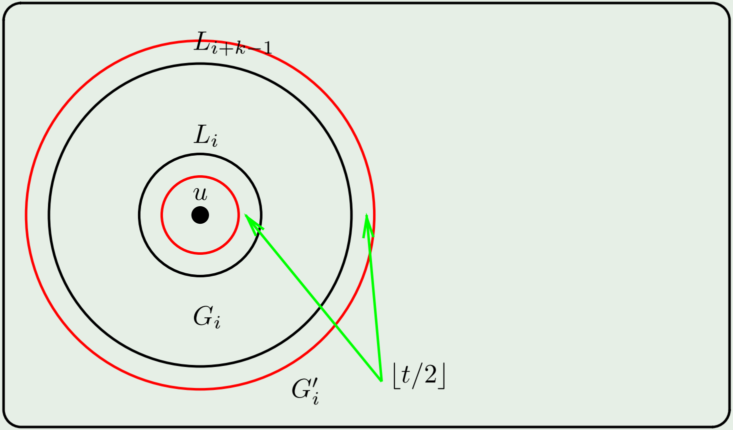

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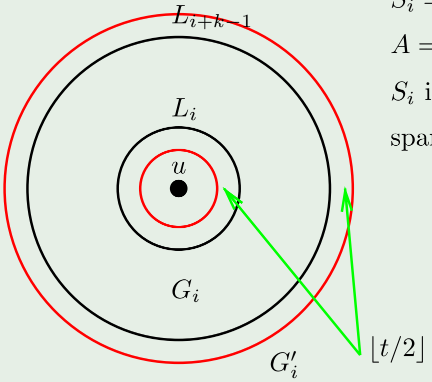

$S_i = S[V(G'_i)]$

$A = E(G_i)$

S_i is a partial spanner for A

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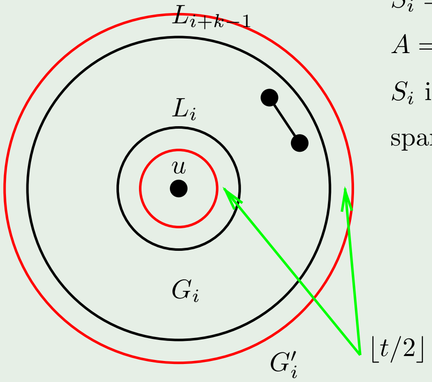

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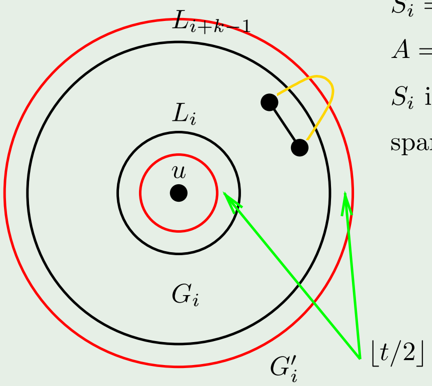

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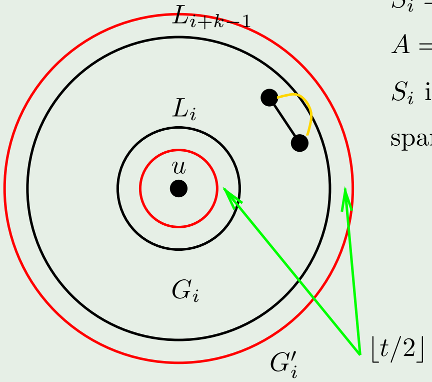

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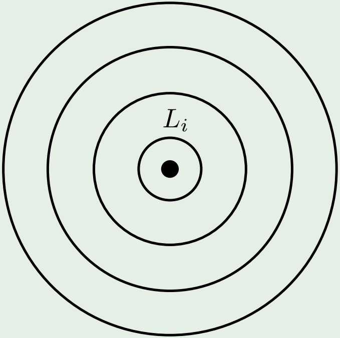
Lemma

Let S be a sparsest t-spanner of G, $S_i = S[V(G_i')]$. If S_i' is a sparsest partial t-spanner in G_i' for $A = E(G_i)$ then $|E(S_i')| \leq |E(S_i)|$.

Idea of the algorithm

Construction of a spanner

$i = 0, 1, \ldots, k-2$



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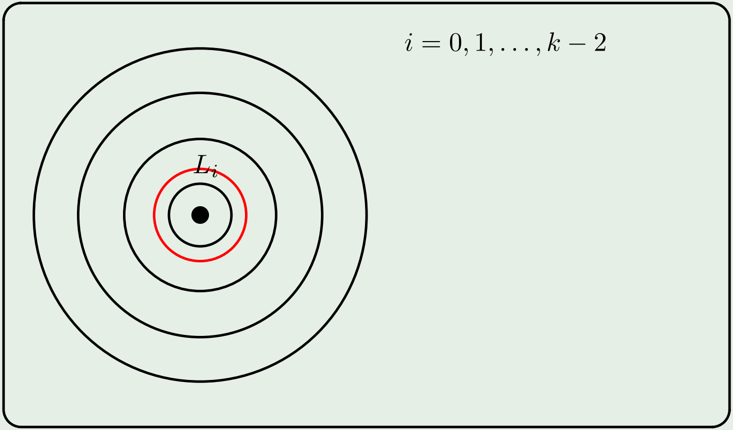

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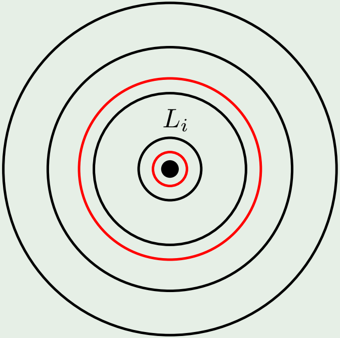

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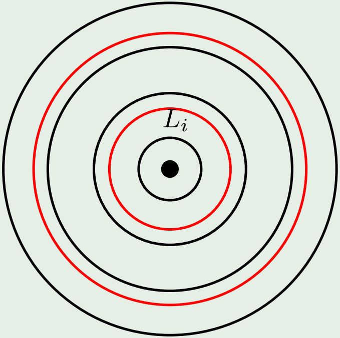

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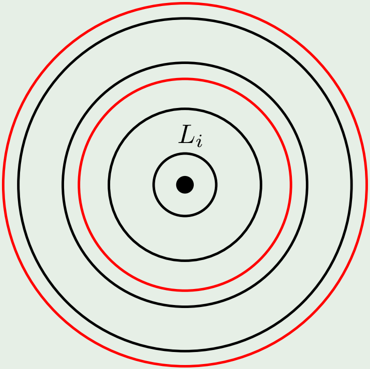

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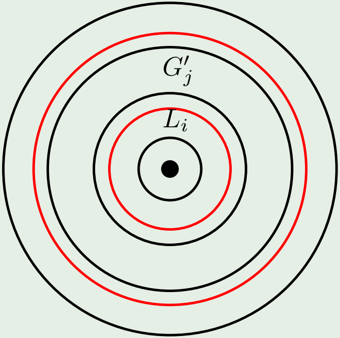

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S'_j - partial spanner
for $A = E(G_j)$

$S' = \cup S'_j$

Idea of the algorithm

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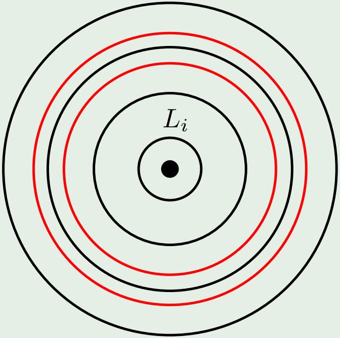

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$|E(S'_j)| \leq |E(S_j)|$

Idea of the algorithm

Lemma

The spanner S' has at most $(1 + \frac{t+1}{k-1})\mathrm{OPT}(G)$ edges, where $\mathrm{OPT}(G)$ is the number of edges in the solution of the SPARSEST *t-*SPANNER *problem on G.*

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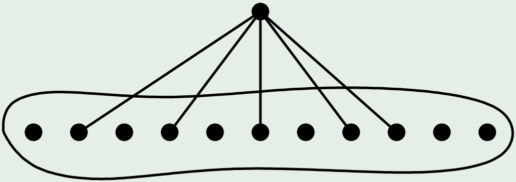

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Theorem (Eppstein)

All minor-closed graph classes that have bounded local treewidth are exactly apex-minor-free graph classes.

Thank you!