New Min-Max Theorems for Weakly Chordal and Dually Chordal Graphs

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Abstract. A distance-k matching in a graph G is matching M in which the distance between any two edges of M is at least k. A distance-2 matching is more commonly referred to as an induced matching. In this paper, we show that when G is weakly chordal, the size of the largest induced matching in G is equal to the minimum number of co-chordal subgraphs of G needed to cover the edges of G, and that the co-chordal subgraphs of a minimum cover can be found in polynomial time. Using similar techniques, we show that the distance-k matching problem for $k > 1$ is tractable for weakly chordal graphs when k is even, and is NP-hard when k is odd. For dually chordal graphs, we use properties of hypergraphs to show that the distance-k matching problem is solvable in polynomial time whenever k is odd, and NP-hard when k is even. Motivated by our use of hypergraphs, we define a class of hypergraphs which lies strictly in between the well studied classes of acyclic hypergraphs and normal hypergraphs.

1 Background and Motivation

In this paper, all graphs are undirected, simple, and finite. That is, a graph $G = (V, E)$ where V is a finite set whose elements are called vertices together with a set E of unordered pairs of vertices. We say $H = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$, and we say that H is an induced subgraph if $E' = \{uv \in E \mid \{u, v\} \subseteq V'\}$. We use P_k to denote an induced path on k vertices and C_k is an induced cycle on k vertices. A graph is chordal if it does not contain any C_k, $k \geq 4$. A graph is co-chordal if its complement is chordal. A graph G is weakly chordal if neither G nor $\overline{G}$ contains any C_k, $k \geq 5$. For background on these and other graph classes referenced below, we refer the interested reader to [6].

An *induced matching* in a graph is a matching that is also an induced subgraph, i.e., no two edges of the matching are joined by an edge in the graph.

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The size of an induced matching is the number of edges in it. Let $\text{im}(G)$ denote the size of a largest induced matching in G. Given G and positive integer k, the problem of deciding whether $\text{im}(G) \geq k$ is NP-complete [8] even when G is bipartite.

For vertices x and y of G, let $\text{dist}_G(x, y)$ be the number of edges on a shortest path between x and y in G. For edges e_i and e_j of G let $\text{dist}_G(e_i, e_j) = \min\{\text{dist}_G(x, y) \mid x \in e_i \textit{ and } y \in e_j\}$. For $M \subseteq E(G)$, M is a *distance-k matching* for a positive integer $k \geq 1$ if for every $e_i, e_j \in M$ with $i \neq j$, $\text{dist}_G(e_i, e_j) \geq k$. For $k = 1$, this gives the usual notion of matching in graphs. For $k = 2$, this gives the notion of induced matching. The *distance-k matching problem* is to find, for a given graph G and an integer $k \geq 1$, a distance-k matching with the largest possible number of edges.

A bipartite graph $G = (X, Y, E)$ is a *chain graph* if it does not have a $2K_2$ as an induced subgraph. Bipartite graph $G' = (X', Y', E')$ is a *chain subgraph* of bipartite graph $G = (X, Y, E)$, if G' is a subgraph of G and G' contains no $2K_2$. For a bipartite graph $G = (X, Y, E)$, let $\text{ch}(G)$ denote the fewest number of chain subgraphs of G the union of whose edge-sets is E. A set of $\text{ch}(G)$ chain subgraphs of bipartite graph $G = (V, E)$ whose edge-sets cover E is a *minimum chain subgraph cover* for G. Yannakakis showed [23] that when $k \geq 3$, deciding whether $\text{ch}(G) \leq k$ for a given bipartite graph G is NP-complete. An efficient algorithm to determine whether $\text{ch}(G) \leq 2$ for a given bipartite graph G is known [16].

It is clear that for any bipartite graph G, $\text{im}(G) \leq \text{ch}(G)$. Families of bipartite graphs where equality holds have been considered in literature. For example, it was shown in [24] that when G is a convex bipartite graph, $\text{im}(G) = \text{ch}(G)$. A bipartite graph is chordal bipartite if it does not contain any induced cycles on 6 or more vertices. It is known that every convex bipartite graph is also chordal bipartite.

The following more general result was recently shown:

Proposition 1. [1] For a chordal bipartite graph G, $\text{im}(G) = \text{ch}(G)$.

Let us move away from the setting of bipartite graphs and consider graphs in general. We say H is a co-chordal subgraph of G if H is a subgraph of G and also H is co-chordal. Let $\text{coc}(G)$ be the minimum number of co-chordal subgraphs of G needed to cover all the edges of G. As a chain subgraph of a bipartite graph G is a co-chordal subgraph of G and vice versa, the parameter $\text{coc}(G)$ when restricted to a bipartite graph G is essentially the same as $\text{ch}(G)$.

Again, it is clear from the definitions that for any graph G, $\text{im}(G) \leq \text{coc}(G)$. In Section 2, we show that when G is weakly chordal, $\text{im}(G) = \text{coc}(G)$ and that the co-chordal subgraphs of a minimum cover can be found in polynomial time. As every chordal bipartite graph is weakly chordal and as a chain subgraph of a bipartite graph is a co-chordal subgraph and vice versa, our result generalizes Proposition 1. In Section 3, we use similar techniques to show that the distance-k matching problem for $k > 1$ is tractable for weakly chordal graphs when k is even, and NP-hard when k is odd. Next, in Section 4 we use techniques from the study of hypergraphs to show that the opposite holds for the class of dually chordal graphs; the distance-k matching problem can be solved in polynomial time for dually chordal graphs if a k is odd, and is NP-hard for all even k. Motivated by our results and by the use of hypergraphs in Section 4, we define a class of hypergraphs in Section 5 which lies strictly in between the well studied classes of acyclic hypergraphs and normal hypergraphs.

2 A Min-Max Theorem for Weakly Chordal Graphs

For a graph G, let G^* denote the square of the line graph of G. More explicitly, vertices of G^* are edges of G. Edges e_i, e_j of G are nonadjacent in G^* if and only if they form a $2K_2$ in G.

It is clear from the construction of G^* that the set of edges of a co-chordal subgraph of G maps to a clique of G^*. Further, $\mathrm{im}(G) = \alpha(G^*)$, where $\alpha(G^*)$ is the size of a largest independent set in G^*.

The following is known:

Proposition 2. [9] If G is weakly chordal, then G^* is weakly chordal.

Also, it is well known that every weakly chordal graph is perfect [13]. Therefore, when G is weakly chordal, $\mathrm{im}(G) = \alpha(G^*) = \theta(G^*)$, where $\theta(G^*)$ is the minimum clique cover number of G^*. Thus, when G is weakly chordal $\theta(G^*) \leq \mathrm{coc}(G)$.

We will show that when G is weakly chordal, $\mathrm{coc}(G) \leq \theta(G^*)$ also holds and therefore we have the following:

Proposition 3. *If G is weakly chordal, then* $\mathrm{coc}(G) = \mathrm{im}(G)$.

The proof of Proposition 3 utilizes the following edge elimination scheme for weakly chordal graphs. Edge xy is a co-pair of graph G, if vertices x and y are not the endpoints of any P_k, $k \geq 4$, in $\overline{G}$.

Proposition 4. [19] Suppose e is a co-pair of graph G. Then, G is weakly chordal if and only if $G - e$ is weakly chordal.

The following is implied by Corollary 2 in [12]:

Proposition 5. [12] Suppose G is a weakly chordal graph that contains a $2K_2$. Then, G contains co-pairs e and f such that e and f form a $2K_2$ in G.

Lemma 6. *If e is a co-pair of a weakly chordal graph G, then $G^* - e = (G - e)^*$.*

Proof. Deleting an edge xy from G will never destroy a $2K_2$, unless it is one of the edges of the $2K_2$. If deleting xy creates a new $2K_2$ then xy must be the middle edge of a P_4 in G, or equivalently, x and y are the end vertices of a P_4 in $\overline{G}$. Thus, when e is a co-pair, two edges form a $2K_2$ in $G - e$ if and only if they form a $2K_2$ in G that does not include the edge e. Since the vertices of $(G^* - e)$ and $(G - e)^*$ both consist of the edges of $G - e$, this guarantees that the edge sets of $(G^* - e)$ and $(G - e)^*$, are identical as well. Hence the graphs are identical. □

In order to establish that when G is weakly chordal, $\mathrm{coc}(G) \leq \theta(G^*)$, first observe that every member of a clique cover of G^* can be assumed to be a maximal clique of G^*. We have the following:

Theorem 7. *Let G be weakly chordal. Then, every maximal clique of G^* is the edge-set of a maximal co-chordal subgraph of G.*

Proof. Proof is by induction on the number of edges in the graph. Clearly, the statement is true when G has no edges.

Assume the statement is true for all weakly chordal graphs with up to $k-1$ edges, and let G be a weakly chordal graph with k edges. If G contains no $2K_2$, then G is co-chordal and G^* is a clique and the theorem holds.

Now, suppose G contains a $2K_2$. Then, from Proposition 5, G contains a $2K_2$ e_1, e_2 each of which is a co-pair of G.

Let M be a maximal clique of G^*. As no maximal clique of G^* contains both e_1 and e_2, we can choose $i \in \{1, 2\}$ such that $e_i \notin M$. As a result, M is a maximal clique of $G^* - e_i$ which equals $(G - e_i)^*$ by Lemma 6. Also, by Proposition 4, $G - e_i$ is weakly chordal. It then follows by the induction hypothesis that M is the edge set of a maximal co-chordal subgraph of $G - e_i$.

Clearly, this subgraph remains co-chordal in G, so it remains to show that this subgraph is, in fact, maximal. If this is not the case, then there exists a co-chordal subgraph M' of G such that $M \subset M'$. As every co-chordal subgraph of G maps to a clique of G^*, it follows that M and M' are cliques of G^* such that $M \subset M'$; this contradicts M being a maximal clique of G^*. □

Thus, $\theta(G^*) = \mathrm{coc}(G)$, establishing Proposition 3. As an efficient algorithm exists [14] to compute a minimum clique cover of a weakly chordal graph, we have the following:

Corollary 8. *When G is weakly chordal,* $\mathrm{coc}(G)$ *and a minimum cover of G by co-chordal subgraphs of G can be found in polynomial time.*

We recently learned of a surprising application of this result: the parameters $\mathrm{coc}(G)$ and $\mathrm{im}(G)$ yield upper and lower bounds, respectively, on the Castelnuovo-Mumford regularity of the edge ideal of G [22]. Thus, when G is weakly chordal, this parameter can be computed efficiently.

Another application of Corollary 8 utilizes the complement of G. As the complement of a weakly chordal graph remains weakly chordal, after taking the complement of each graph in a cover by co-chordal subgraphs, we have a set of chordal graphs whose edge-intersection is the edge-set of a weakly chordal graph. The study of a variety of similar parameters, known as the *intersection dimension* of a graph G with respect to a graph class $\mathcal{A}$, was introduced in [15]. The problem when $\mathcal{A}$ is the set of chordal graphs was termed the *chordality* of G in [17]. We use $\dim_{\mathrm{CH}}(G)$ to denote the chordality or chordal dimension of a graph G. In this context, we have another corollary of Theorem 7.

Corollary 9. *When G is weakly chordal,* $\dim_{CH}(G) = \mathrm{im}(\overline{G})$ *and a minimum set of chordal graphs whose edge-intersection give the edge-set of G can be found in polynomial time.*

A chordal graph that does not contain a $2K_2$ is a split graph, and it has been shown in [8] that a split graph cover of a chordal graph can be computed in polynomial time.

Proposition 10. [8] Let G be a chordal graph. Then, a minimum cover of edges of G by split subgraphs of G can be found in polynomial time.

The proof of Proposition 10 in [8] utilizes the clique tree of a chordal graph G and the Helly property. An alternate proof can be given by showing that the edges referred to in Proposition 5 can be chosen so that each edge is incident with a simplicial vertex of G. Since no such edge is the only chord of a cycle, this guarantees that $G - e$ will be chordal whenever G is chordal. As every chordal graph is also weakly chordal, Lemma 6 and a slightly modified version of Theorem 7 then imply Proposition 10.

3 Distance-k Matchings in Weakly Chordal Graphs

In this section, we observe that the correspondence between maximal cliques of G^* and maximal co-chordal subgraphs of G can be adapted to find maximum a distance-k matching in a weakly chordal graph G for any positive even integer k. We then show that finding a largest distance-k matching when k is odd and $k \geq 3$ is NP-hard.

We begin by noting the fundamental connection between distance-k matchings in a graph G and independent sets in the k^{th} power of the line graph of G, which we denote $L^k(G)$.

Proposition 11. [7] For $k \geq 1$ and graph G, the edge set M is a distance-k matching in G if and only if M is an independent vertex set in $L^k(G)$.

As a result, identifying a largest distance-k matching in a graph G is no more difficult than constructing the k^{th} power of the line graph of G and finding a maximum independent set in $L^k(G)$. Clearly, for any edge e, the set of edges within distance k of e can be computed in linear time, and as a result a polynomial time algorithm exists for the distance-k matching problem whenever an efficient algorithm exists for finding a largest independent set in $L^k(G)$. Proposition 2 guarantees that such an efficient algorithm exists for induced matchings, as efficient algorithms to compute the largest independent sets of weakly chordal graphs are well known. For $k > 1$, the existence of a polynomial algorithm for computing distance-$2k$ matchings in weakly chordal graphs is guaranteed by combining Proposition 2 with the following result.

Proposition 12. [5] Let G be a graph and $k \geq 1$ be a fixed integer. If G^2 is weakly chordal, then so is G^{2k}.

For distance-k matchings when k is odd, we note that the case $k = 3$ was recently shown to be NP-complete for the class of chordal graphs, which is properly contained in the class of weakly chordal graphs.

Proposition 13. [7] The largest distance-3 matching problem is NP-hard for chordal graphs.

We will extend this result to distance-$(2k+1)$ matchings for every positive integer k. This extension is done by showing that the distance-$(2k+1)$ matching problem can be transformed into the distance-$(2k+3)$ matching problem in polynomial time, for any positive integer k.

Theorem 14. *For any positive integer k, there exists a polynomial time transformation from the distance-$(2k+1)$ matching problem to the distance-$(2k+3)$ matching problem.*

Proof. Let $G = (V, E)$ be a graph, and let k be a positive integer. We will define the graph $G^+ = (V^+, E^+)$ from G as follows. For each edge $e = uv$ of G, we introduce two new vertices x_e and y_e and add edges to make the subgraph induced by $\{u, v, x_e, y_e\}$ a clique. Formally, $V^+ = V \cup \{x_e,\ y_e \mid e \in E\}$, and

$$E^+ = E \cup \left(\bigcup_{e=uv \in E} \{x_e y_e,\ x_e u,\ x_e v,\ y_e u,\ y_e v\} \right).$$

We will show that a distance-$(2k+1)$ matching of size p exists in G if and only if a distance-$(2k+3)$ matching of size p exists in G^+. Since G^+ can clearly be constructed from G in linear time, this will establish the theorem.

First, we note that every matching M in G corresponds to a matching $M^+ = \{x_e y_e \mid e \in M\}$ in G^+, and that the distance between two edges $x_e y_e$ and $x_f y_f$ from M^+ is clearly $\mathrm{dist}_G(e, f) + 2$. Thus, if M is a distance-$(2k+1)$ matching of size p in G, M^+ is a distance-$(2k+3)$ matching of size p in G^+. Next, note that if M is a distance-$(2k+3)$ matching of G^+, then for each $e = uv$, at most one edge of M is incident with any vertex of $\{u, v, x_e, y_e\}$ as these vertices induce a clique in G^+. We construct a set M^- by removing from M any edge incident with a vertex x_e or y_e of $V^+ \setminus V$ and replacing it with the associated edge e. Since no other edge of M is incident with either vertex of e, we conclude that $|M^-| = |M|$ and that M^- is a matching in G. The pairwise distance between any two edges in M^- is at least $(2k+3) - 2 = 2k+1$. Thus, if a distance-$(2k+3)$ matching of size p exists in G^+, then a distance-$(2k+1)$ matching of size p exists in G. □

Combining Proposition 13 and Theorem 14 we conclude the following:

Corollary 15. *For $k > 0$, the distance-$(2k+1)$ matching problem is NP-hard for chordal graphs.*

Proof. We use the well known result of Dirac [10] that a graph is chordal if and only if it has a simplicial elimination ordering. Since each vertex of $V^+ \setminus V$ is simplicial in G^+, we can easily extend any simplicial elimination ordering of G to a simplicial elimination ordering of G^+. Therefore G^+ is chordal whenever G is chordal. The corollary now follows immediately from Proposition 13 and repeated application of Theorem 14. □

We summarize the consequences of Propositions 2, 11, 12 and Corollary 15 below.

Proposition 16. *Suppose G is a weakly chordal graph and k is a positive integer. Then, when k is even, a largest distance-k matching in G can be found in polynomial time. When k is odd, finding a largest distance-k matching in G is NP-hard for any $k > 1$.*

4 A Min-Max Theorem for Distance-(2k+1) Matching in Dually Chordal Graphs

As we saw in the previous section, for a weakly chordal graph G and any integer $k \geq 1$, a largest distance-$(2k)$ matching in G can be found in polynomial time, whereas computing a largest distance-$(2k+1)$ matching in G is an NP-hard problem. In this section, we show that for the class of dually chordal graphs the opposite holds.

Dually chordal graphs were introduced in [11] as a generalization of strongly chordal graphs (which are the hereditary dually chordal graphs) where the Steiner tree problem and many domination-like problems still (as in strongly chordal graphs) have efficient solutions. It also was shown recently in [7] that the distance-k matching problem is solvable in polynomial time for strongly chordal graphs for any integer k. Here, we extend this result to all doubly chordal graphs by showing that the distance-k matching problem is solvable in polynomial time for any odd k for all dually chordal graphs.

To define dually chordal graphs and doubly chordal graphs, we need some notions from the theory of hypergraphs [2]. Let $\mathcal{E}$ be a hypergraph with underlying vertex set V, i.e. $\mathcal{E}$ is a collection of subsets of V (called *hyperedges*) The *dual hypergraph* $\mathcal{E}^*$ has $\mathcal{E}$ as its vertex set and for every $v \in V$ a hyperedge $\{e \in \mathcal{E} : v \in e\}$. The *line graph* $L(\mathcal{E}) = (\mathcal{E}, E)$ of $\mathcal{E}$ is the intersection graph of $\mathcal{E}$, i.e. $ee' \in E$ if and only if $e \cap e' \neq \emptyset$. A *Helly hypergraph* is one whose edges satisfy the Helly property, that is, any subfamily $\mathcal{E}' \subseteq \mathcal{E}$ of pairwise intersecting edges has a nonempty intersection. A hypergraph $\mathcal{E}$ is a *hypertree* if there is a tree T with vertex set V such that every edge $e \in \mathcal{E}$ induces a subtree in T. Equivalently, $\mathcal{E}$ is a hypertree if and only if the line graph $L(\mathcal{E})$ is chordal and $\mathcal{E}$ is a Helly hypergraph. A hypergraph $\mathcal{E}$ is a *dual hypertree* (also known as *α–acyclic hypergraph*) if there is a tree T with vertex set $\mathcal{E}$ such that, for every vertex $v \in V$, $T_v = \{e \in \mathcal{E} : v \in e\}$ induces a subtree of T. Observe that $\mathcal{E}$ is a hypertree if and only if $\mathcal{E}^*$ is a dual hypertree.

For a graph $G = (V, E)$ by $\mathcal{C}(G) = \{C : C \text{ is a maximal clique in } G\}$ we denote the *clique hypergraph*. Let also $\mathcal{D}(G) = \{D_r(v) : v \in V, r \text{ a non-negative integer}\}$ be the *disk hypergraph* of G. Recall that $D_r(v) = \{u \in V : \text{dist}_G(u, v) \leq r\}$ is a disk of radius r centered at vertex v. A graph G is called *dually chordal* if the clique hypergraph $\mathcal{C}(G)$ is a hypertree [4, 11, 20]. In [4, 11] it is shown that dually chordal graphs are exactly the graphs G whose disk hypergraphs $\mathcal{D}(G)$ are hypertrees (see [4, 11] for other characterizations, in particular in terms of certain elimination schemes, and [3] for their algorithmic use). From the definition of hypertrees we deduce that for dually chordal graphs the line graphs of the clique and disk hypergraphs are chordal. Conversely, if G is a chordal graph then $\mathcal{C}(G)$ is a dual hypertree, and, therefore, the line graph $L(\mathcal{C}(G))$ is a dually chordal graph, justifying the term "dually chordal graphs". Hence, the dually chordal graphs are exactly the intersection graphs of maximal cliques of chordal graphs (see [20, 4]). Finally note that graphs being both chordal and dually chordal were dubbed *doubly chordal* and investigated in [4, 11, 18]. The class of dually chordal graphs contains such known graph families as interval graphs, ptolemaic graphs, directed path graphs, strongly chordal graphs, doubly chordal graphs, and others.

Let $G = (V, E)$ be an arbitrary graph and r be a non-negative integer. For an edge $uv \in E$, let $D_r(uv) := \{w \in V : \mathrm{dist}_G(u, w) \leq r \text{ or } \mathrm{dist}_G(v, w) \leq r\} = D_r(v) \cup D_r(u)$ be a disk of radius r centered at edge uv. For the edge-set E of graph $G = (V, E)$, we define a hypergraph $\mathcal{D}_{E,r}(G)$ as follows:

$$\mathcal{D}_{E,r}(G) = \{D_r(uv) : uv \in E\}.$$

Along with the distance-k matching problem we will consider also a problem which generalizes the (minimum) vertex cover problem. A *distance-k vertex cover* for a non-negative integer $k \geq 0$ in an undirected graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that for every edge $uv \in E$ there exists a vertex $x \in S$ with $D_k(x) \cap \{u, v\} \neq \emptyset$. For $k = 0$, this gives the usual notion of vertex cover in graphs. The *distance-k vertex cover problem* is to find, for a given graph G and an integer $k \geq 0$, a distance-k vertex cover with the fewest number of vertices.

Using the hypergraph $\mathcal{D}_{E,k}(G)$ of G (where $k \geq 0$ is an integer), the distance-$(2k + 1)$ matching problem and the distance-k vertex cover problem on G can be formulated as the transversal and matching problems, respectively, on the hypergraph $\mathcal{D}_{E,k}(G)$. Recall that a *transversal* of a hypergraph $\mathcal{E}$ is a subset of vertices which meets all edges of $\mathcal{E}$. A *matching* of $\mathcal{E}$ is a subset of pairwise disjoint edges of $\mathcal{E}$. For a hypergraph $\mathcal{E}$, the *transversal problem* is to find a transversal with minimum size $\tau(\mathcal{E})$ and the *matching problem* is to find a matching with maximum size $\nu(\mathcal{E})$.

Denote by $\mathrm{dm}_k(G)$ the size of a largest distance-k matching in G, and by $\mathrm{dvc}_k(G)$ the size of a smallest distance-k vertex cover in G. From the definitions we obtain

Lemma 17. *Let $G = (V, E)$ be an arbitrary graph and $k \geq 0$ be a non-negative integer. S is a distance-k vertex cover of G if and only if S is a transversal of $\mathcal{D}_{E,k}(G)$. M is a distance-$(2k + 1)$ matching of G if and only if $\{D_k(uv) : uv \in M\}$ is a matching of $\mathcal{D}_{E,k}(G)$. Thus, $\tau(\mathcal{D}_{E,k}(G)) = \mathrm{dvc}_k(G)$ and $\nu(\mathcal{D}_{E,k}(G)) = \mathrm{dm}_{2k+1}(G)$ hold for every graph G and every non-negative integer k.*

The parameters $\mathrm{dm}_{2k+1}(G)$ and $\mathrm{dvc}_k(G)$ are always related by a min–max duality inequality $\mathrm{dvc}_k(G) \geq \mathrm{dm}_{2k+1}(G)$. The next result shows that for dually chordal graphs and $k \geq 1$ equality holds.

Theorem 18. *Let* $G = (V, E)$ *be a dually chordal graph and* $k \geq 1$ *be an integer. Then, the hypergraph* $\mathcal{D}_{E,k}(G)$ *is a hypertree and, as a consequence,* $\mathrm{dvc}_k(G) = \mathrm{dm}_{2k+1}(G)$ *holds.*

Proof. Since G is a dually chordal graph, the disk hypergraph $\mathcal{D}(G)$ of G is a hypertree. That is, there is a tree T with vertex set V such that every disk $D_k(v)$, $v \in V$, induces a subtree in T. Consider an arbitrary edge uv of G. Since both disks $D_k(v)$ and $D_k(u)$ induce subtrees in T and $k \geq 1$ (i.e., $v \in D_k(u)$ as well as $u \in D_k(v)$), vertices of $D_k(v) \cup D_k(u)$ induce a subtree in T (the union of subtrees induced by $D_k(v)$ and $D_k(u)$). Hence, disk $D_k(uv)$ of G centered at any edge $uv \in E$ induces a subtree in T, implying that the hypergraph $\mathcal{D}_{E,k}(G)$ of G is a hypertree.

It is well known [2] that the equality $\tau(\mathcal{E}) = \nu(\mathcal{E})$ holds for every hypertree. Hence, by Lemma 17, we obtain the required equality. □

For any hypertree $\mathcal{E}$ with underlying set V, a transversal with minimum size $\tau(\mathcal{E})$ and a matching with maximum size $\nu(\mathcal{E})$ can be found in time $O(|V| + \sum_{e \in \mathcal{E}} |e|)$ [21]. Thus, we conclude.

Corollary 19. *Let* G *be a dually chordal graph and* $k \geq 1$ *be an integer. Then, a smallest distance-*k *vertex cover of* G *and a largest distance-*$(2k+1)$ *matching of* G *can be found in* $O(|V||E|)$ *time.*

We notice now that any graph G can be transformed into a dually chordal graph G' by adding to G two new, adjacent to each other, vertices x and y and making one of them, say vertex x, adjacent to all vertices of G (and leaving y as a pendant vertex in G'). Using this transformation, it is easy to show that the classical vertex cover problem (i.e., the distance-k vertex cover problem with $k = 0$) is NP-hard for dually chordal graphs. It is enough to notice that $\mathrm{dvc}_0(G) = \mathrm{dvc}_0(G') - 1$. Adding a new vertex z to G' and making it adjacent to only y transforms dually chordal graph G' into a dually chordal graph G''. It is easy to see also that $\mathrm{dm}_2(G) = \mathrm{dm}_2(G'') - 1$, implying that the induced matching problem (i.e., the distance-2 matching problem) is NP-hard on dually chordal graphs.

Proposition 20. *The vertex cover problem (i.e., the distance-*k *vertex cover problem with* $k = 0$*) and the induced matching problem (i.e., the distance-2 matching problem) both are NP-hard on dually chordal graphs.*

We can prove also a more general result.

Theorem 21. *For every integer* $k \geq 1$*, the distance-*$(2k)$ *matching problem is NP-hard on dually chordal graphs.*

Proof. By Proposition 20, we can assume that $k \geq 2$. Let $G = (V, E)$ be a dually chordal graph. We construct a new graph G^+ by attaching to each vertex $v \in V$ an induced path $P_v = (v = v_0, v_1, \ldots, v_k)$ of length k. Note that each path P_v shares only vertex v with G and $P_v \cap P_u = \emptyset$ if $v \neq u$. Denote by e_v the last edge $v_{k-1}v_k$ of P_v. Since adding a pendant vertex to a dually chordal graph results in a dually chordal graph (see [4, 3]), the graph G^+ is dually chordal. It is enough to show now that $\alpha(G) = \mathrm{dm}_{2k}(G^+)$, where $\alpha(G)$ is the size of a largest independent set of G. It is known that the maximum independent set problem is NP-hard on dually chordal graphs [3].

Let $S \subseteq V$ be a largest independent set of G, i.e., $|S| = \alpha(G)$. Consider two arbitrary vertices x and y from S. We have $\mathrm{dist}_G(x, y) \geq 2$. But then, by construction of G^+, $\mathrm{dist}_{G^+}(e_x, e_y) \geq k - 1 + 2 + k - 1 = 2k$. Hence, $\alpha(G) \leq \mathrm{dm}_{2k}(G^+)$. Let now M be a set of edges of G^+ forming a largest distance-$(2k)$ matching of G^+. We can assume that $M \subseteq \{e_v : v \in V\}$ since for any edge $e \in M$ there must exist a vertex $v \in V$ such that $P_v \cap e \neq \emptyset$ and therefore e can be replaced in M by e_v. Now, if e_x and e_y $(x \neq y)$ are in M, then $\mathrm{dist}_{G^+}(e_x, e_y) \geq 2k$ implies $\mathrm{dist}_G(x, y) \geq 2$. That is $\alpha(G) \geq \mathrm{dm}_{2k}(G^+)$. □

Figure 1 shows the containment relationships between the classes of Weakly Chordal, Chordal, Dually Chordal, Strongly Chordal and Interval Graphs. Since doubly chordal graphs are both chordal and dually chordal, and all chordal graphs are weakly chordal, we conclude this section with the following corollary of Proposition 16 and Corollary 19.

Corollary 22. *A maximum distance-k matching of a doubly chordal graph G can be found in polynomial time for every integer $k \geq 1$.*

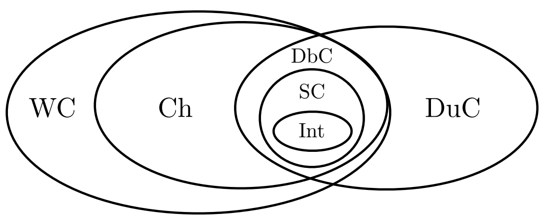

Fig. 1. The containment relationships between the classes of Weakly Chordal (WC), Chordal (Ch), Dually Chordal (DuC), Doubly Chordal (DbC), Strongly Chordal (SC) and Interval (Int) Graphs

5 A Class of Hypergraphs

As we have seen in the previous section, the hypertrees were very useful in obtaining the duality result between distance-k vertex cover and distance-$(2k+1)$ matching on dually chordal graphs. Recall that the dual hypertrees (called also α–acyclic hypergraphs) are exactly the clique hypergraphs of the chordal graphs (and the hypertrees are exactly the clique hypergraphs of dually chordal graphs). The duality result obtained for weakly chordal graphs in Section 2 can also be interpreted in terms of transversal and matching of a special class of hypergraphs.

We will need few more definitions from hypergraph theory. Let $\mathcal{E}$ be a hypergraph with underlying vertex set X. The *2–section graph* $2SEC(\mathcal{E})$ of the hypergraph $\mathcal{E}$ has vertex set X and two distinct vertices are adjacent if and only if they are contained in a common edge of $\mathcal{E}$. A hypergraph $\mathcal{E}$ is *conformal* if every clique C in $2SEC(\mathcal{E})$ is contained in an edge $e \in \mathcal{E}$.

Three well–known properties of hypergraphs will be helpful (for these and other properties cf. [2]).

(i) For a hypergraph $\mathcal{E}$, the graphs $L(\mathcal{E})$ and $2SEC(\mathcal{E}^*)$ are isomorphic;
(ii) A hypergraph $\mathcal{E}$ is conformal if and only if its dual hypergraph $\mathcal{E}^*$ has the Helly property;
(iii) Taking the dual of a hypergraph twice is isomorphic to the hypergraph itself.

Using these properties, α–acyclic hypergraphs can be equivalently defined as conformal hypergraphs with chordal 2–section graphs.

We now define a hypergraph $\mathcal{E}$ (with underlying vertex set X) to be *weakly acyclic* if $2SEC(\mathcal{E})$ is weakly chordal and $\mathcal{E}$ is conformal. In other words, weakly acyclic hypergraphs are exactly the clique hypergraphs of weakly chordal graphs. Note that since weakly chordal graphs are perfect, $\alpha(2SEC(\mathcal{E})) = \theta(2SEC(\mathcal{E}))$ and, as a consequence, $\alpha(L(\mathcal{E}^*)) = \theta(L(\mathcal{E}^*))$. The latter implies that equality $\nu(\mathcal{E}^*) = \tau(\mathcal{E}^*)$ holds for every weakly acyclic hypergraph $\mathcal{E}$. Indeed, $\alpha(L(\mathcal{E}^*)) = \nu(\mathcal{E}^*)$ holds for every hypergraph $\mathcal{E}^*$, and $\theta(L(\mathcal{E}^*)) = \tau(\mathcal{E}^*)$ holds for every Helly hypergraph $\mathcal{E}^*$ (equivalently, for every conformal hypergraph $\mathcal{E}$) [2]. For hypergraph $\mathcal{E}$, $\tau(\mathcal{E}^*)$ is equal to the minimum number of hyperedges needed to cover entire underlying vertex set X of $\mathcal{E}$, $\nu(\mathcal{E}^*)$ is equal to the maximum number of vertices $S \subseteq X$ such that no two of them can be covered by an hyperedge $e \in \mathcal{E}$.

Let now $G = (V, E)$ be a weakly chordal graph. Define a hypergraph $\mathcal{E}$ with underlying vertex set E as follows (note that the vertices of $\mathcal{E}$ are exactly the edges of G). Let $\mathcal{E}$ be the set of all maximal co-chordal subgraphs of G. Firstly, $2SEC(\mathcal{E}) = G^*$. Therefore, by Proposition 2, $2SEC(\mathcal{E})$ is weakly chordal. Further, from Theorem 7, $\mathcal{E}$ is conformal. Therefore, $\mathcal{E}$ is a weakly acyclic hypergraph. Consequently, since $\text{coc}(G) = \tau(\mathcal{E}^*)$, $\text{im}(G) = \nu(\mathcal{E}^*)$ and $\nu(\mathcal{E}^*) = \tau(\mathcal{E}^*)$, we obtain $\text{coc}(G) = \text{im}(G)$.

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