COPS + ROBBERS ON GRAPHS & WINNING STRATEGY

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Outline

- Simplified cops vs. robbers in real life
- Cop-win graphs and strategy
  - examples of cops winning/losing
  - corners
  - retracts
  - dismantling
  - strategy and solution
- Variations
- Bounds on cop-number

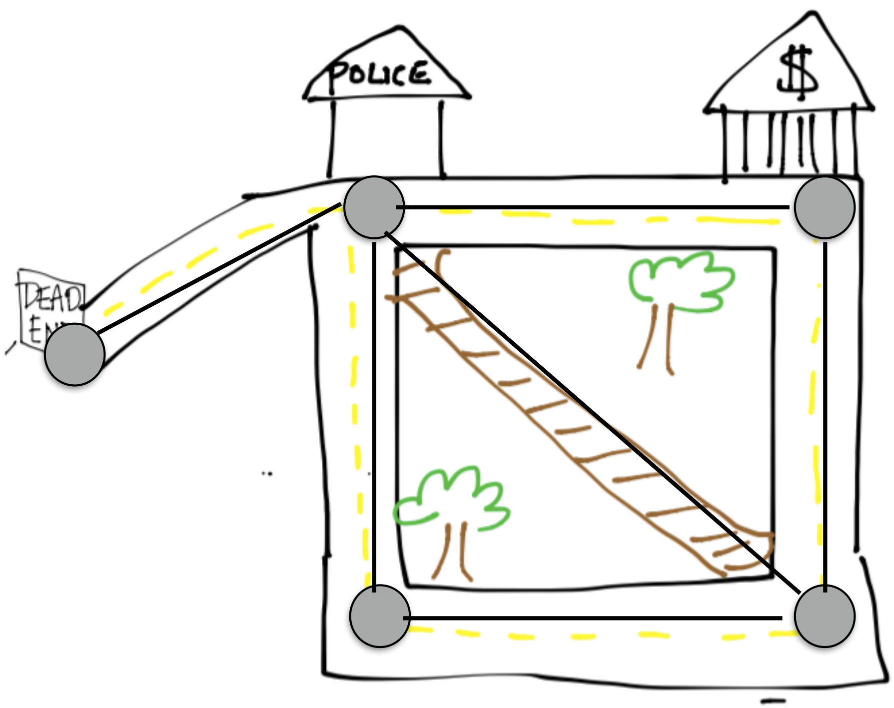
Problem: Cops vs. Robbers

- Given a map, a cop, and a robber - **can the cop catch the robber,** and if so, **how**?
  - The cop and robber always see each other
  - The cop and robber take turns moving (but can pass), starting with the cop
  - The cop and robber move at the same speed
  - If the cop catches the robber, the cop wins

Similar problems

- More applications in other pursuit/evasion scenarios
  - search and rescue
  - modeling network security problems
  - surveillance and tracking
  - artificial intelligence in games

Map and Graph Construction



- Cops and robber each occupy a **node** on a graph, and take turns moving to adjacent nodes via the **edges**.
- Graph is reflexive
- Cops win if they can occupy the same space as the robber
- Robber wins if he never gets caught (indefinitely evades)

Graph Problem

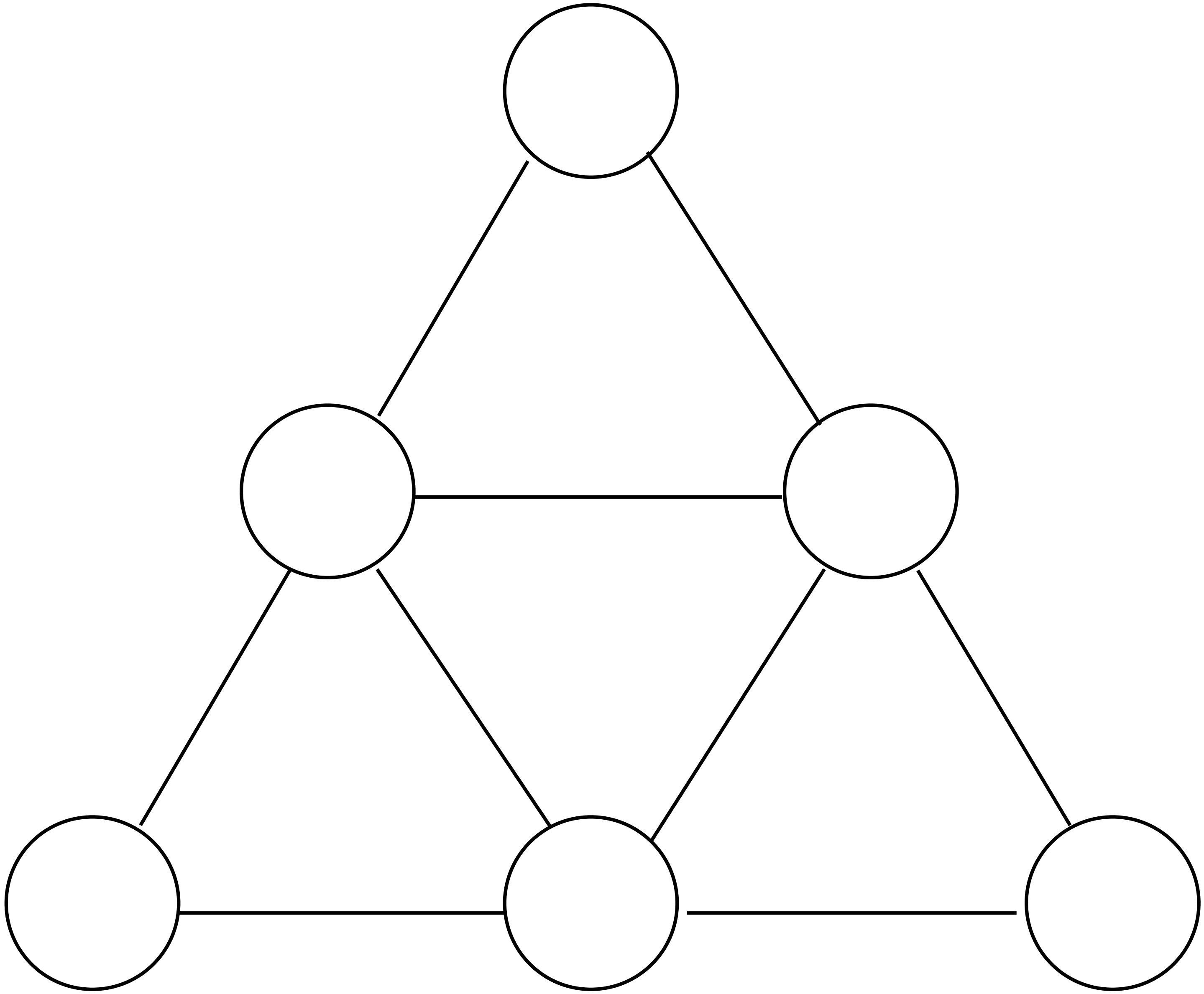
Is the constructed graph a cop-win graph?

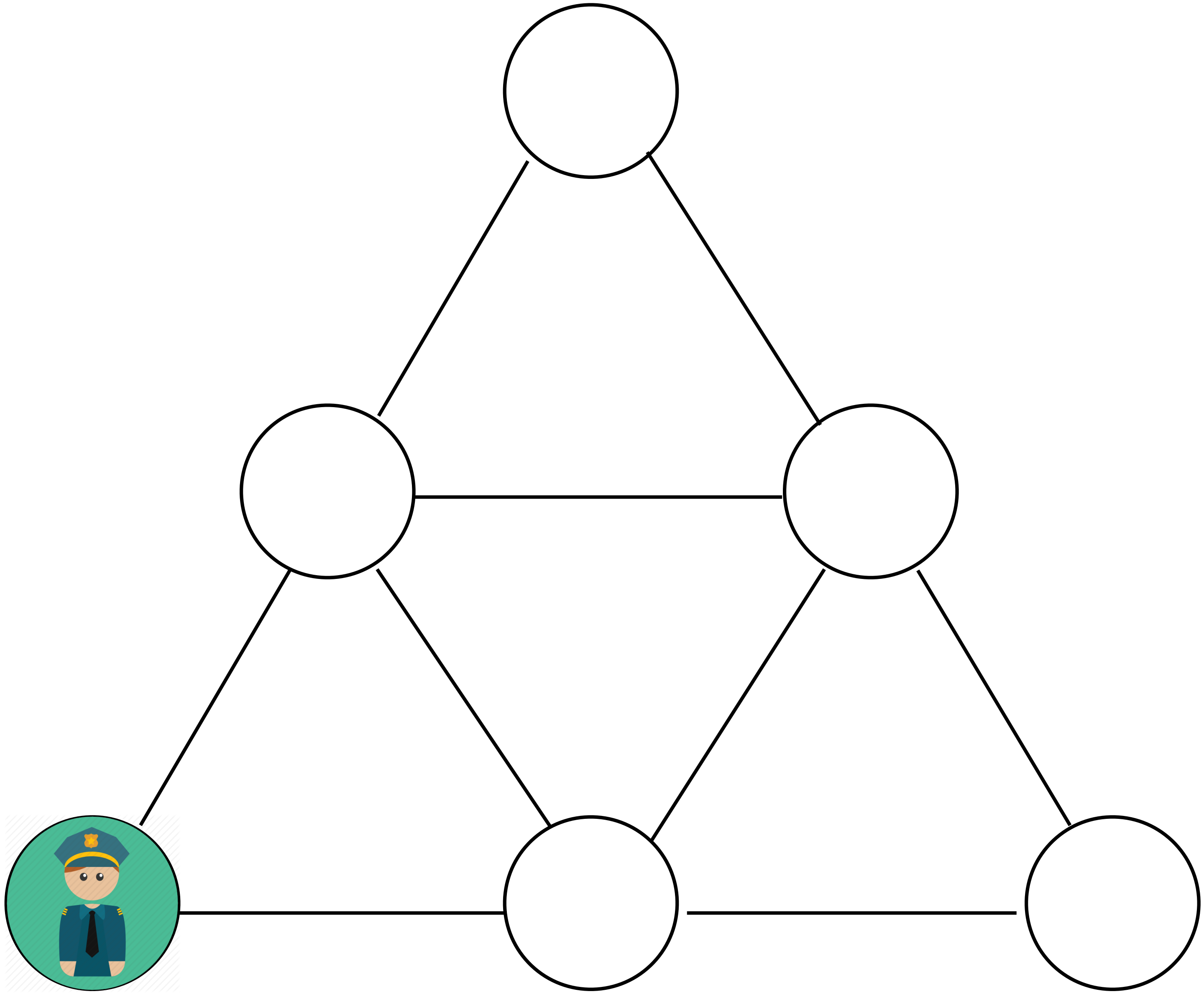
If yes, what strategy should the cop use to win?

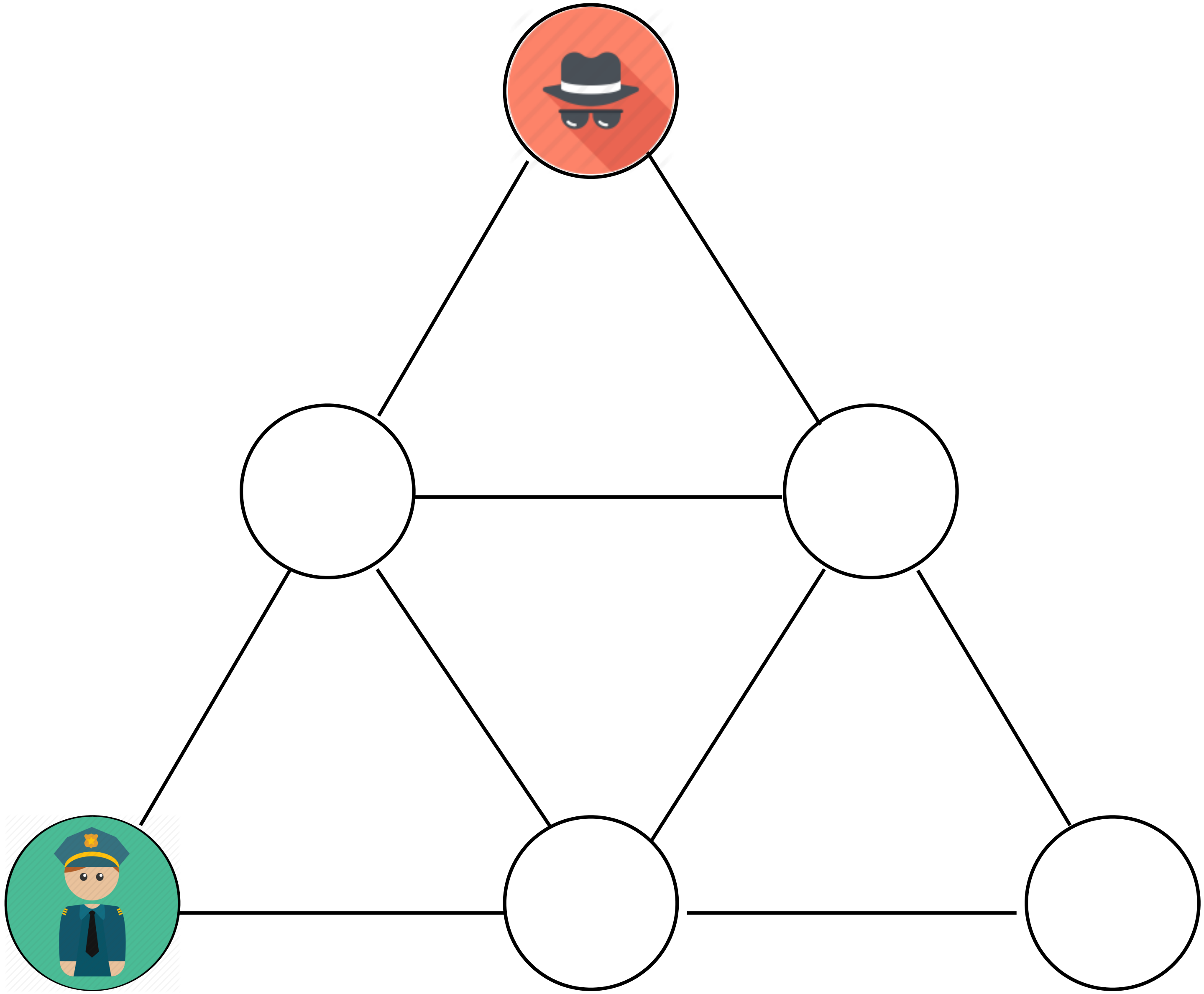
Cops and Robbers on graphs introduced independently by Nowakowski and Winkler (1983) and Quilliot (1978)

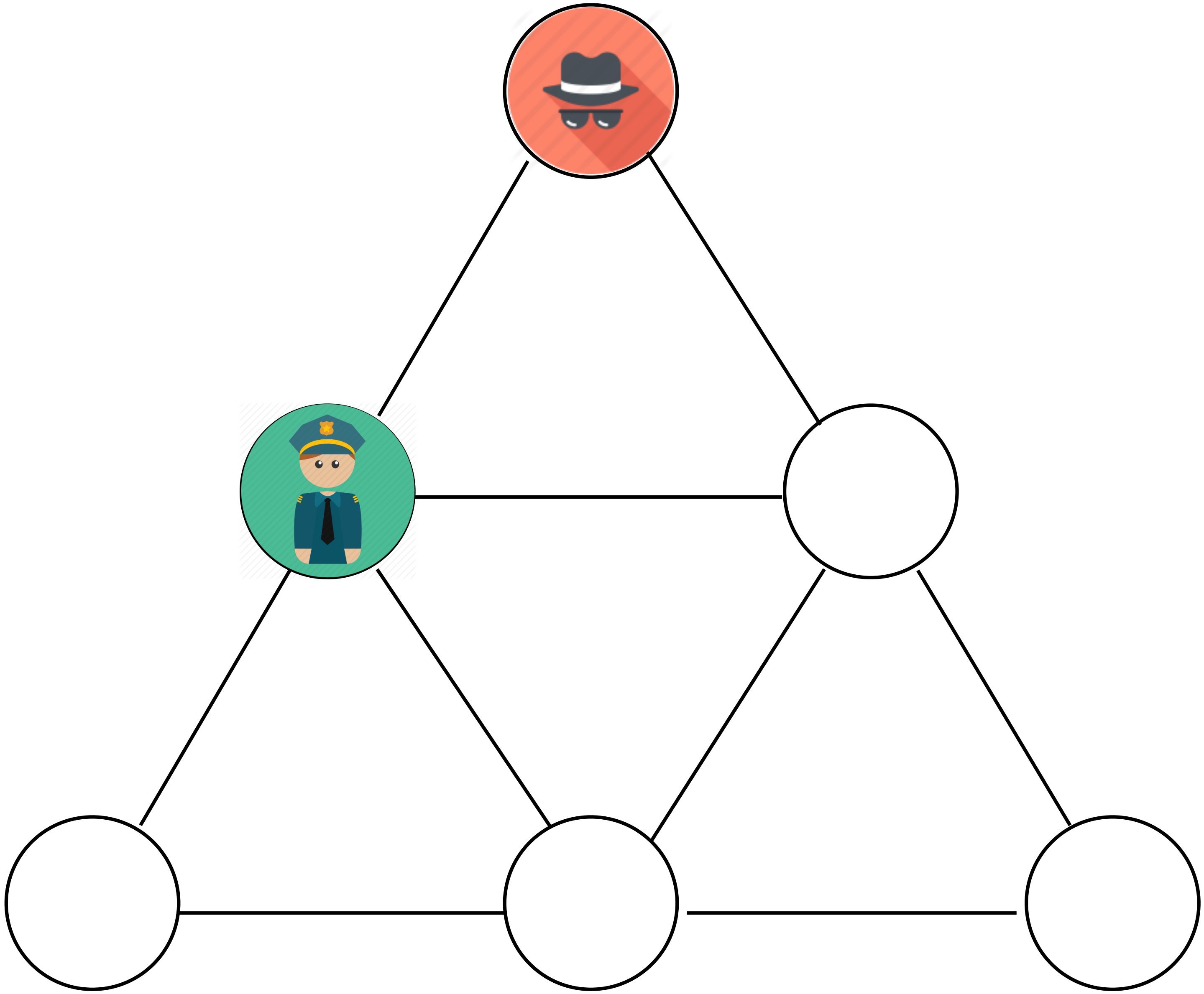
Preliminaries

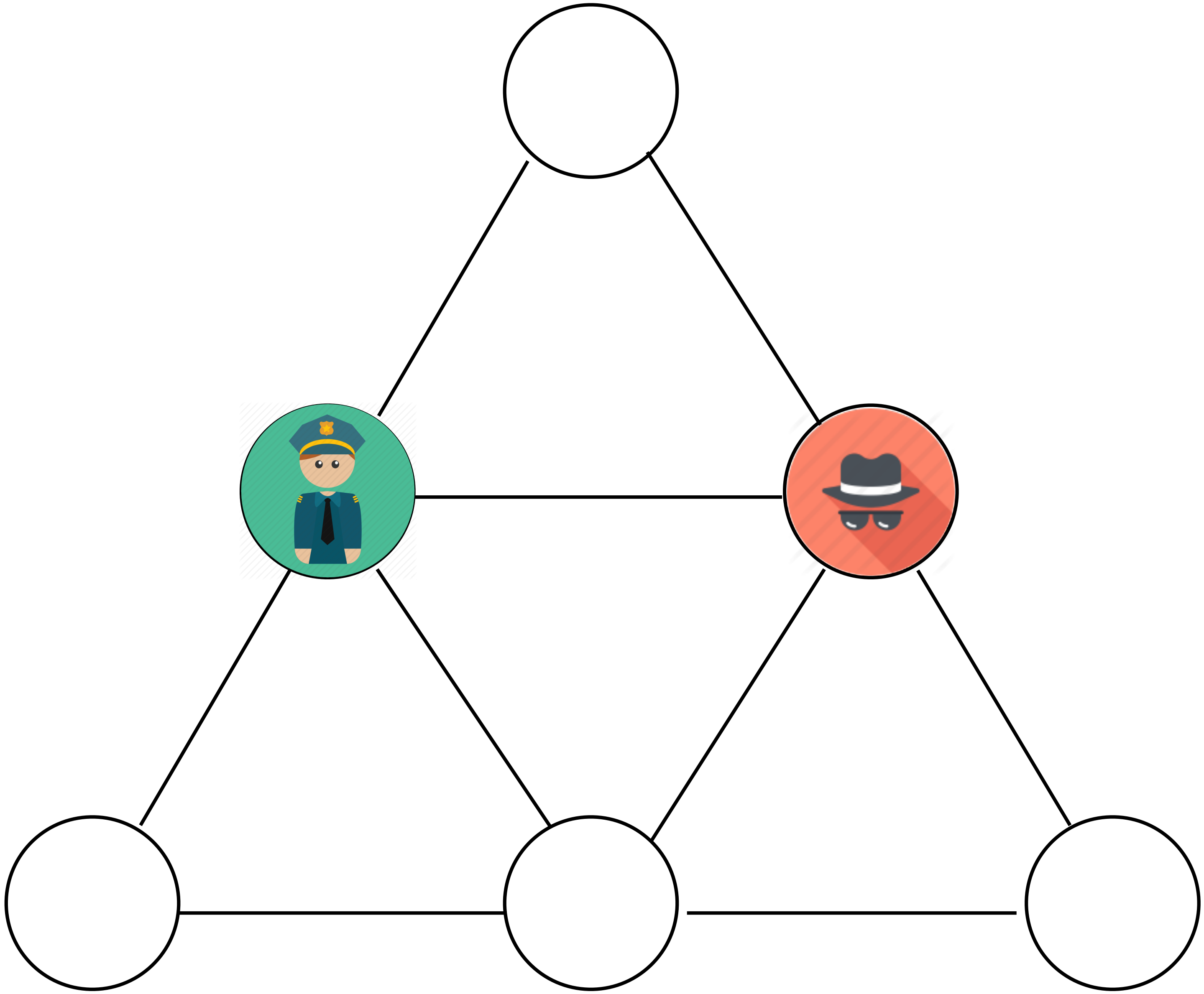
- The cop-number of a graph, $c(G)$, is the minimum number of cops to catch a robber in G.
    - If $c(G)=k$, then G is *k*-cop-win.
    - If $c(G)=1$, then G is cop-win.
- Examples…

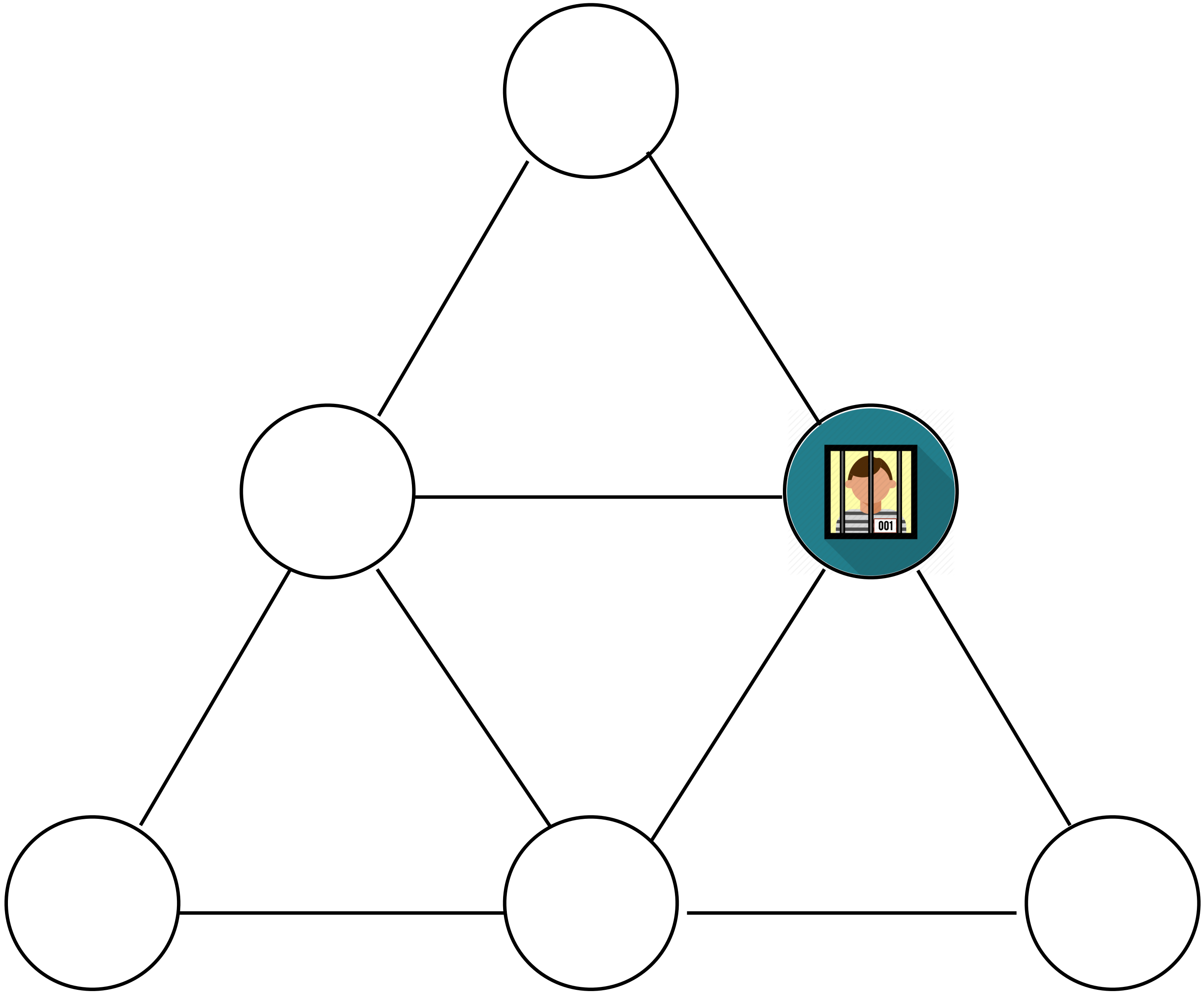








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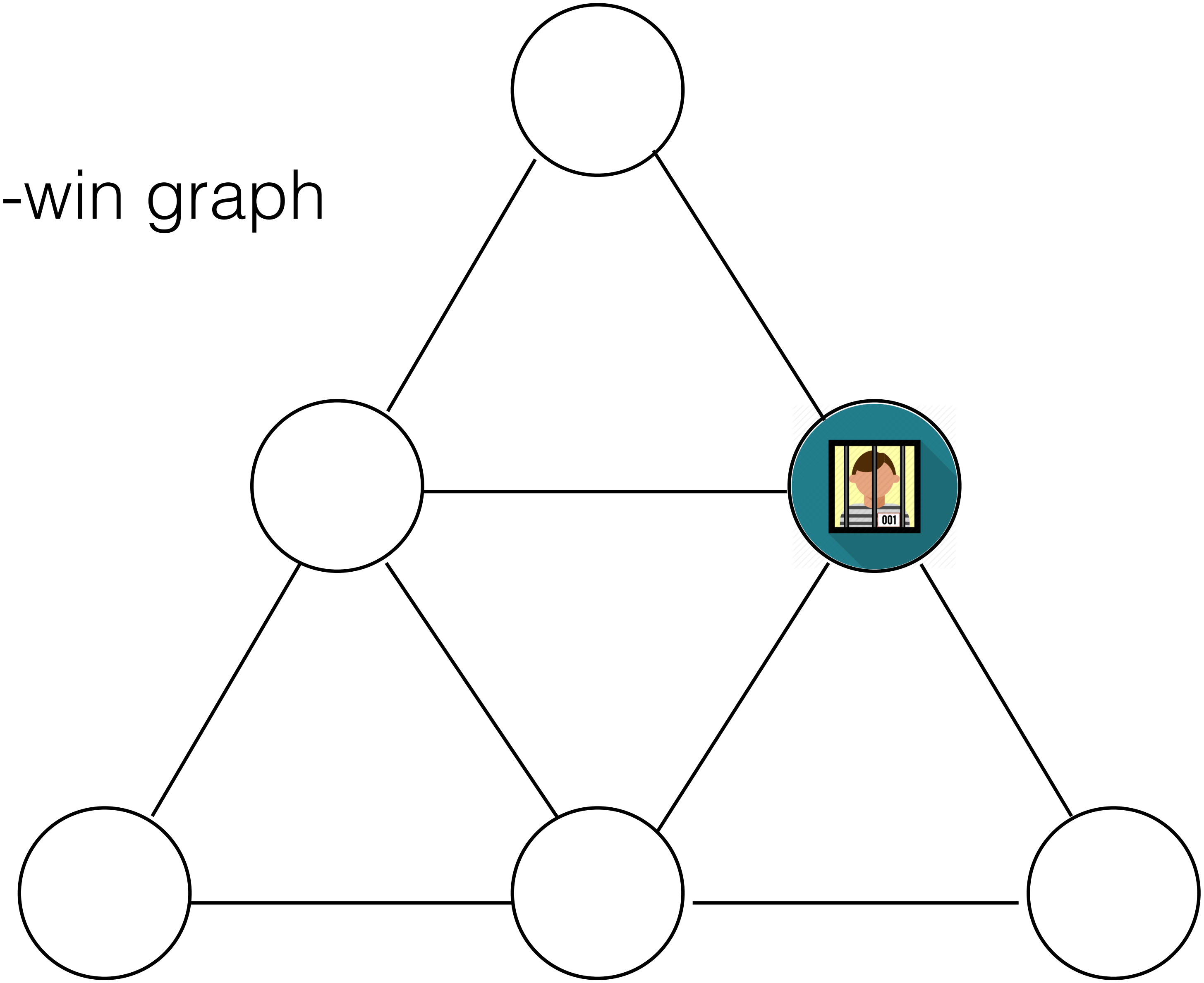
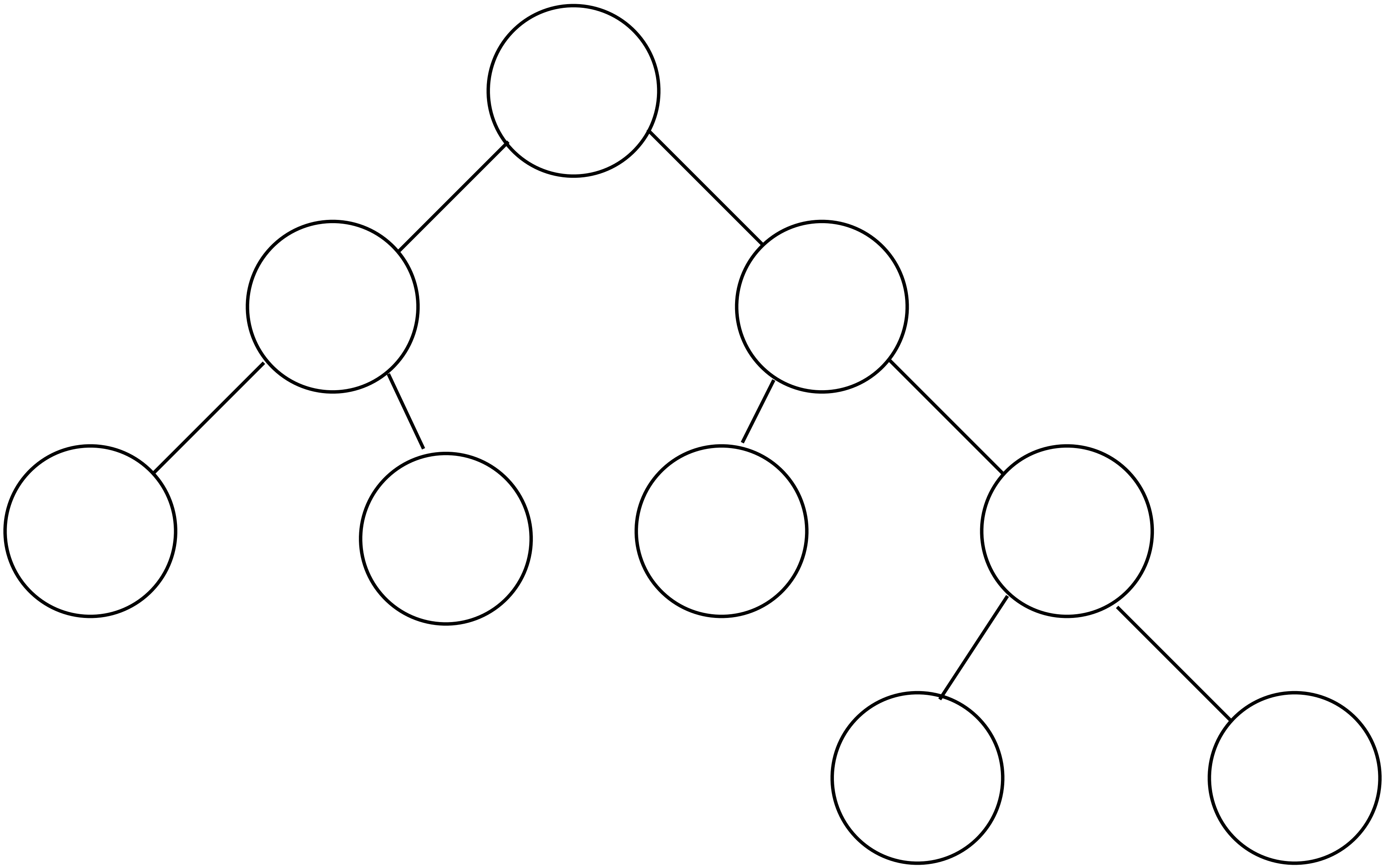
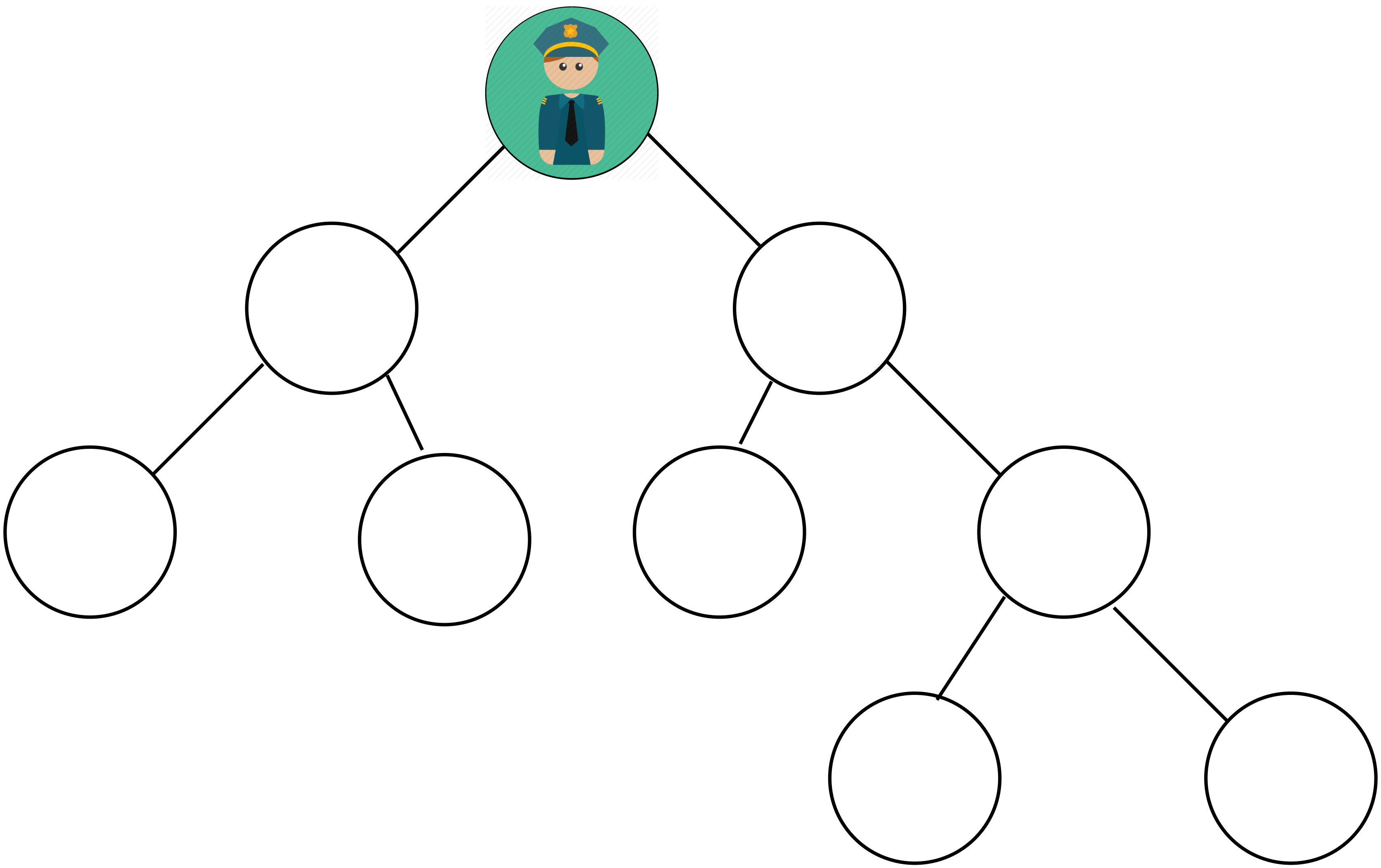
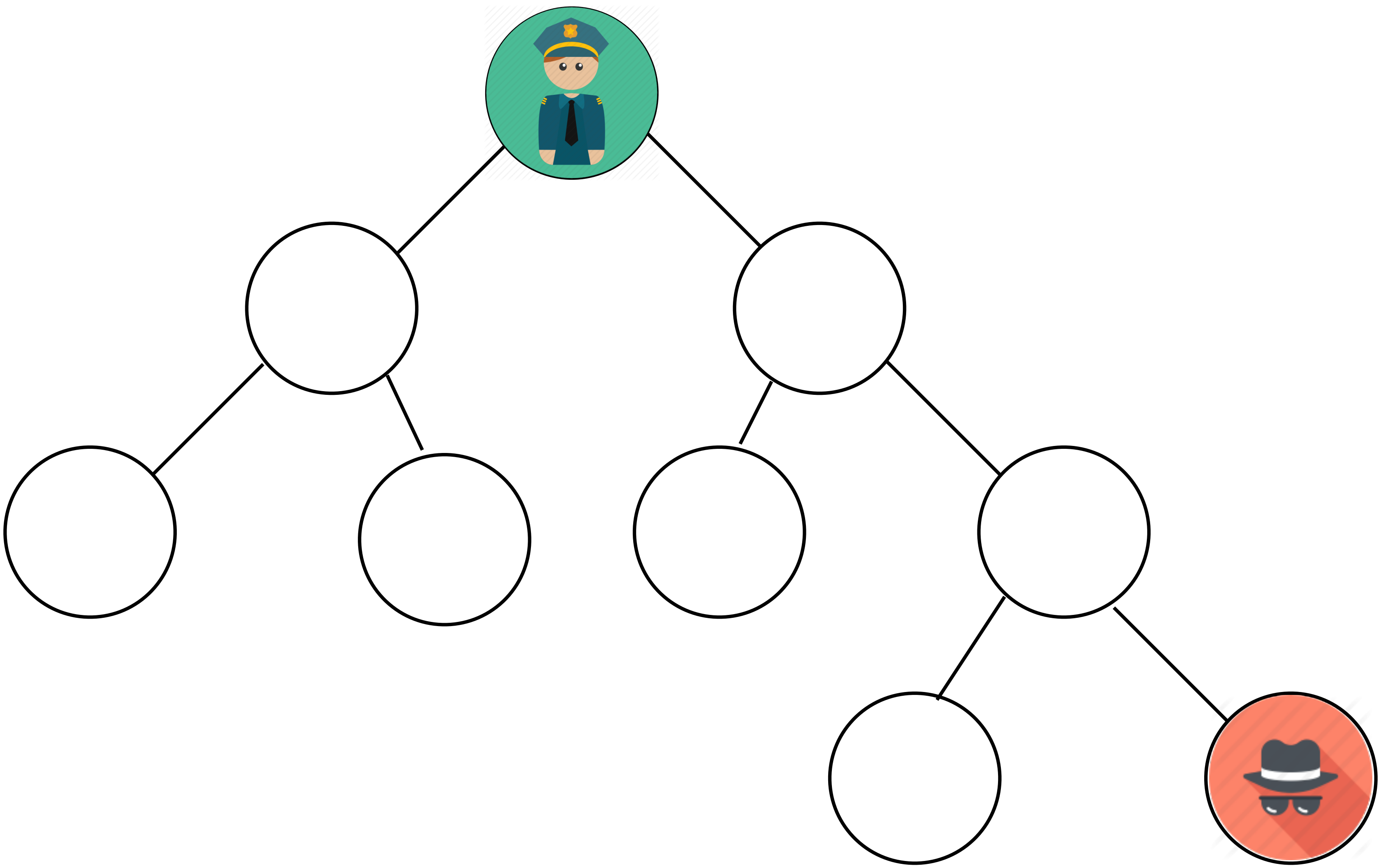
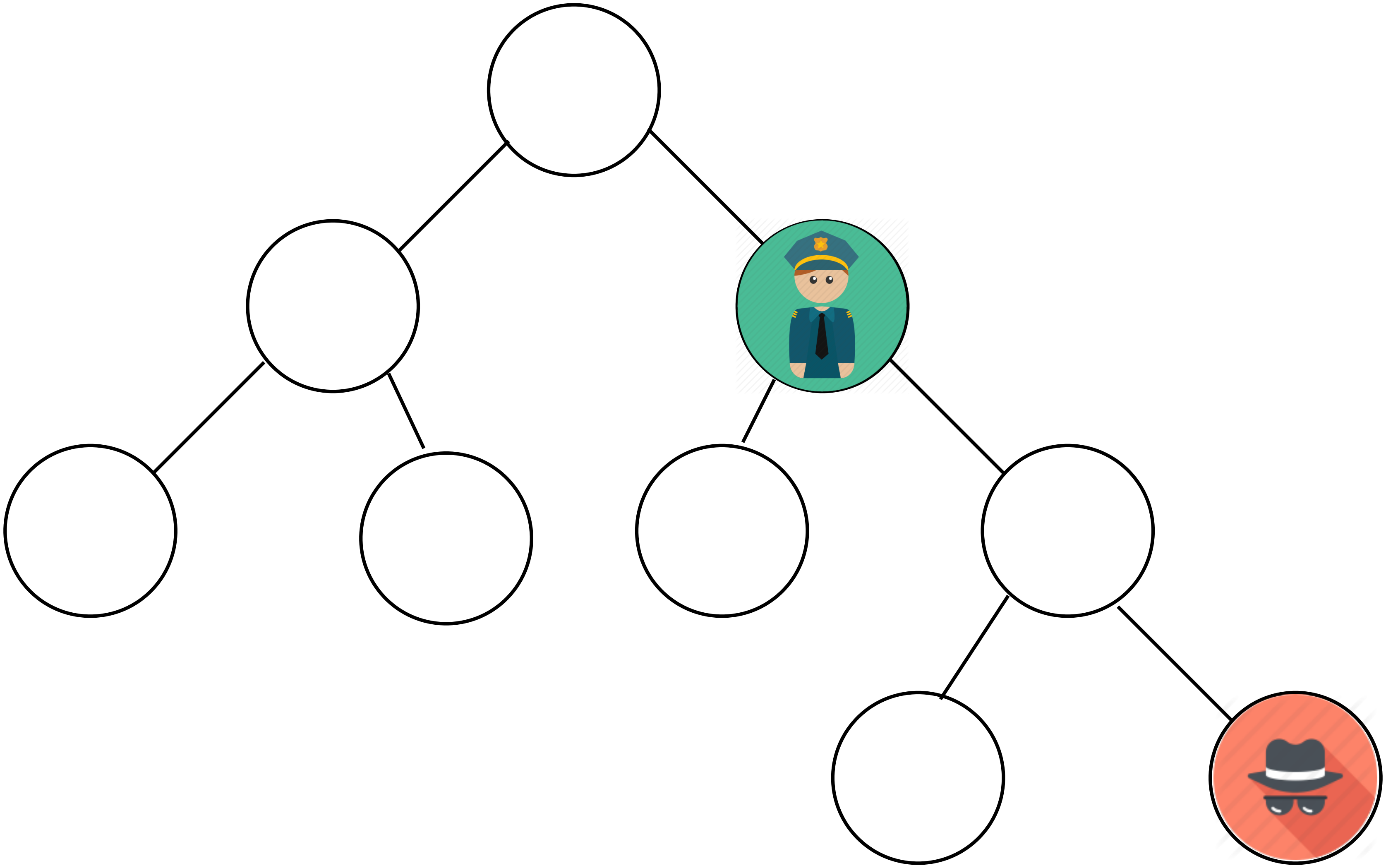
cop-win graph

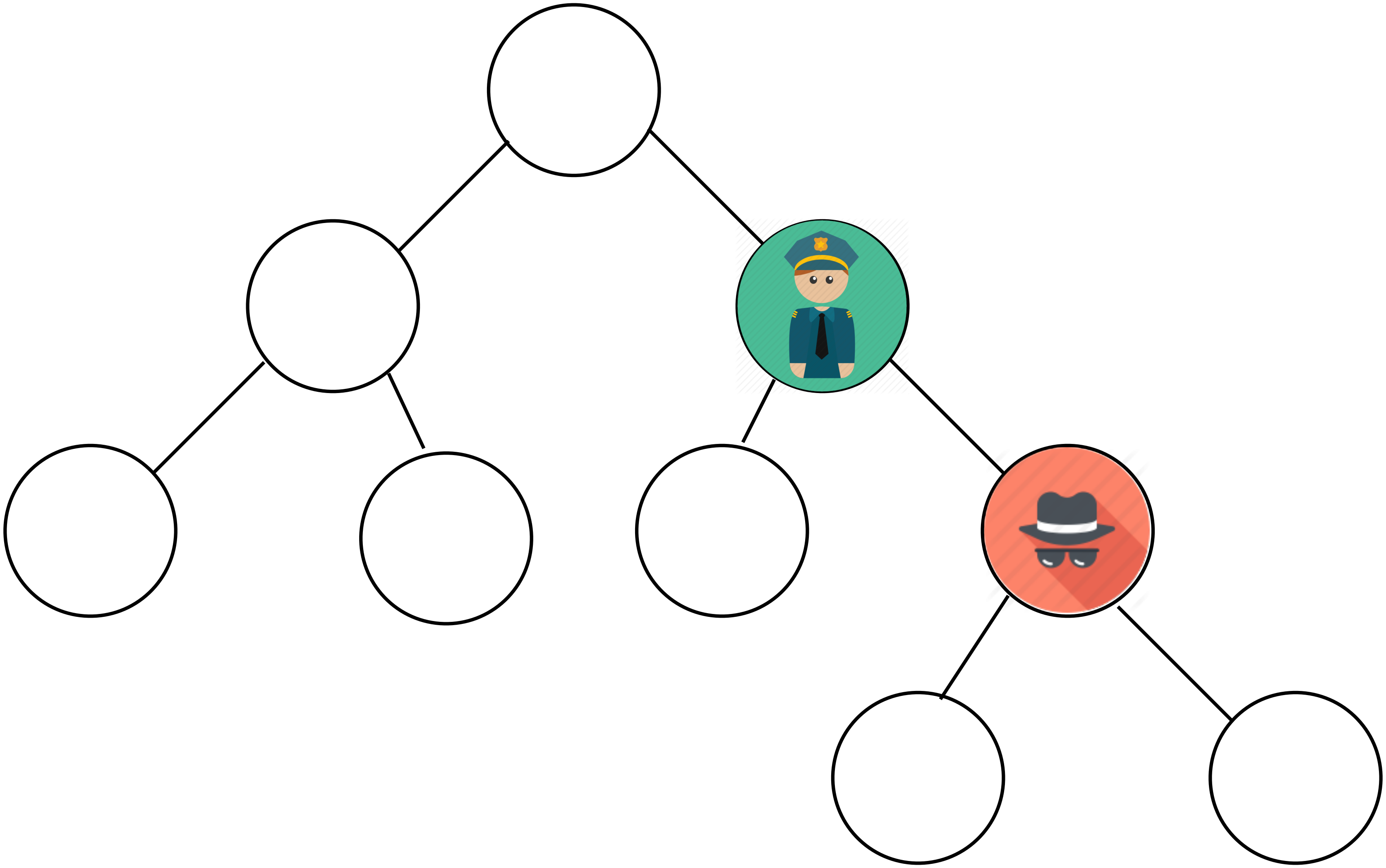


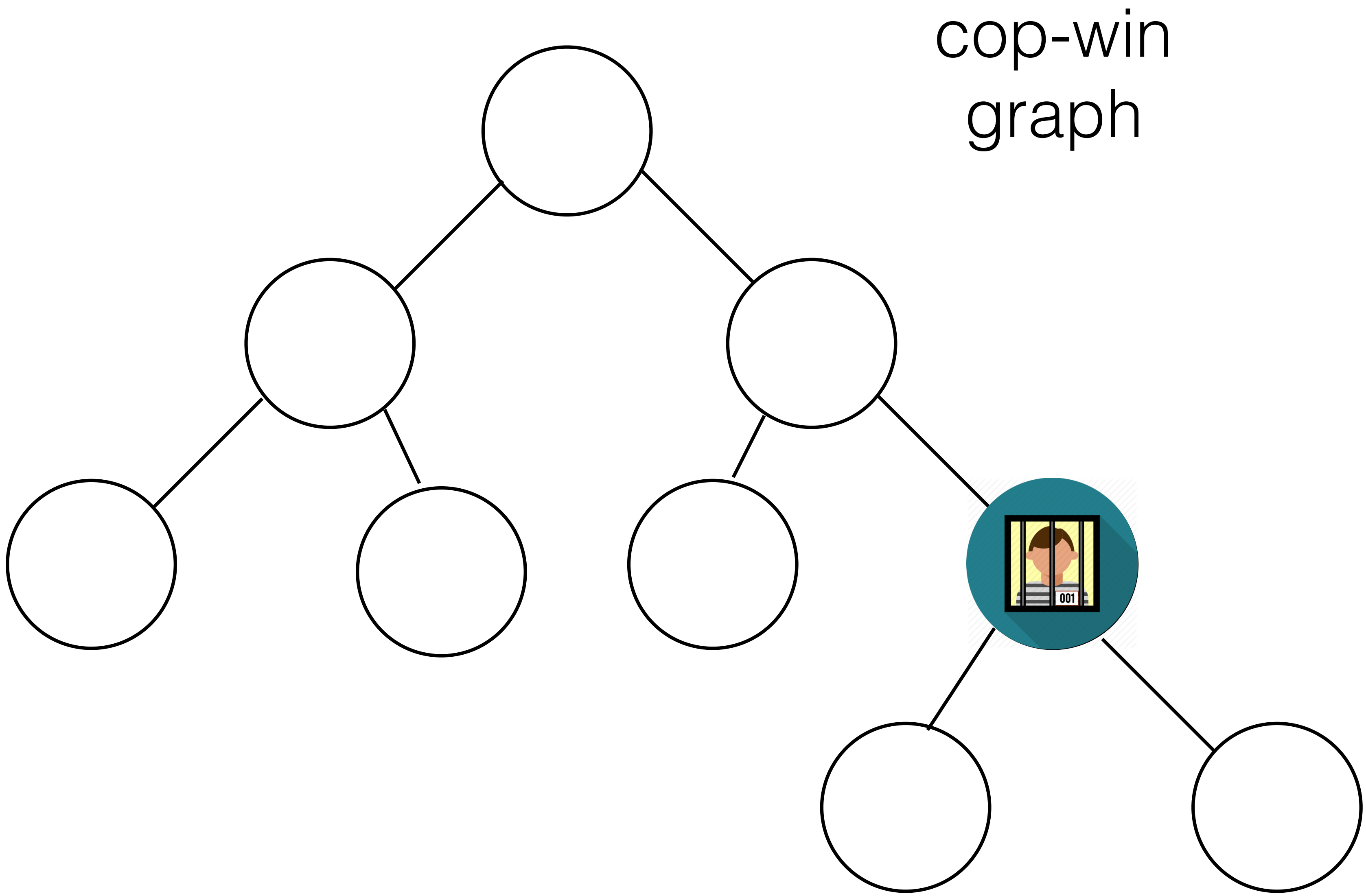








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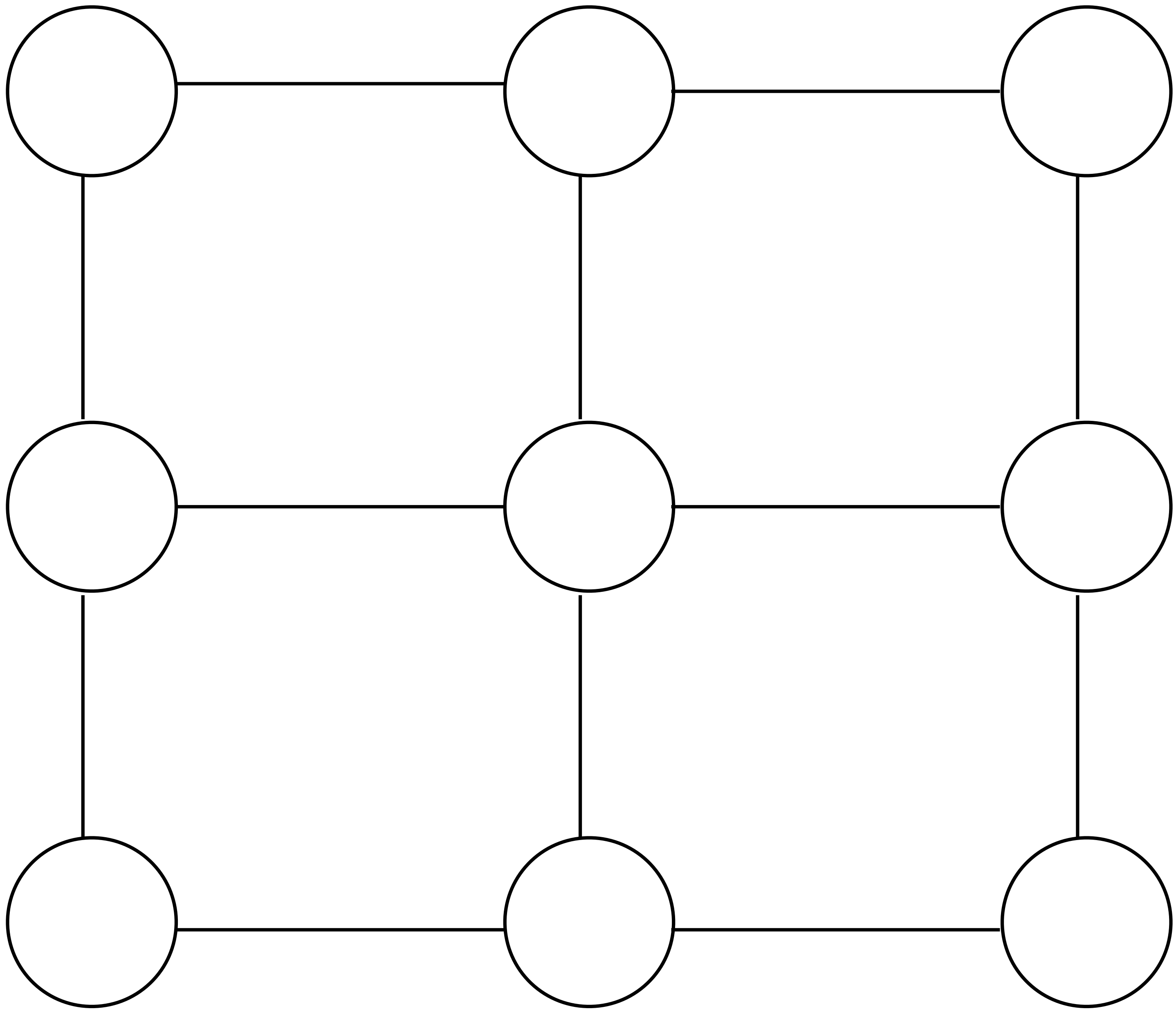

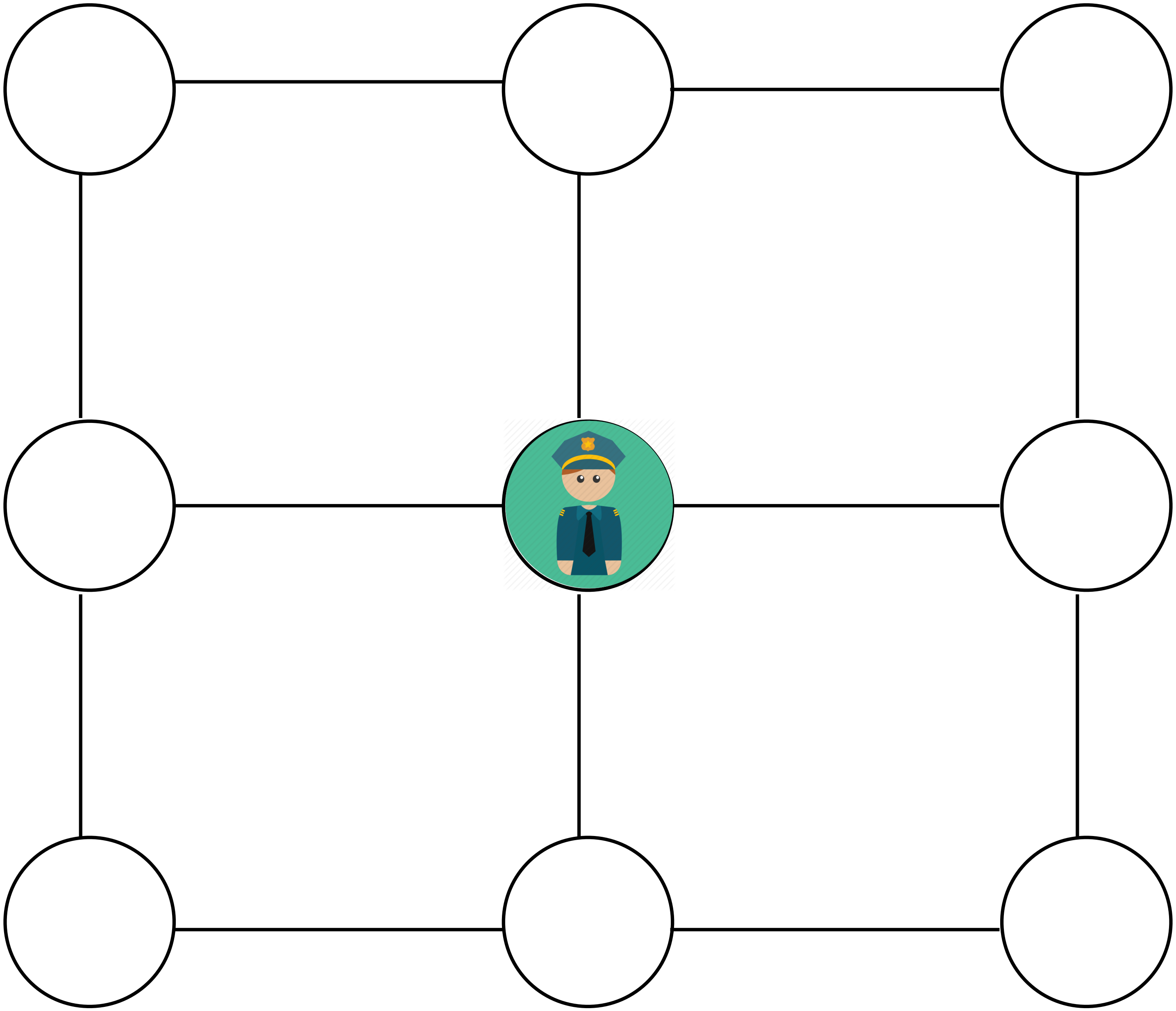

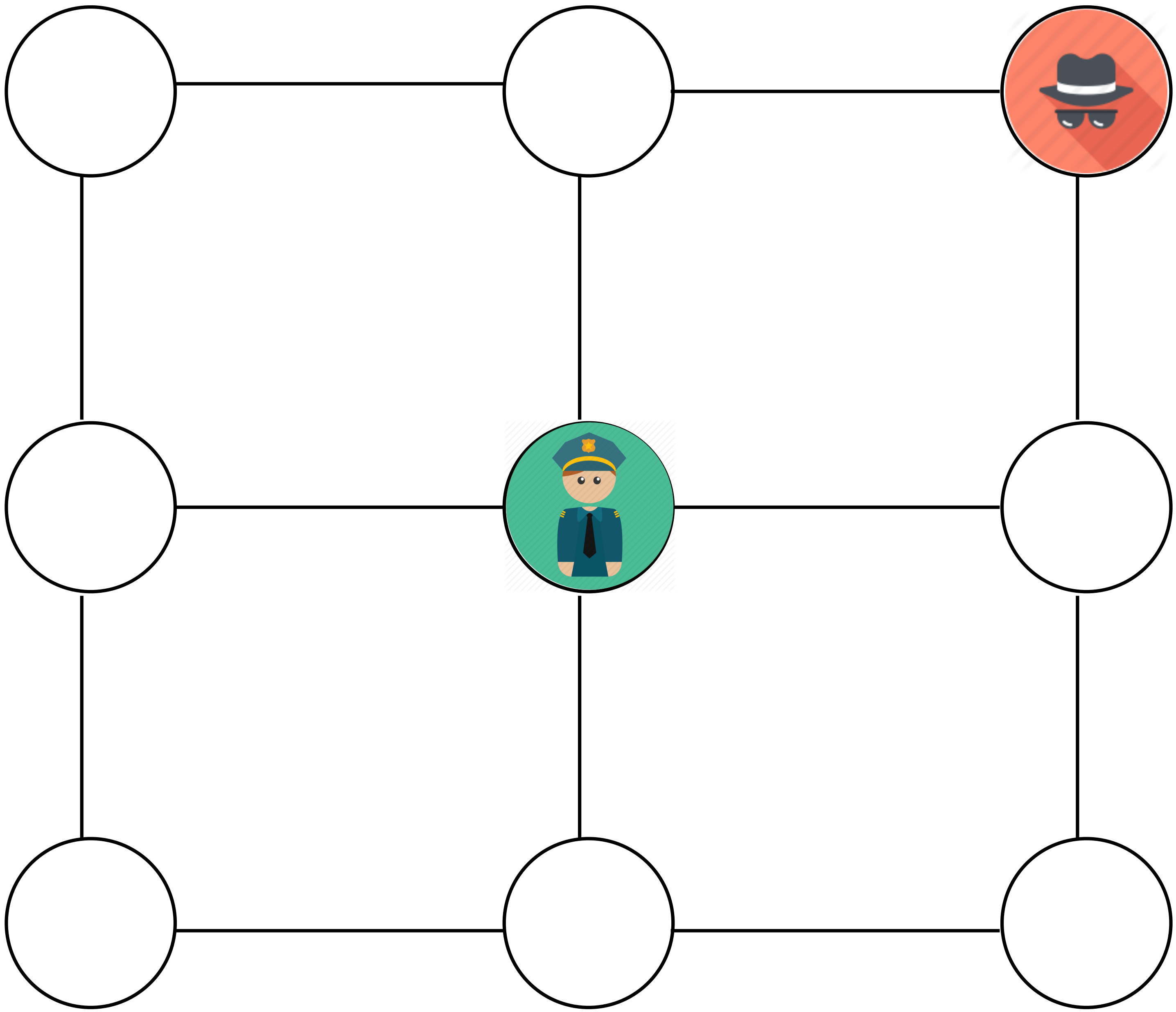

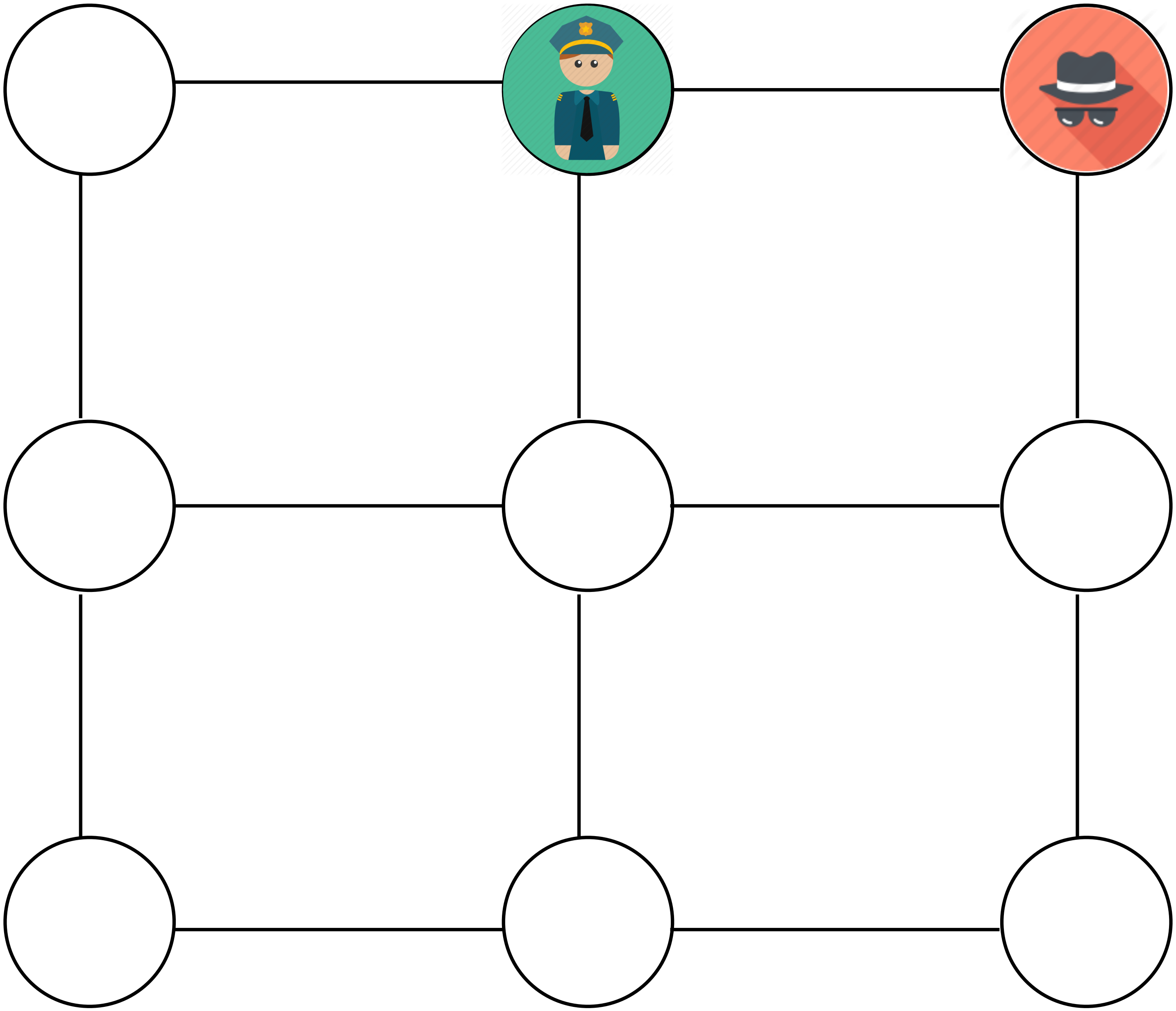

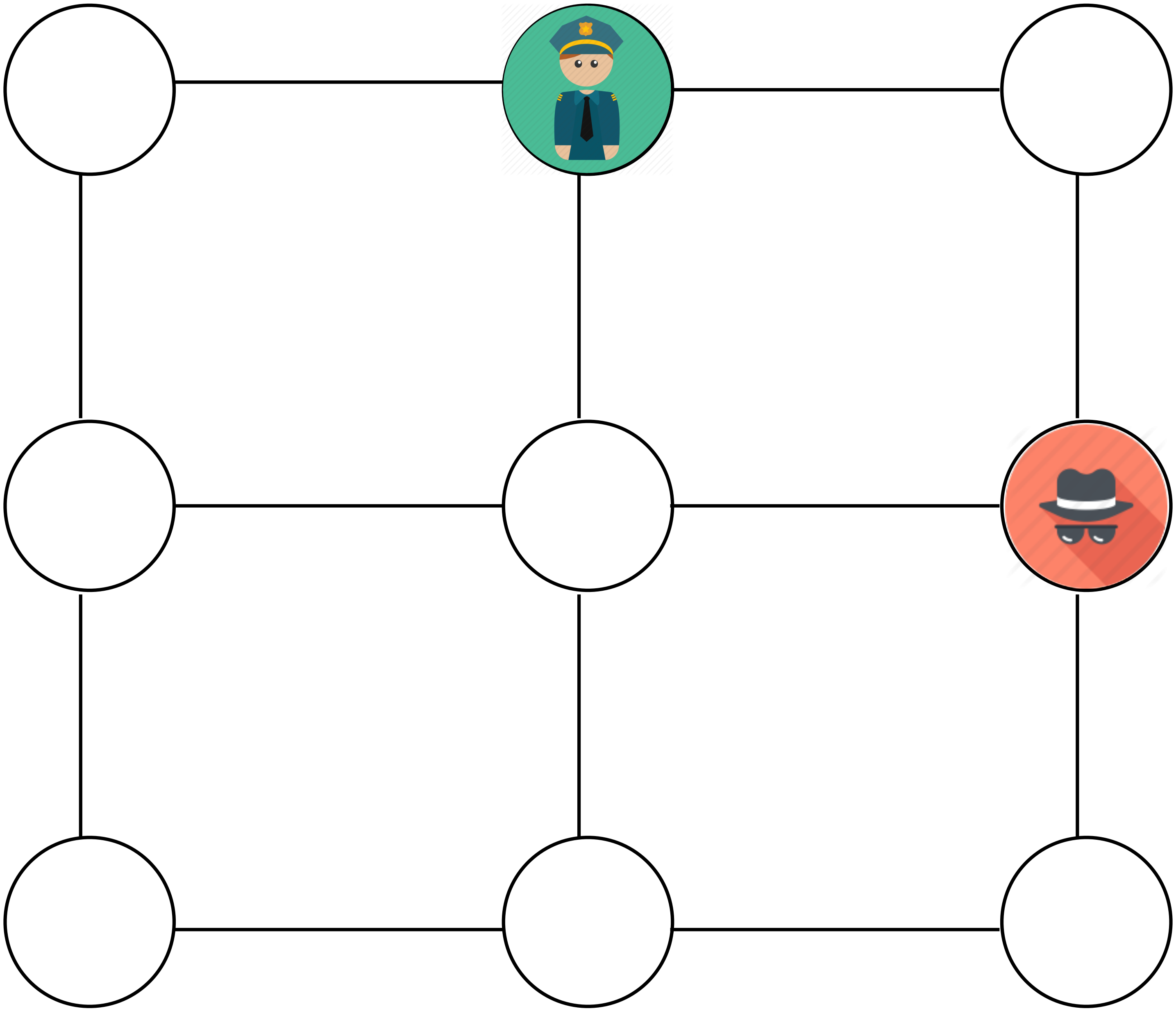

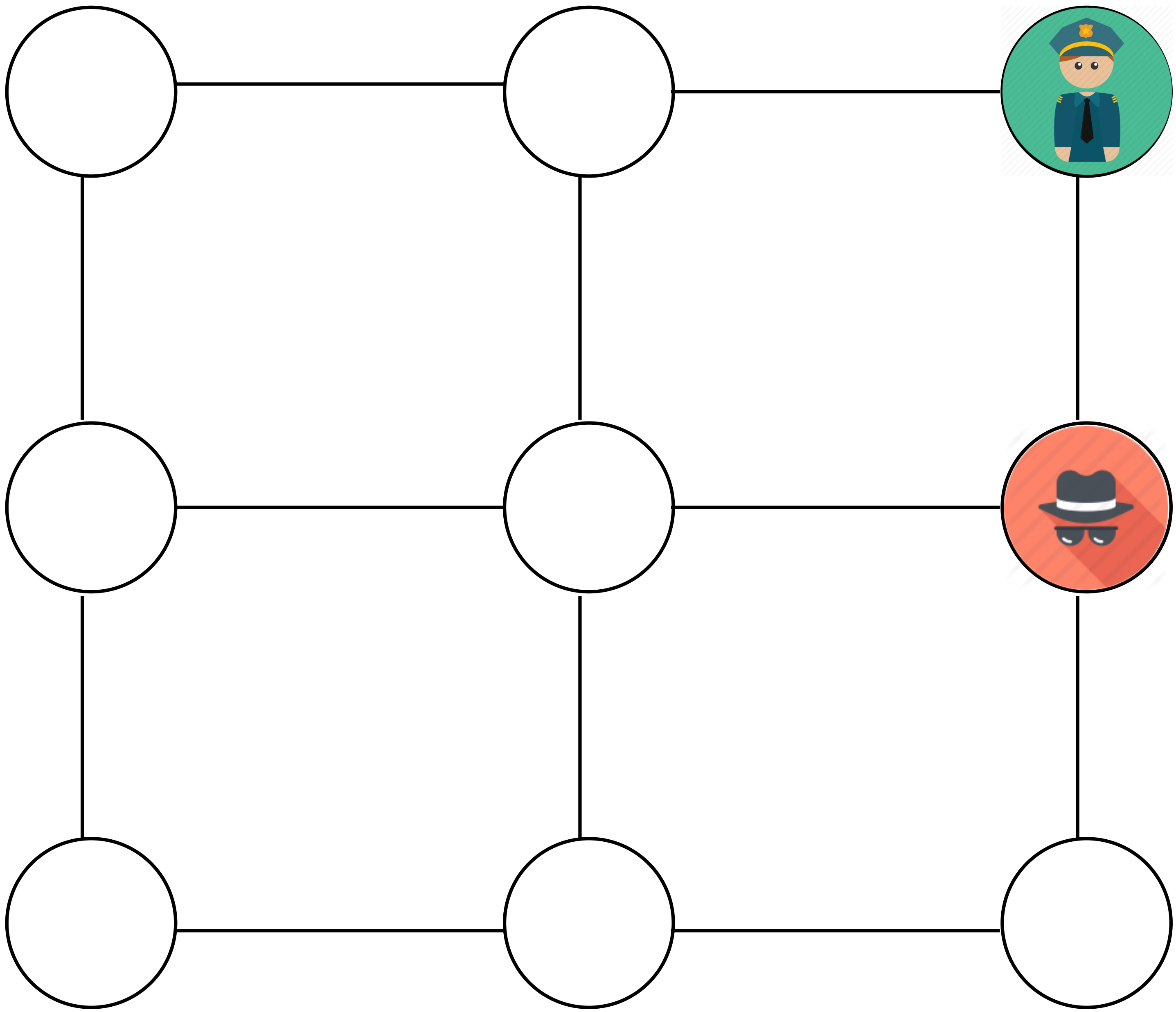

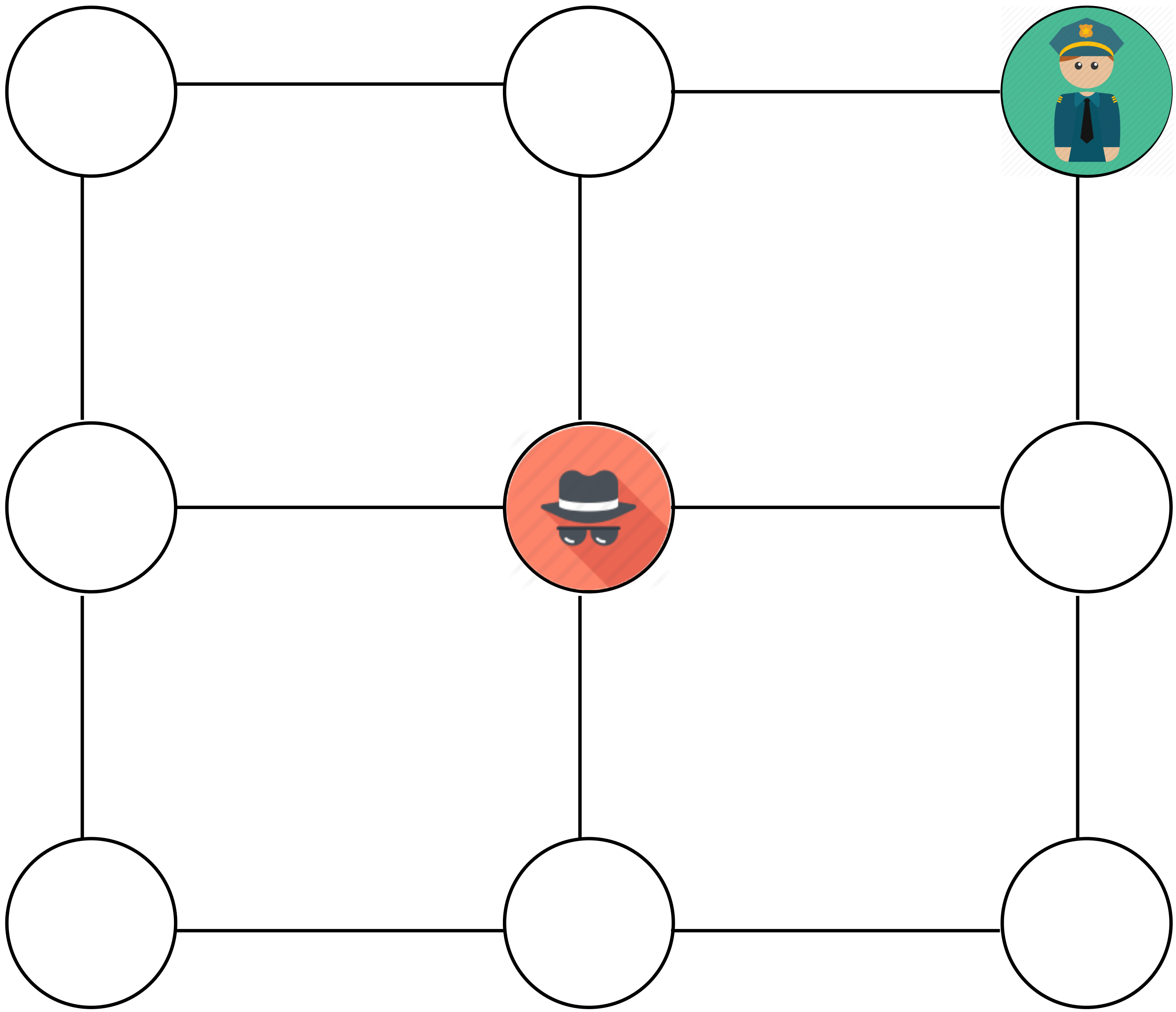

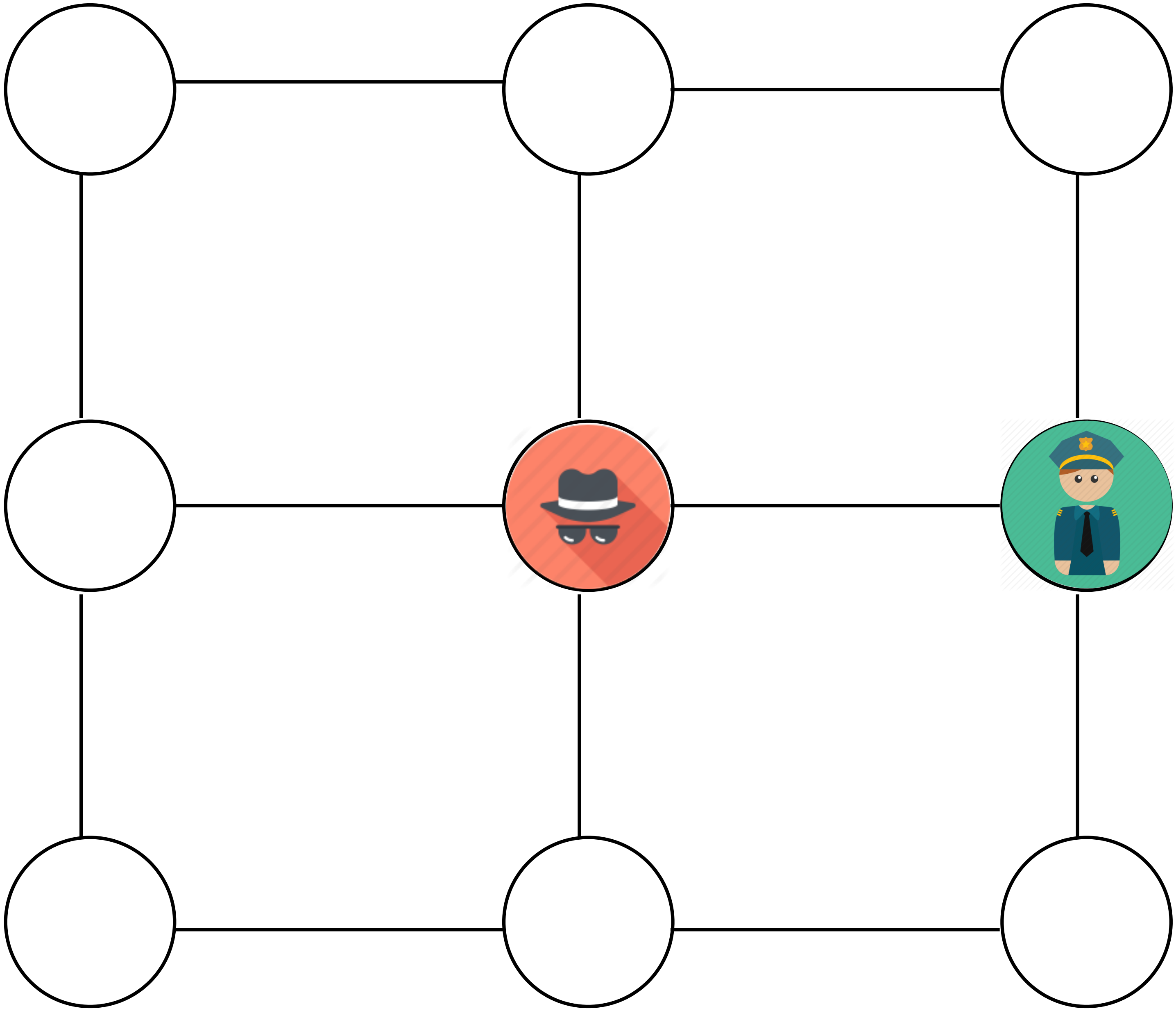

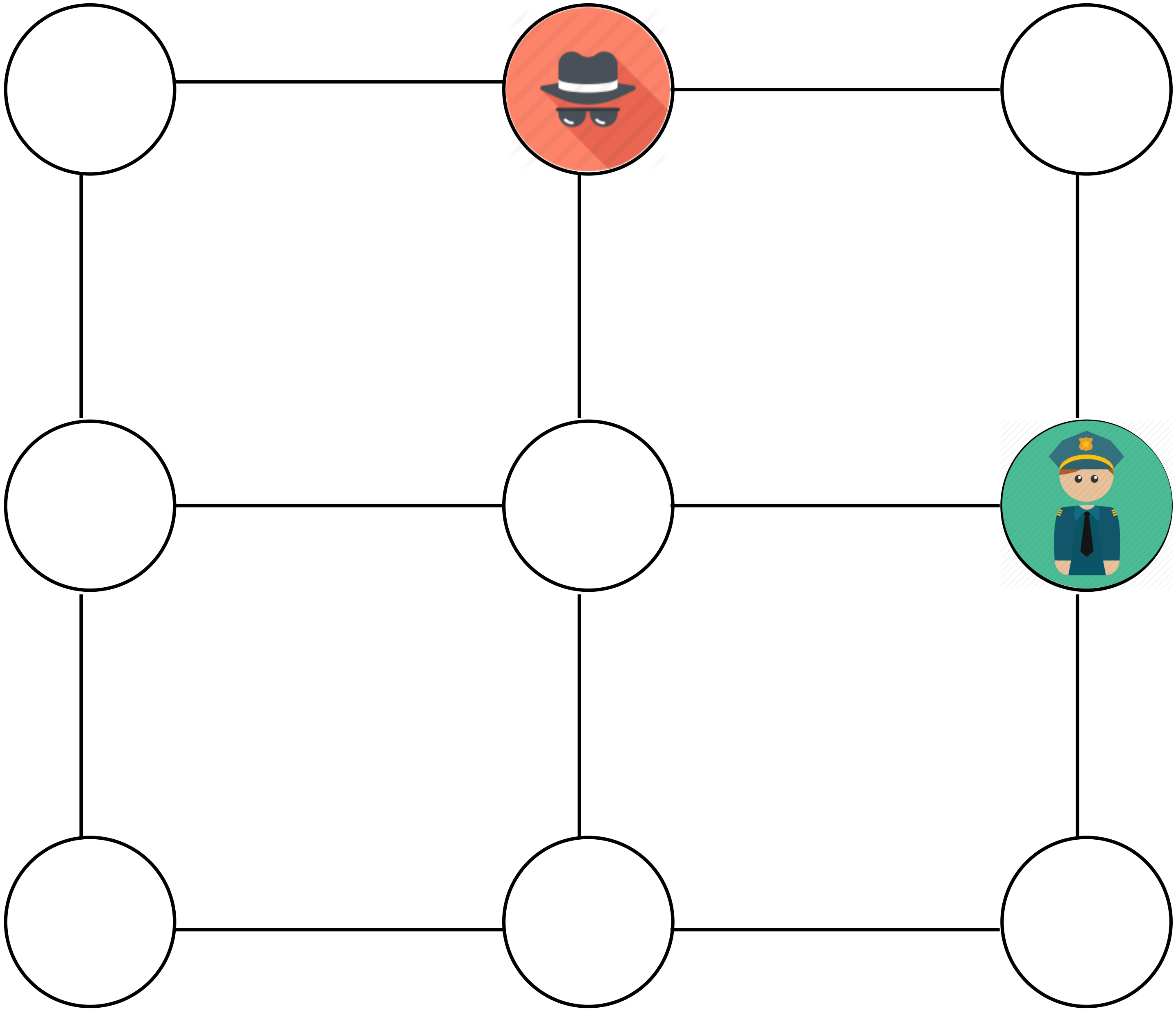

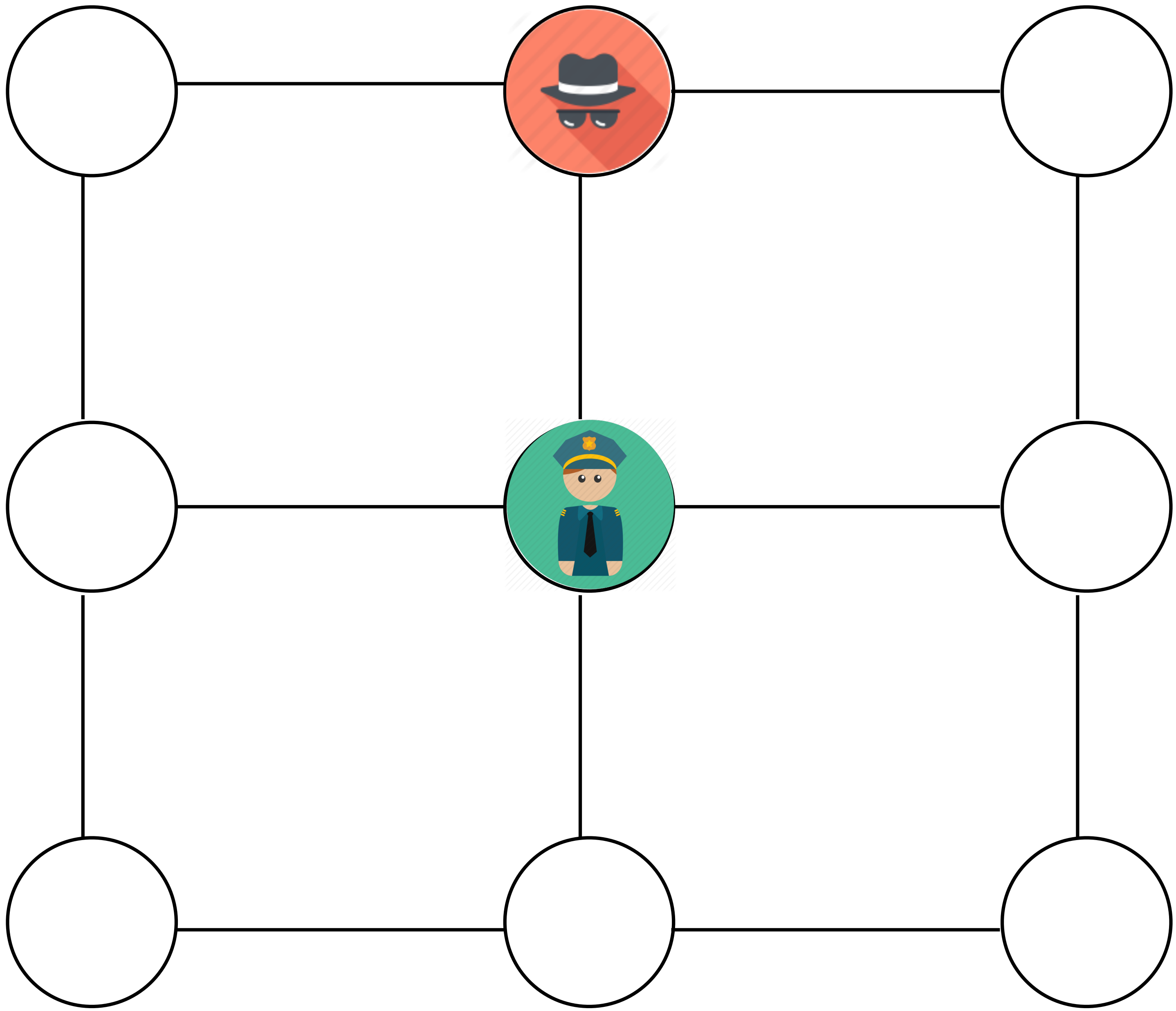

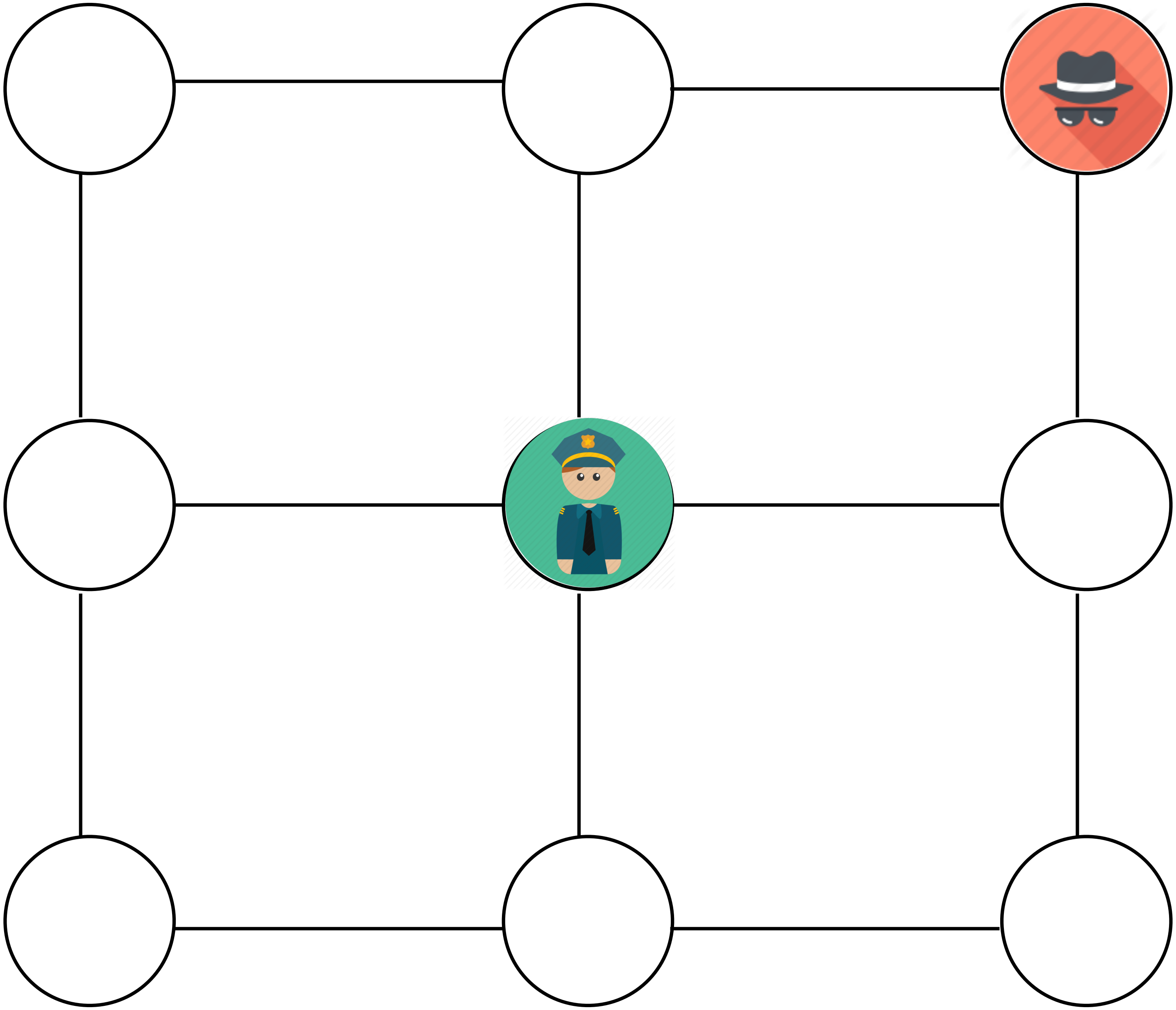

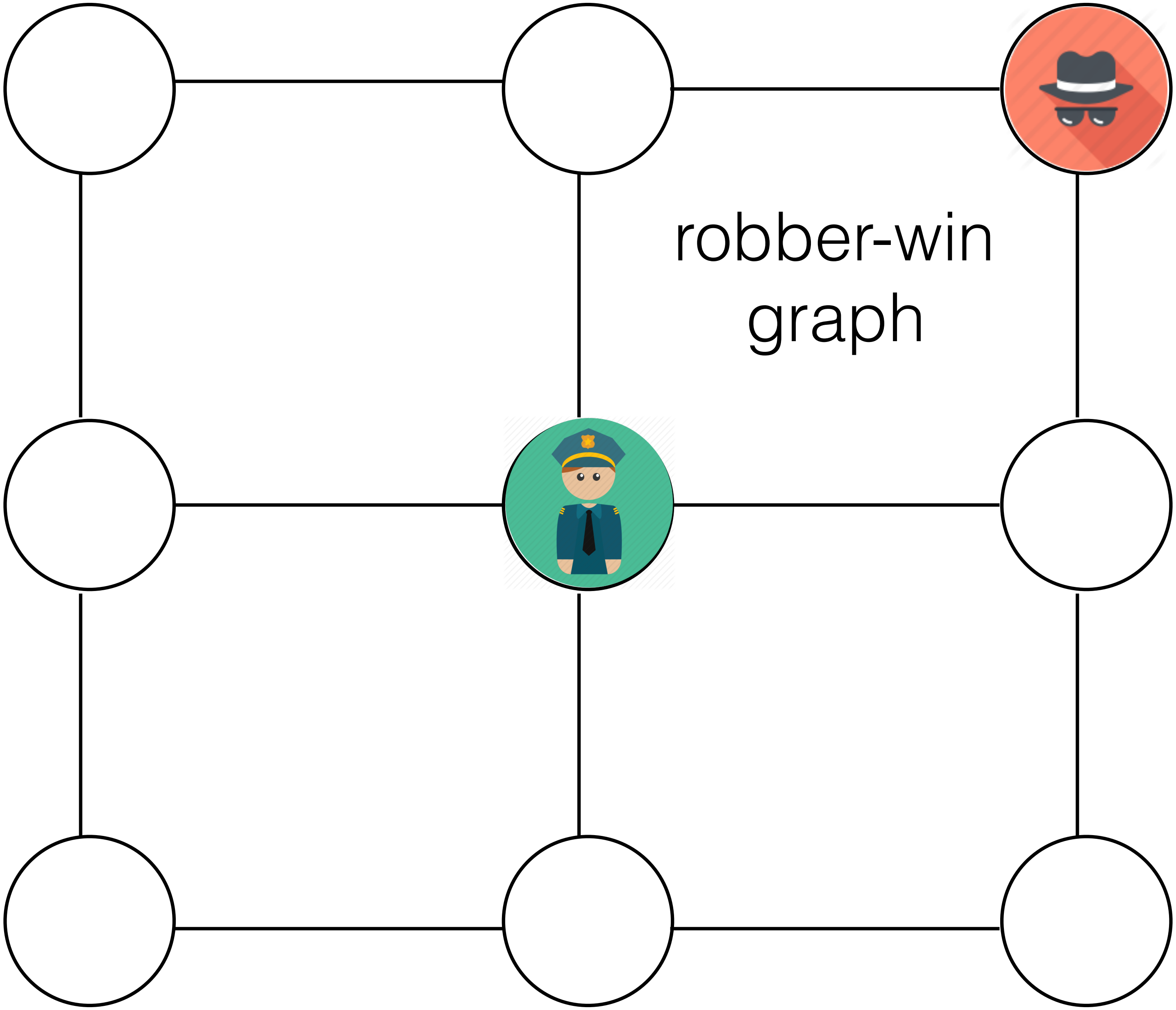
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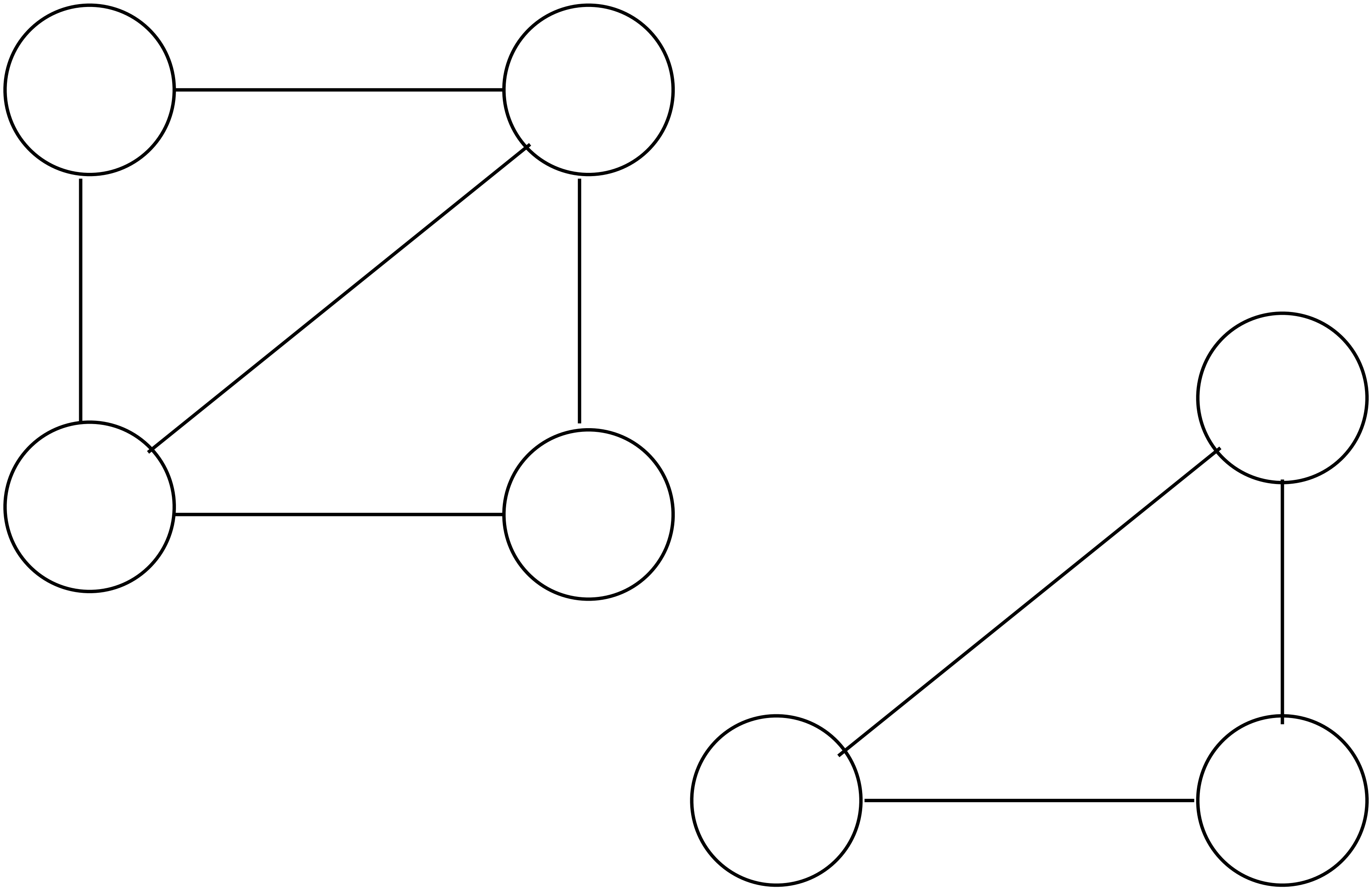

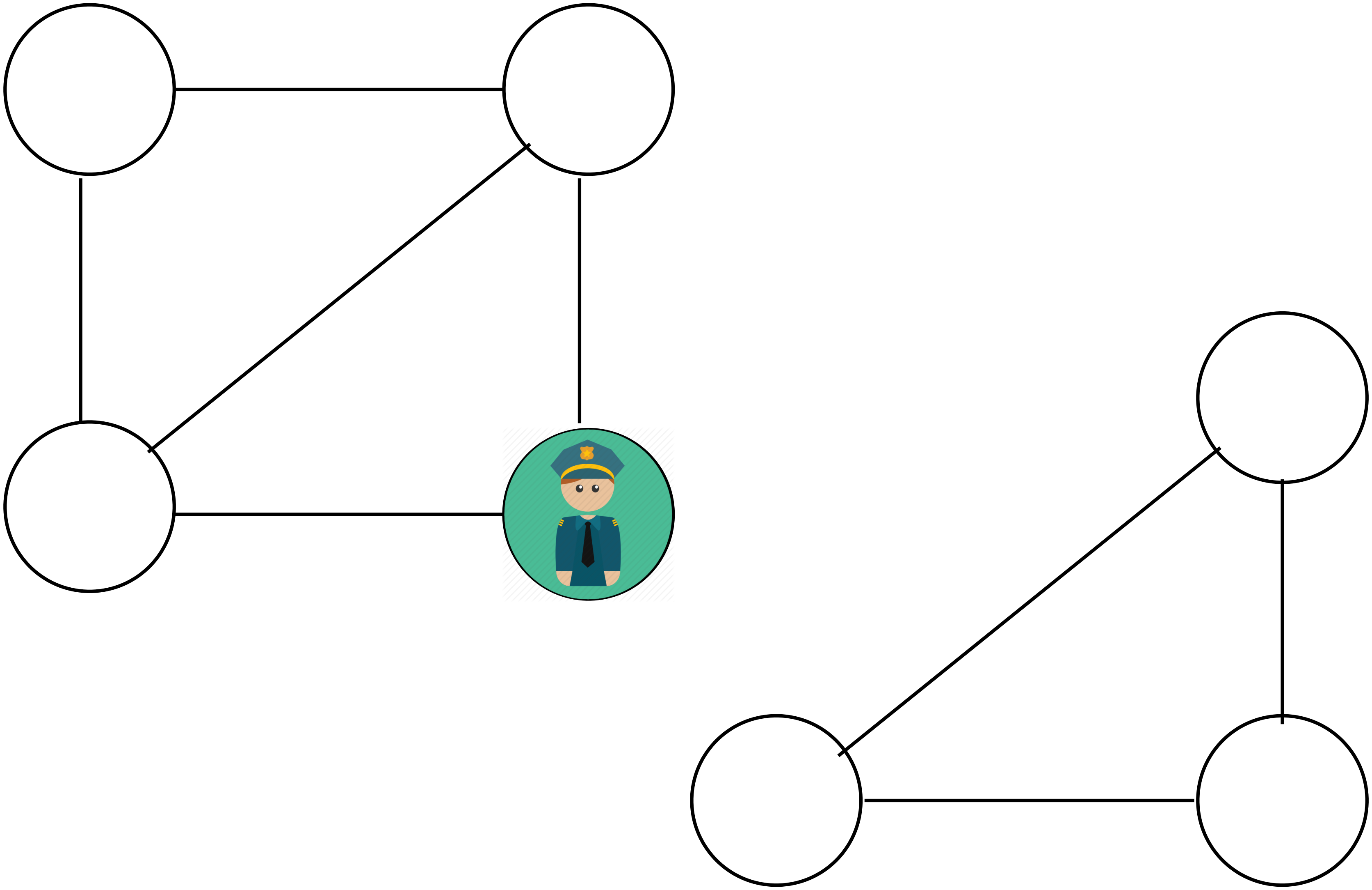

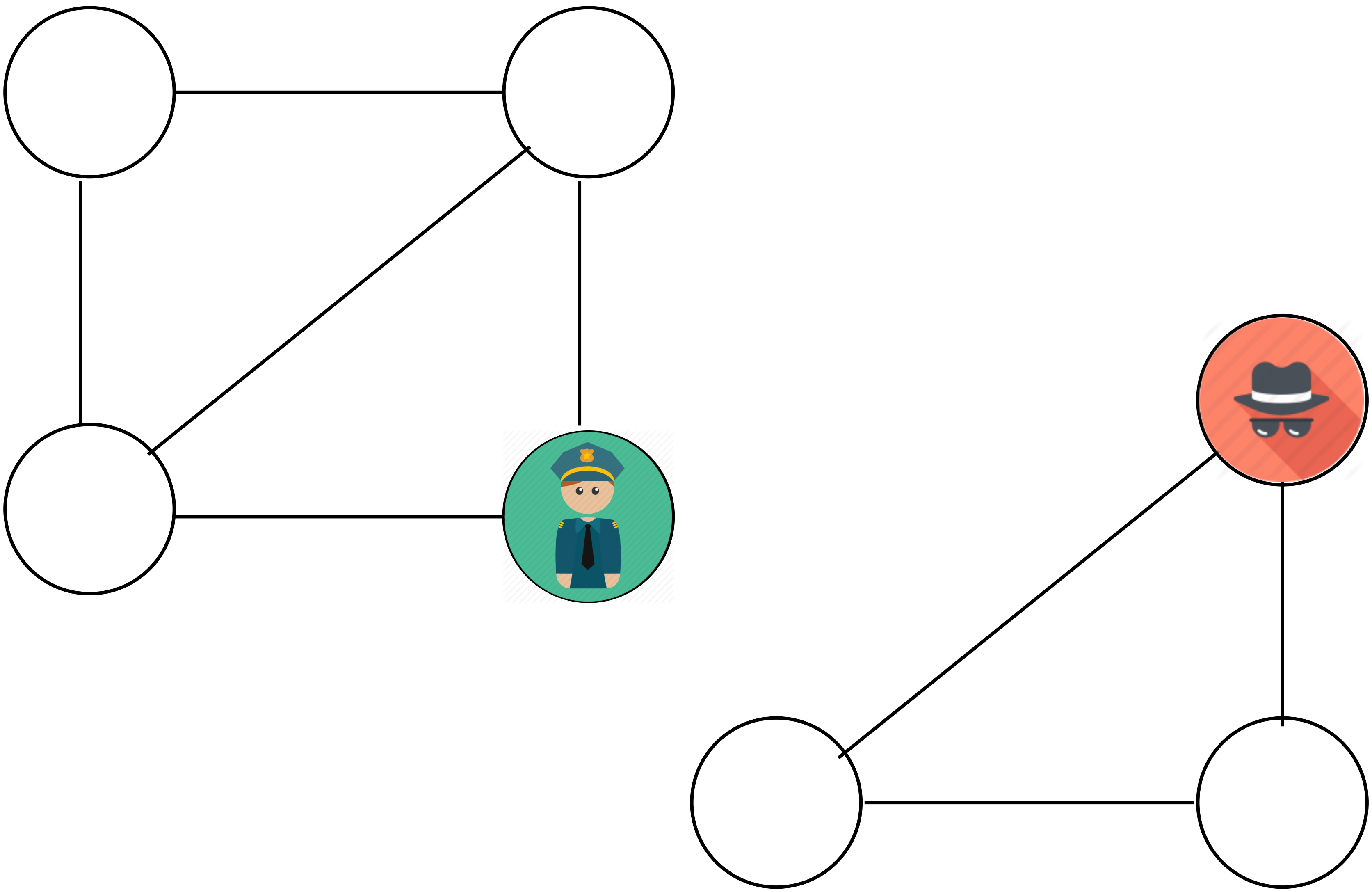

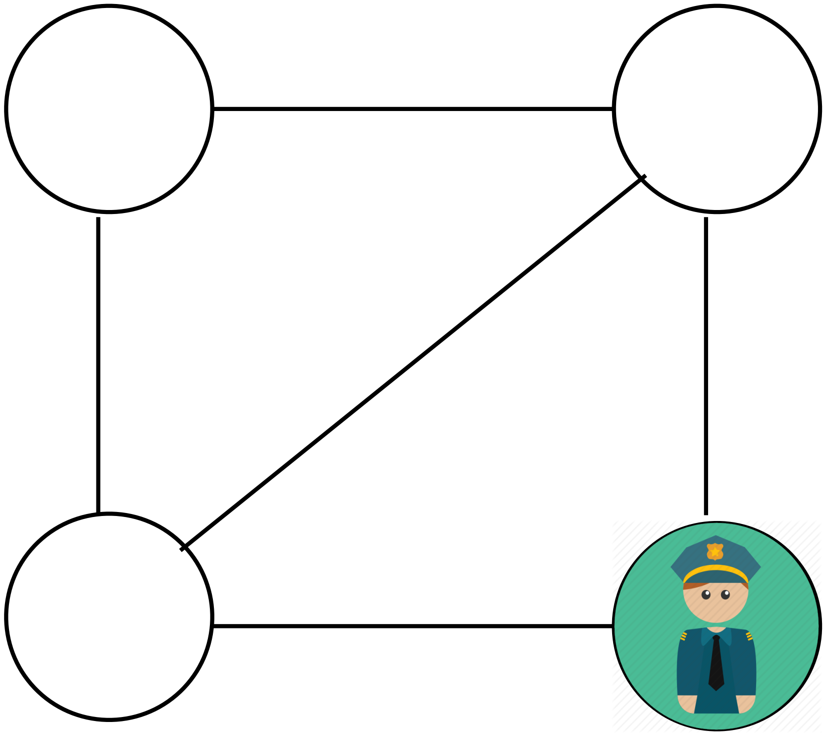

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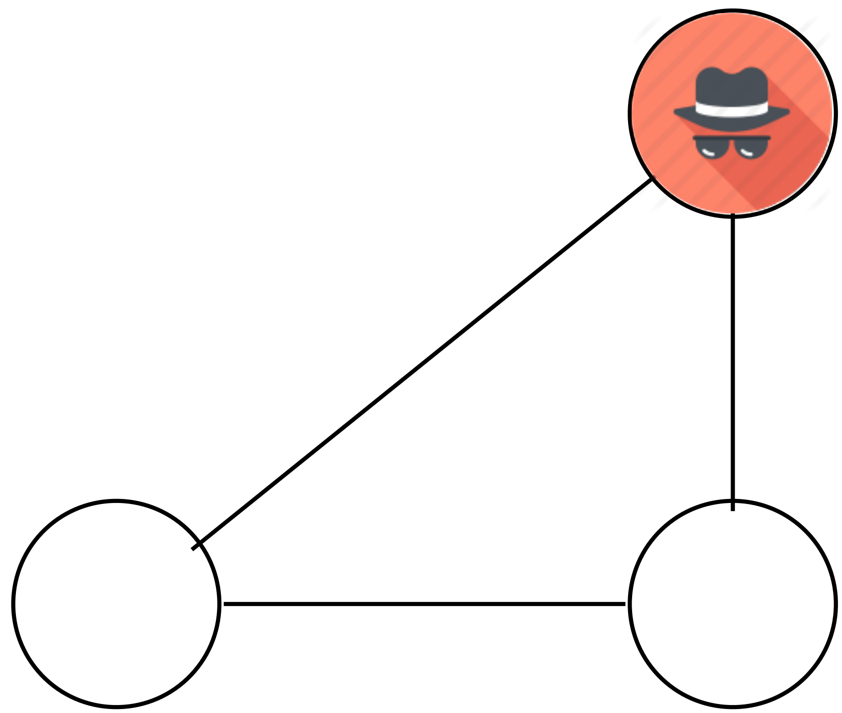
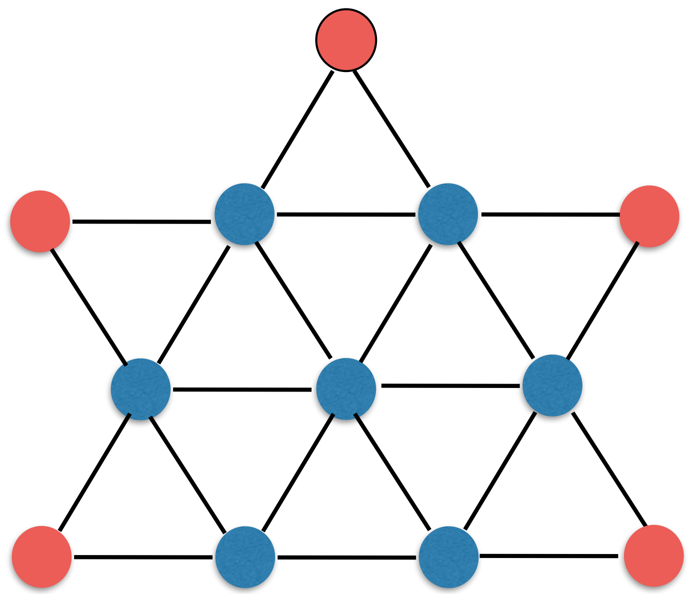


Corners

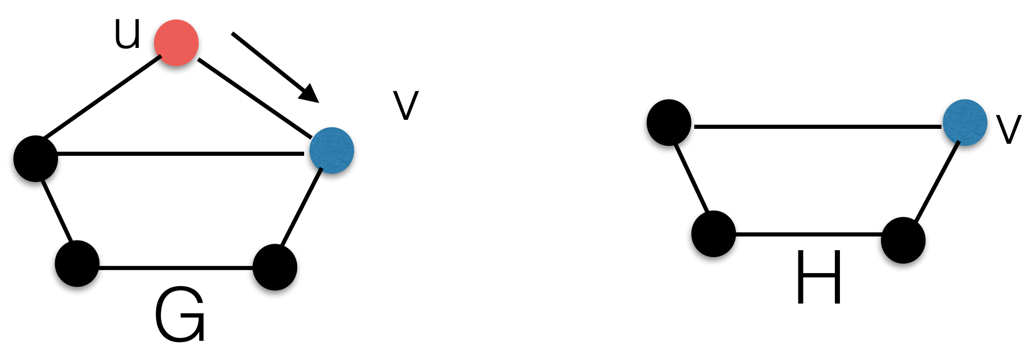
- **Definition**: A vertex u is a **corner** (or a trap, pitfall, or irreducible) if there is some vertex v such that N[u]⊆N[v].



- **Lemma 1**: If G is cop-win, then it has at least one corner.

Retracts

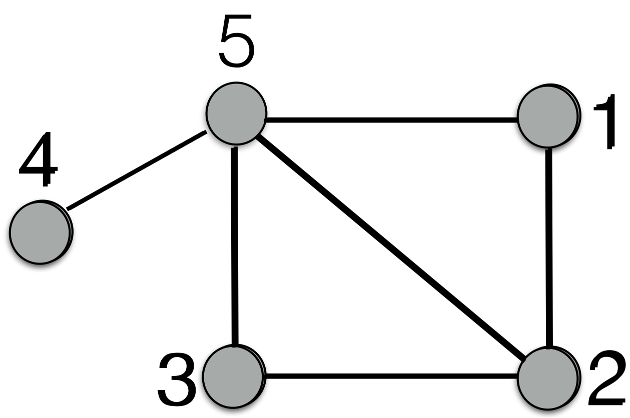
- **Definition**: Let H be an induced subgraph of G formed by deleting one vertex. We say that H is a **retract** of G if there is a homomorphism *f* from G onto H so that f(x)=x for x∈V(H)
- The subgraph formed by deleting a corner u is a retract, given by the mapping $f(x) = \begin{cases} v & \text{if } x=u \\ x & \text{otherwise} \end{cases}$



- **Theorem 2:** If H is a **retract** of G, then c(H) ≤ c(G).
- **Corollary 3**: If G is cop-win, then so is each retract of G.

Dismantling

- **Definition**: A graph is dismantlable if some sequence of deleting corners results in the graph K_1.

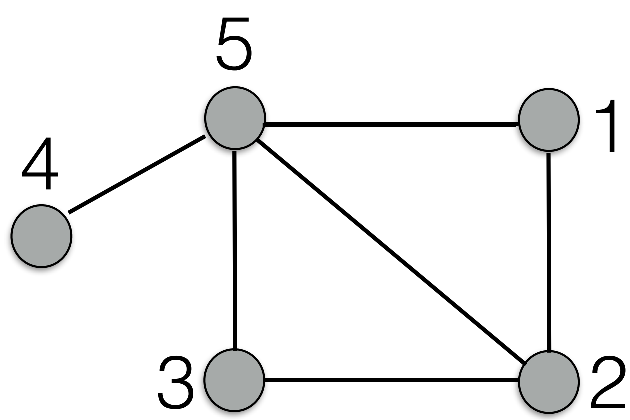


- **Theorem 4:** A graph G is cop-win if and only if it is dismantlable.

Cop-win Ordering

- **Definition:** A cop-win ordering is a sequence of positive integers [n] such that for each $i<n$, the vertex i is a corner in the subgraph induced by {i, i+1, …, n}.

cop-win ordering: | i, | i+1, ……., | j, | j+1, …., n |



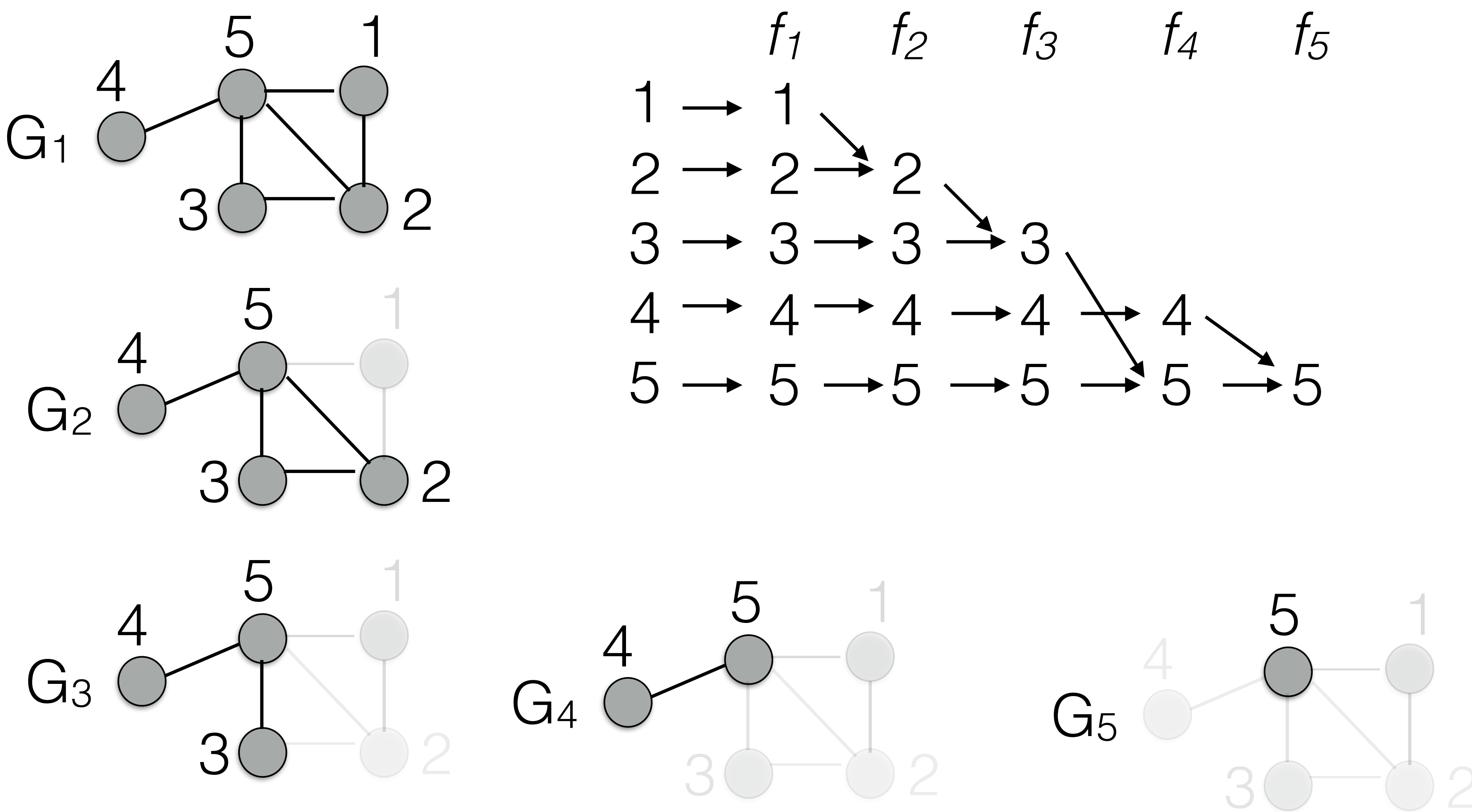
- A cop-win ordering for the above graph is {1,2,3,4,5}

Cop-win Strategy (Preliminaries)

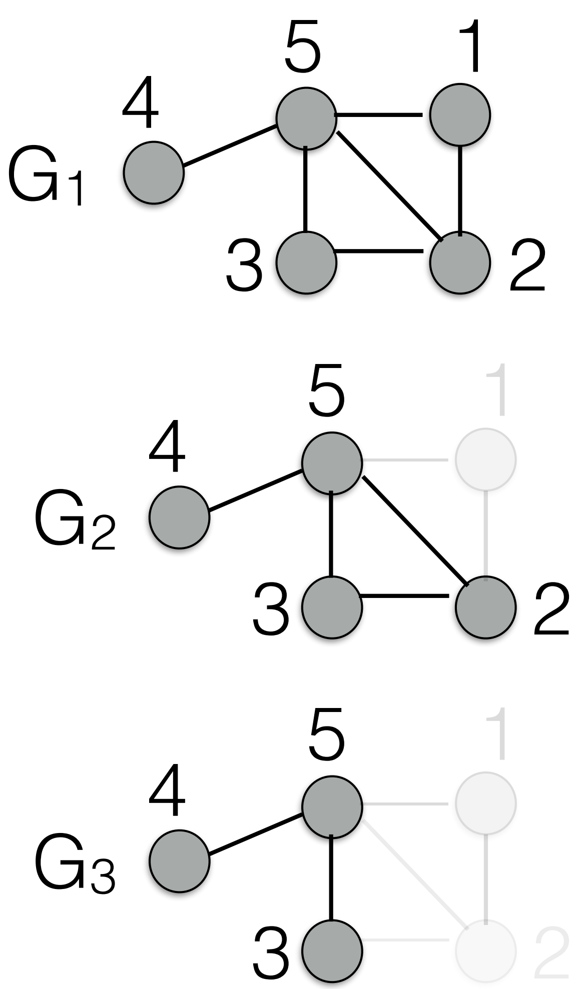
Assume [n] is a cop-win ordering of G

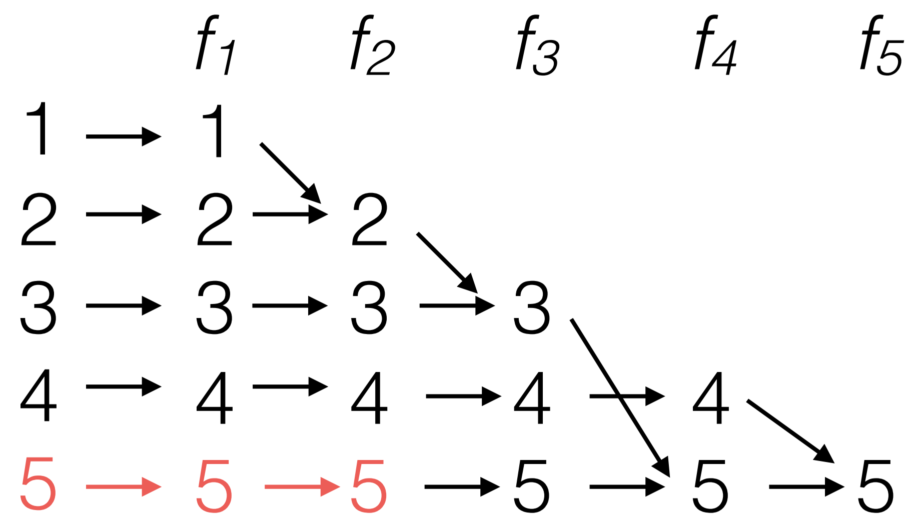
- The winning strategy involves the cop "shadowing" the movements of the robber in increasingly larger induced subgraphs of G
- for $1 \leq i \leq n$ define $G_i = G \upharpoonright \{n, n-1, \ldots, i\}$ (the increasingly smaller subgraphs of G)
- for each $1 \leq i \leq n-1$, let $f_i : G_i \rightarrow G_{i+1}$ (the retractions which remove a corner to make a smaller subgraph)
- $F_i = f_{i-1} \circ \ldots \circ f_2 \circ f_1$ (a composition of retractions - this will be the "shadow" that the cop will follow)

Cop-win Strategy (Preliminaries)

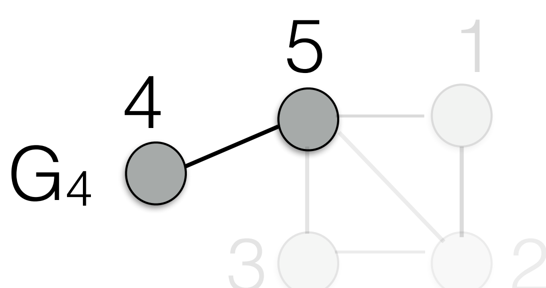


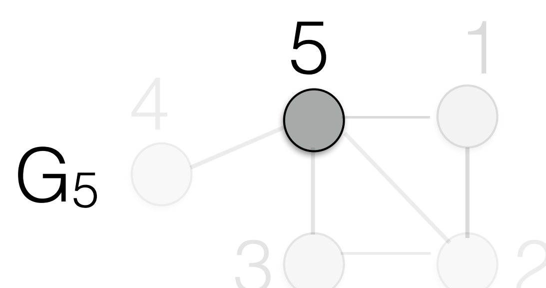
Cop-win Strategy (Preliminaries)



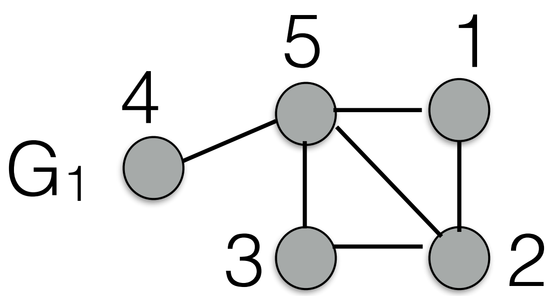


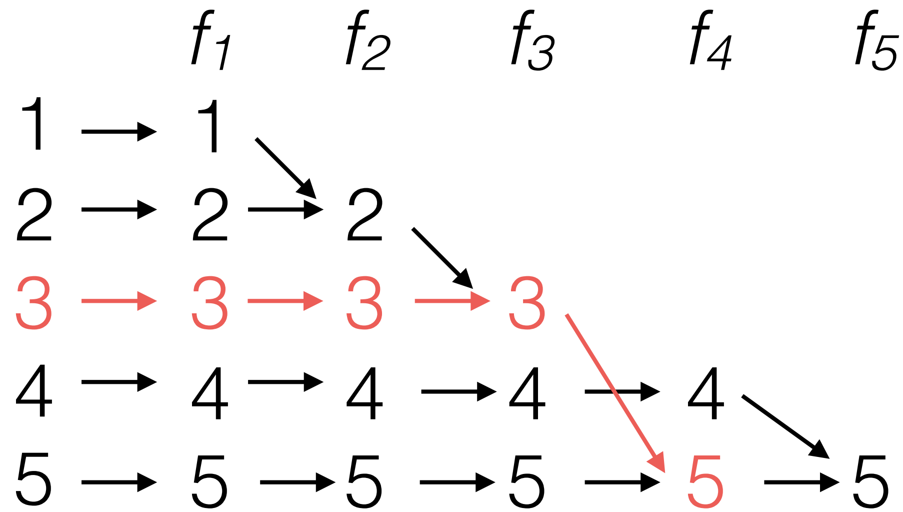
Example: $F_2(5)=5$ (the shadow of 5 in G_2)

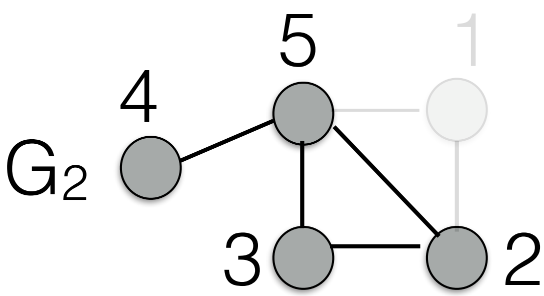




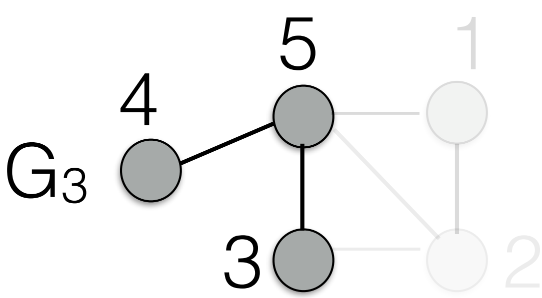
Cop-win Strategy (Preliminaries)

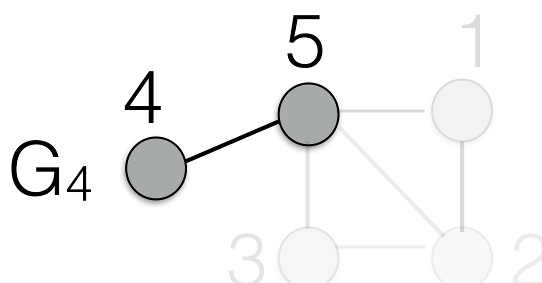


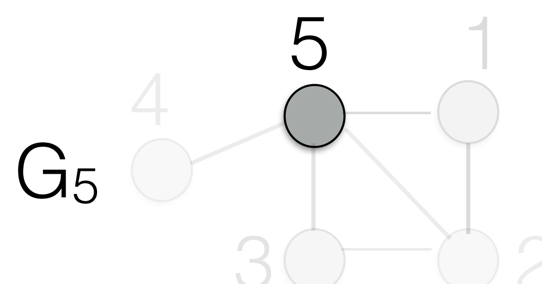




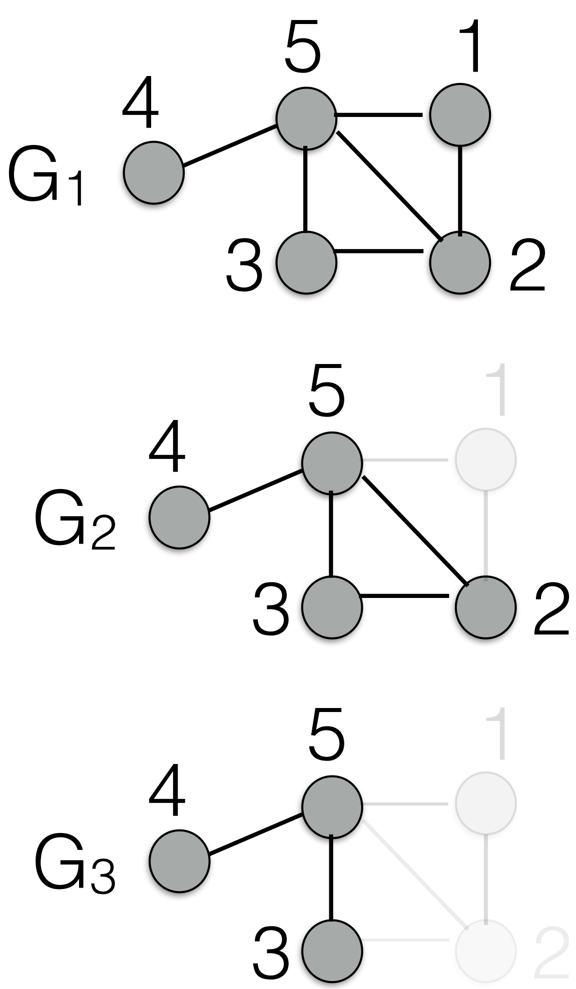
Example: $F_4(3)=5$ (the shadow of 3 in G_4)

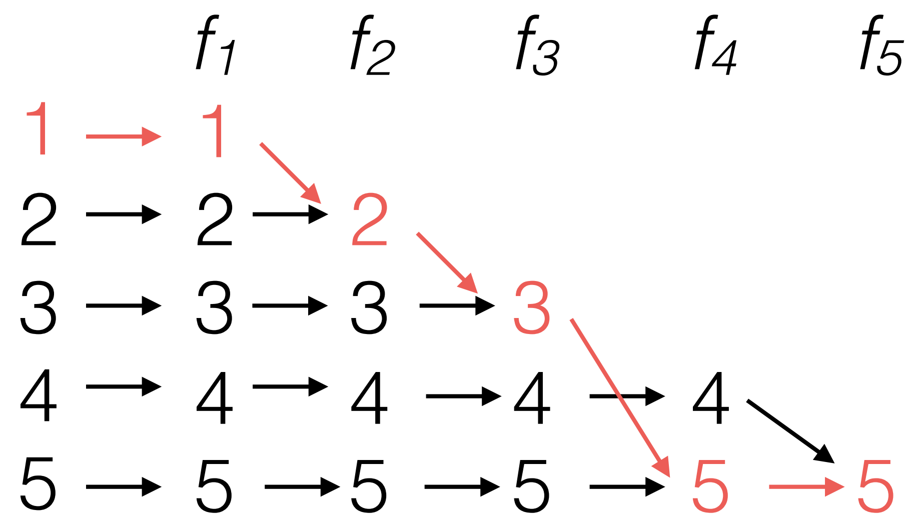




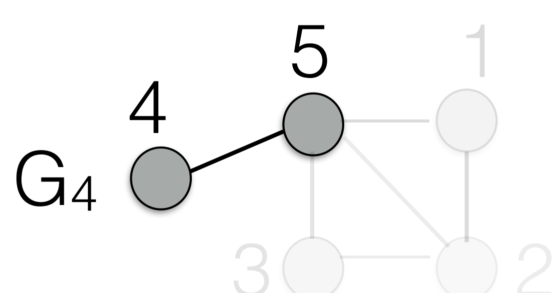


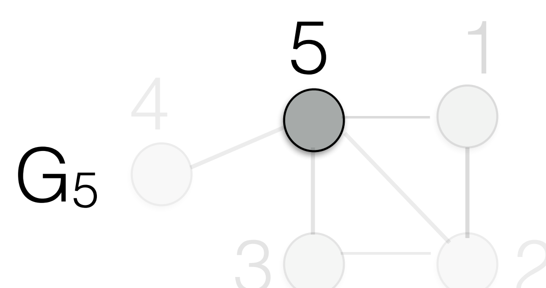
Cop-win Strategy (Preliminaries)





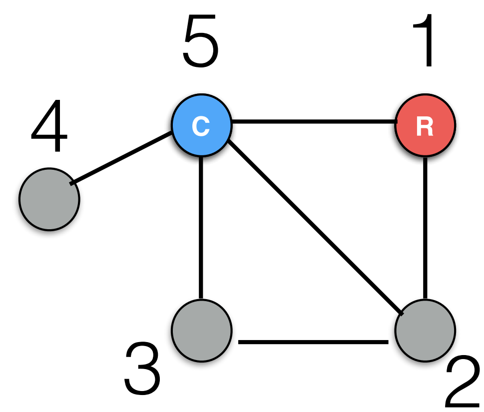
Example: $F_5(1)=5$ (the shadow of 1 in G_5)





Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.



Turn 0 C:5 R:1

Cop-win Strategy

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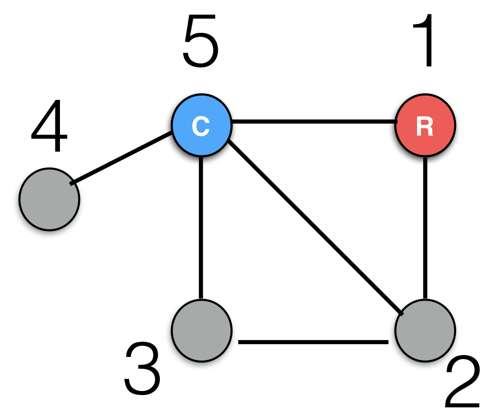

Turn 0 C:5 R:1
Turn 1… (in progress)

Cop shadows robber.
$F_4(1)=5$

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
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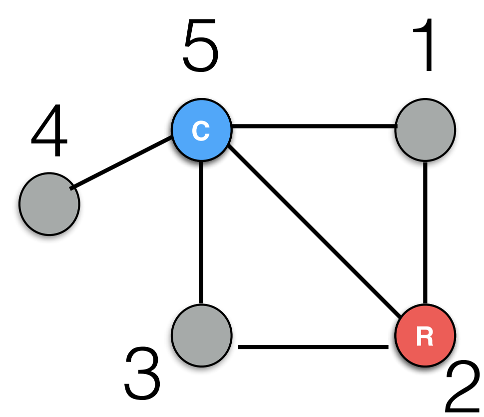


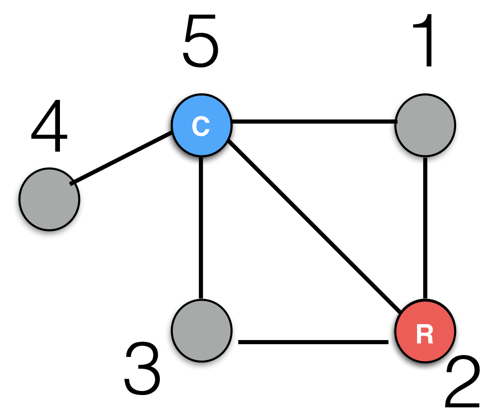
Turn 0 C:5 R:1
Turn 1… (in progress)

Cop shadows robber.
$F_4(1)=5$

Robber runs away.

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.



Turn 0 C:5 R:1
Turn 1 C:5 R:2

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
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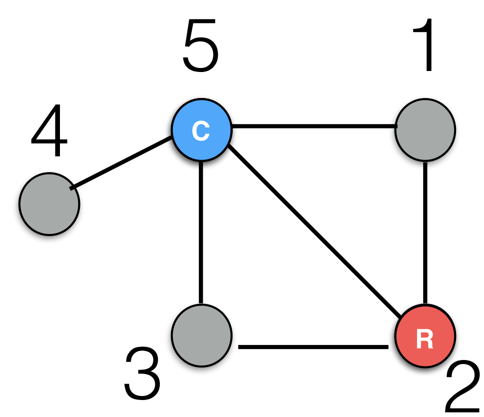

Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2… (in progress)

Cop-win Strategy

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- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
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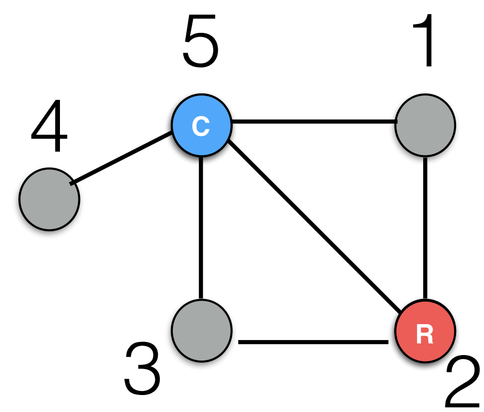
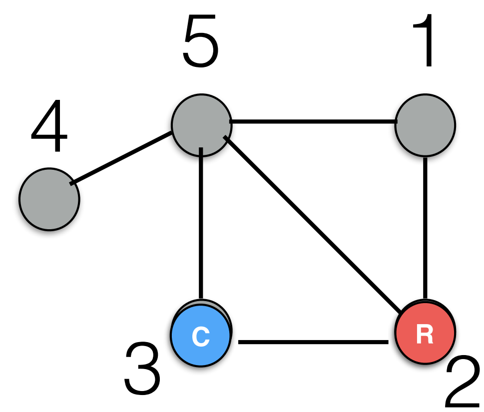


Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2… (in progress)

Cop shadows robber.
$F_3(2)=3$

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.

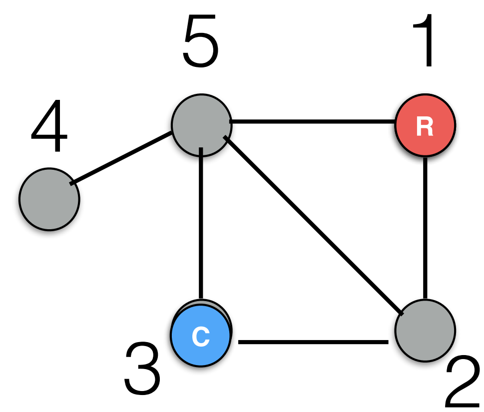


Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2… (in progress)

Cop shadows robber.
$F_3(2)=3$
Robber runs away.

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.



Turn 0	C:5	R:1
Turn 1	C:5	R:2
Turn 2	C:3	R:1

Cop-win Strategy

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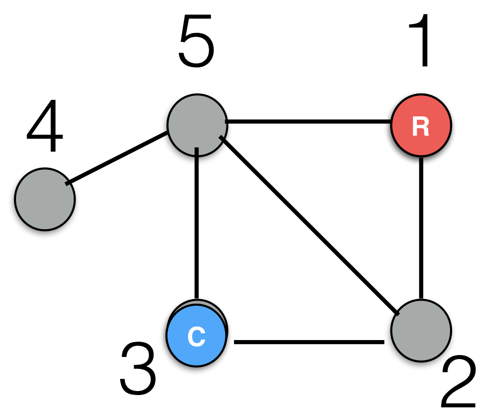

Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2 C:3 R:1
Turn 3… (in progress)

Cop-win Strategy

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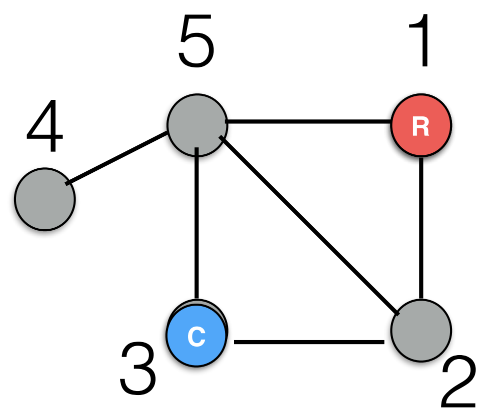

Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2 C:3 R:1
Turn 3… (in progress)

Cop shadows robber.

$F_2(1)=2$

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
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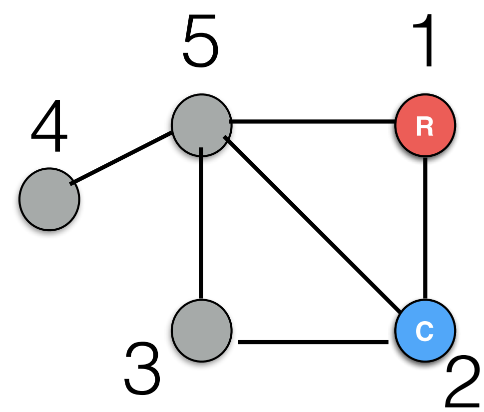


Turn 0 C:5 R:1

Turn 1 C:5 R:2

Turn 2 C:3 R:1

Turn 3… (in progress)

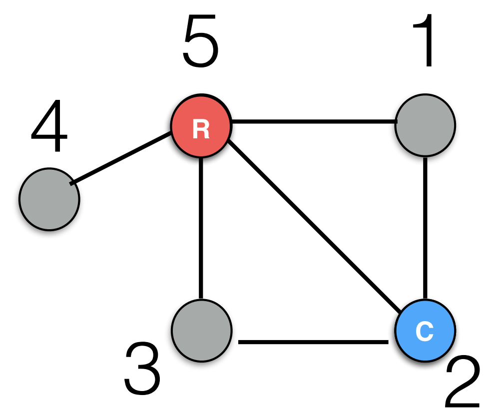
Cop shadows robber.

$F_2(1)=2$

Robber runs away.

Cop-win Strategy

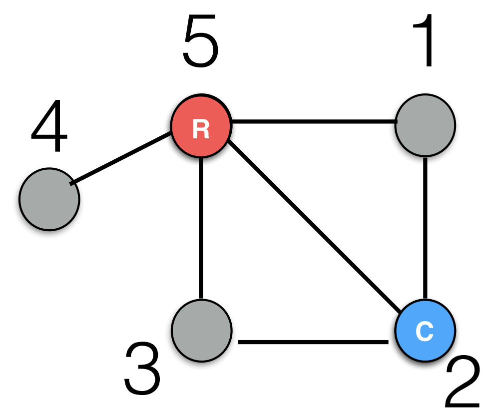
- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.



Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2 C:3 R:1
Turn 3 C:2 R:5
Turn 4… (in progress)

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.

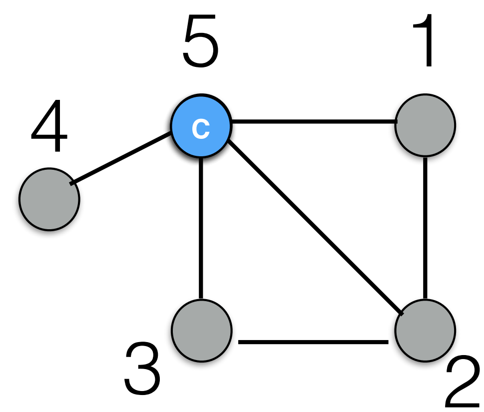


Turn 0 C:5 R:1
Turn 1 C:5 R:2
Turn 2 C:3 R:1
Turn 3 C:2 R:5
Turn 4… (in progress)

Cop shadows robber.
$F_1(5)=5$

Cop-win Strategy

- Cop starts at G_n, (the shadow of all other positions, including the robber's, under F_n)
- Suppose the robber is on u and the cop occupies the shadow of the robber, $F_i(u)$, in G_i
- If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ in the larger graph G_{i-1}.



Turn 0	C:5	R:1
Turn 1	C:5	R:2
Turn 2	C:3	R:1
Turn 3	C:2	R:5
Turn 4	C:5	

Cop-win Strategy

- May not be the fastest strategy, but guarantees the robber will be captured in at most n moves.
- Also not a unique for a graph because it relies on the cop-win ordering, which is not unique.

Hardness

- Checking if c(G)=k for some fixed k can be done in polynomial time.
- Determining k in which c(G)=k, when k is not fixed is NP-hard
- Determining the cop-number of a general graph is NP-hard, but for cop-win graphs, the cop-number is exactly 1.
    - We can check in polynomial time if a graph is cop-win

Problem Solution

- Given a map, a cop, and a robber - (1) **can the cop catch the robber,** and if so, (2) **how**?

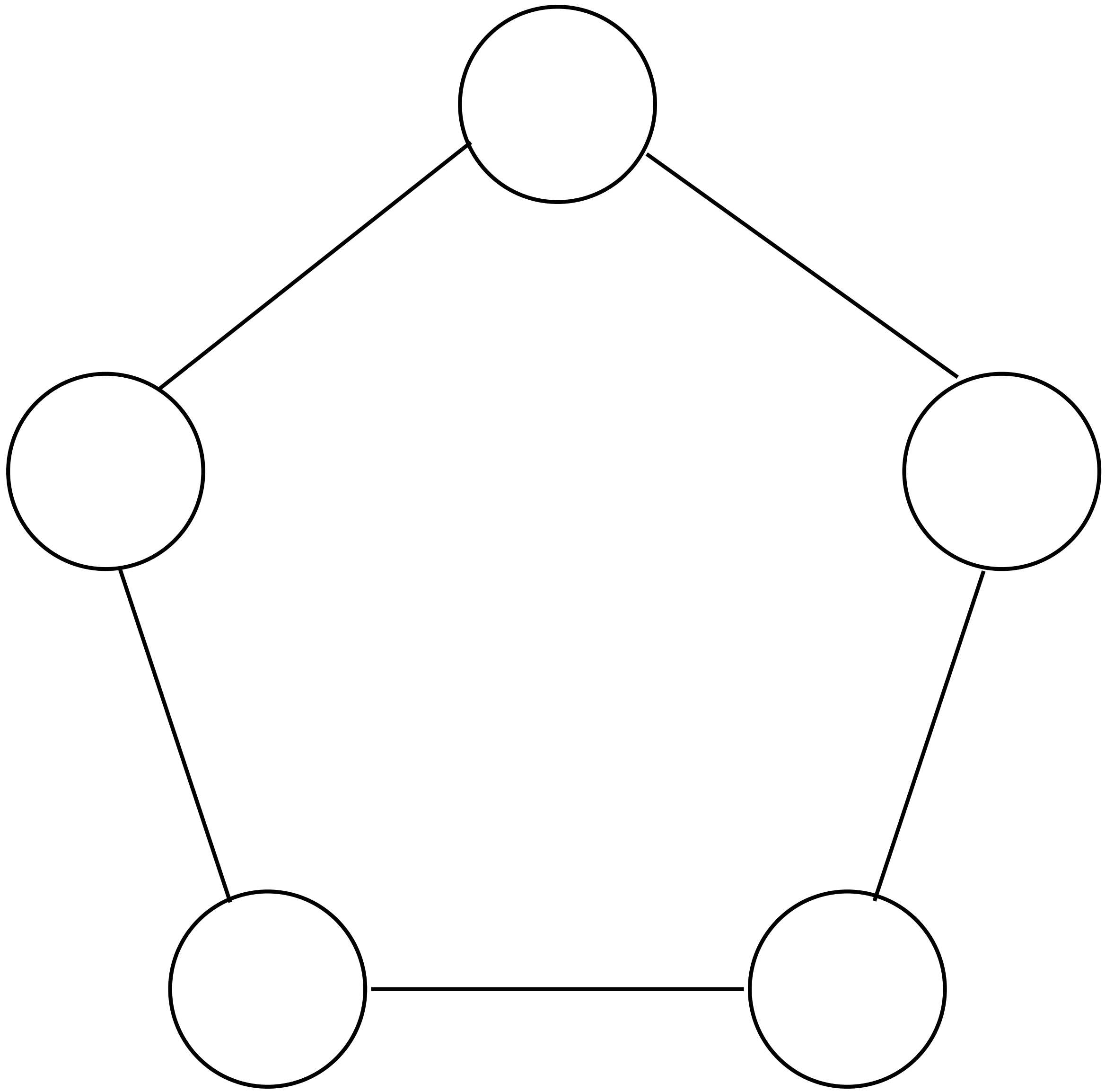
  (1) In our example, yes! Determine the graph's cop-win ordering in polynomial time

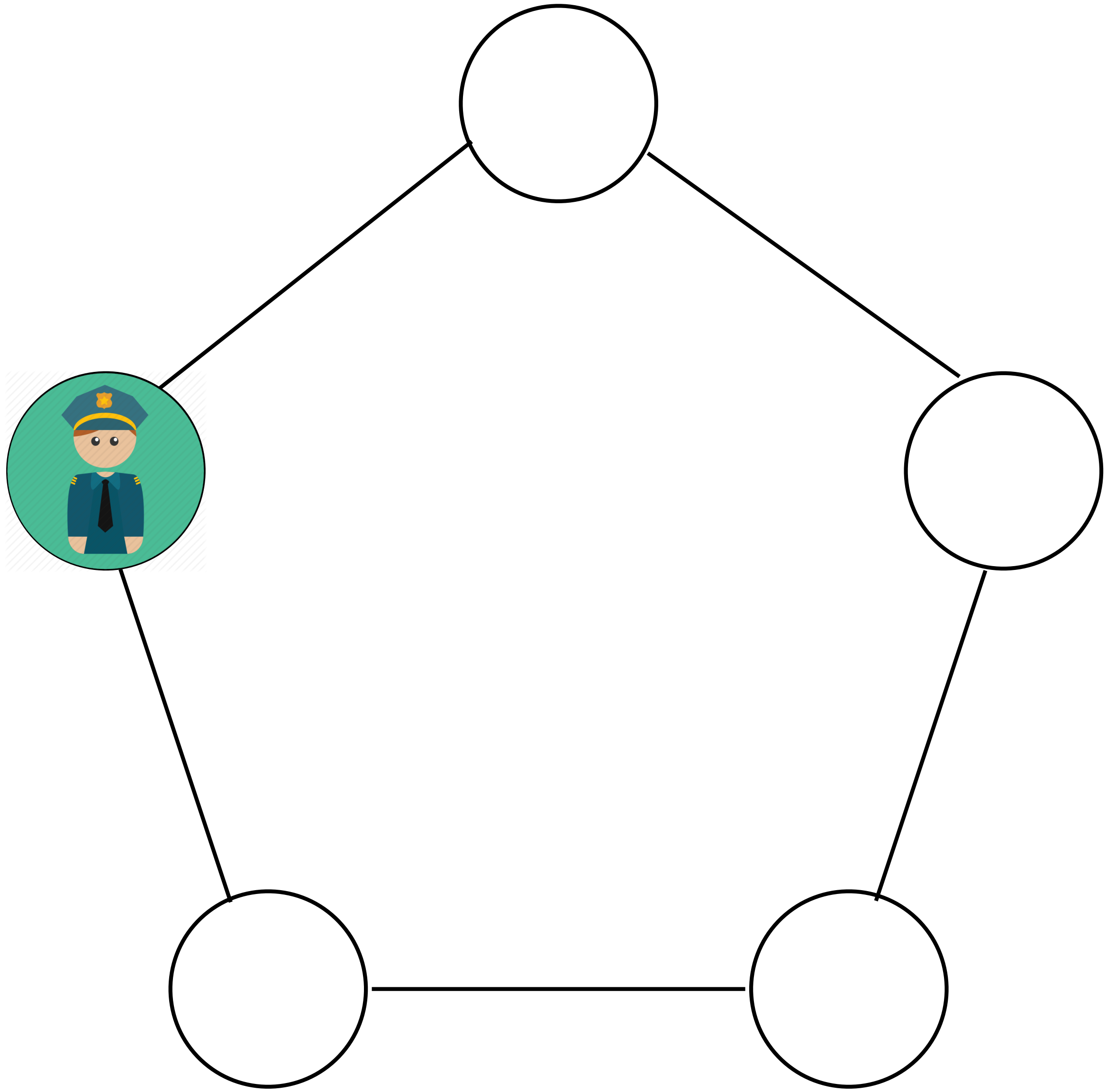
  (2) Given the cop-win ordering, find a guaranteed win strategy for the cop using this ordering

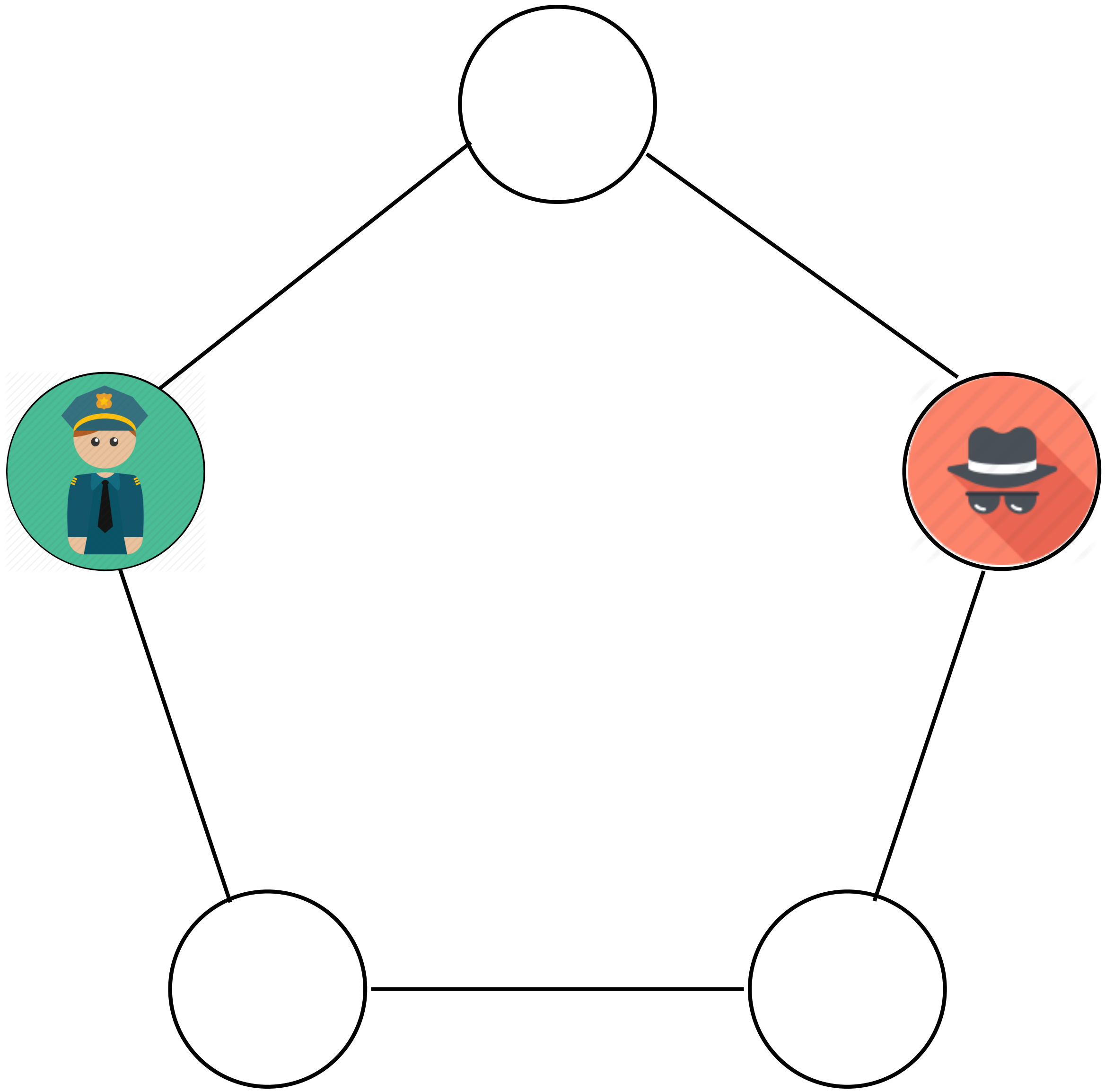
    - (we make some assumptions about where the robber will go to finish the example, but the strategy will work regardless of where robber picks to go next)
    - Cop follows the shadow of the robber in maps with increasingly more corners, until he finally reaches him

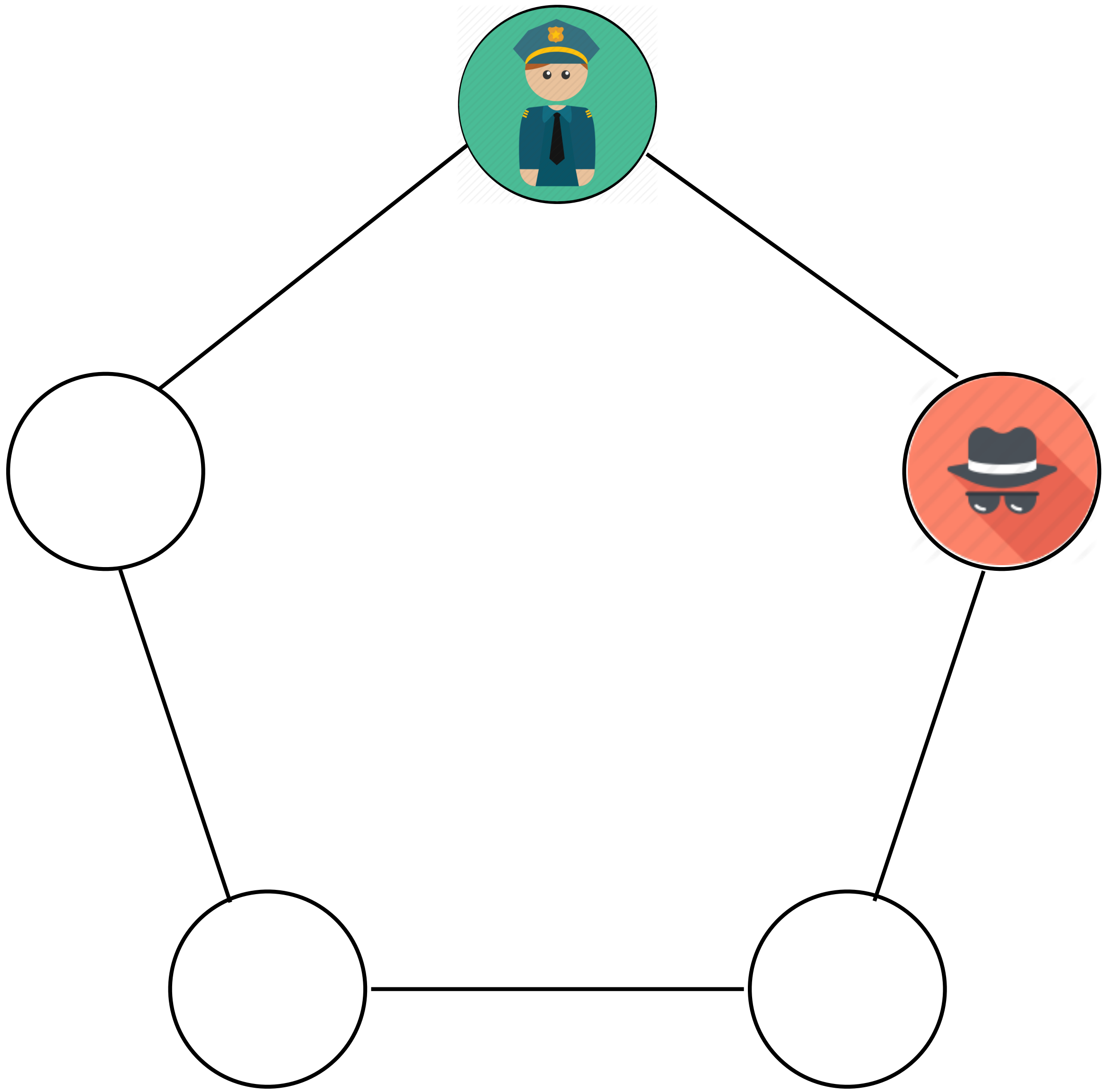
Variations

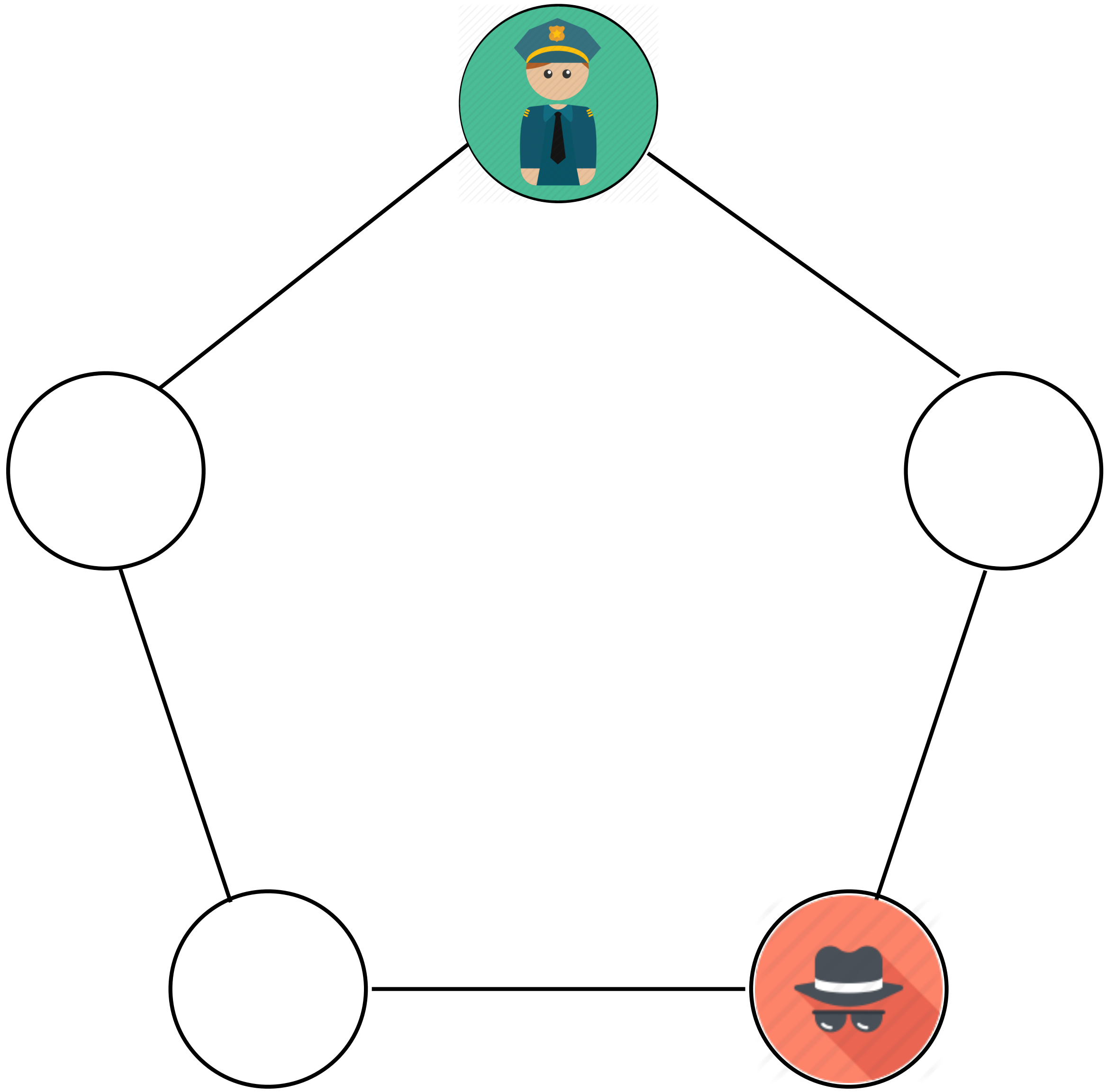
- Imperfect information (limited visibility): alarm indicates general location of robber, blind-spots of security system
- Cops set traps to catch or impede robbers
- Tandem-win: two cops patrol together within close proximity of each other
- A cop only needs to “see” a robber within distance k to catch/shoot him
- Minimize capture time (steps for cops to win)
- Varying speeds of cops and robbers
- Minimize number of cops to win

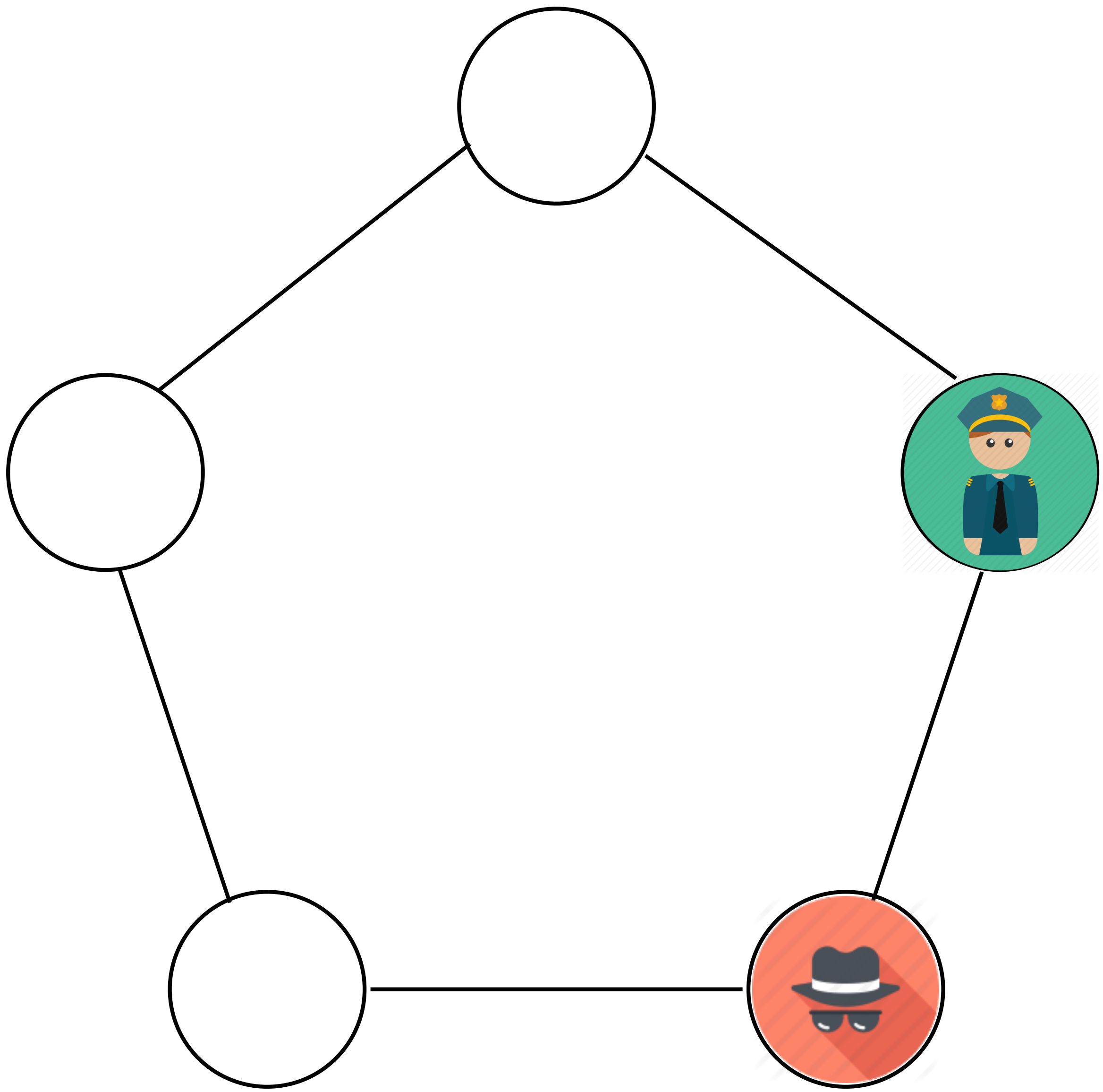


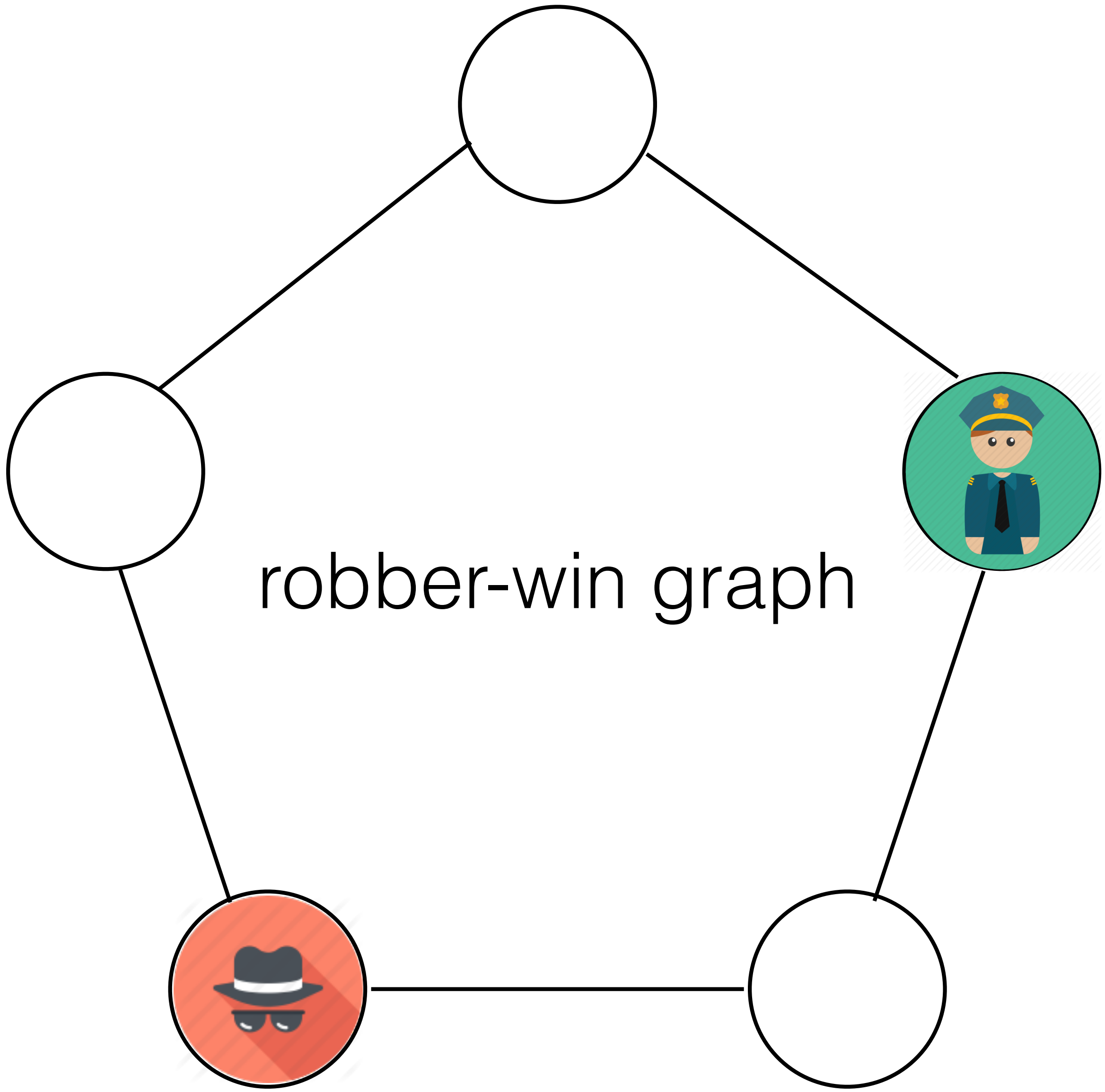




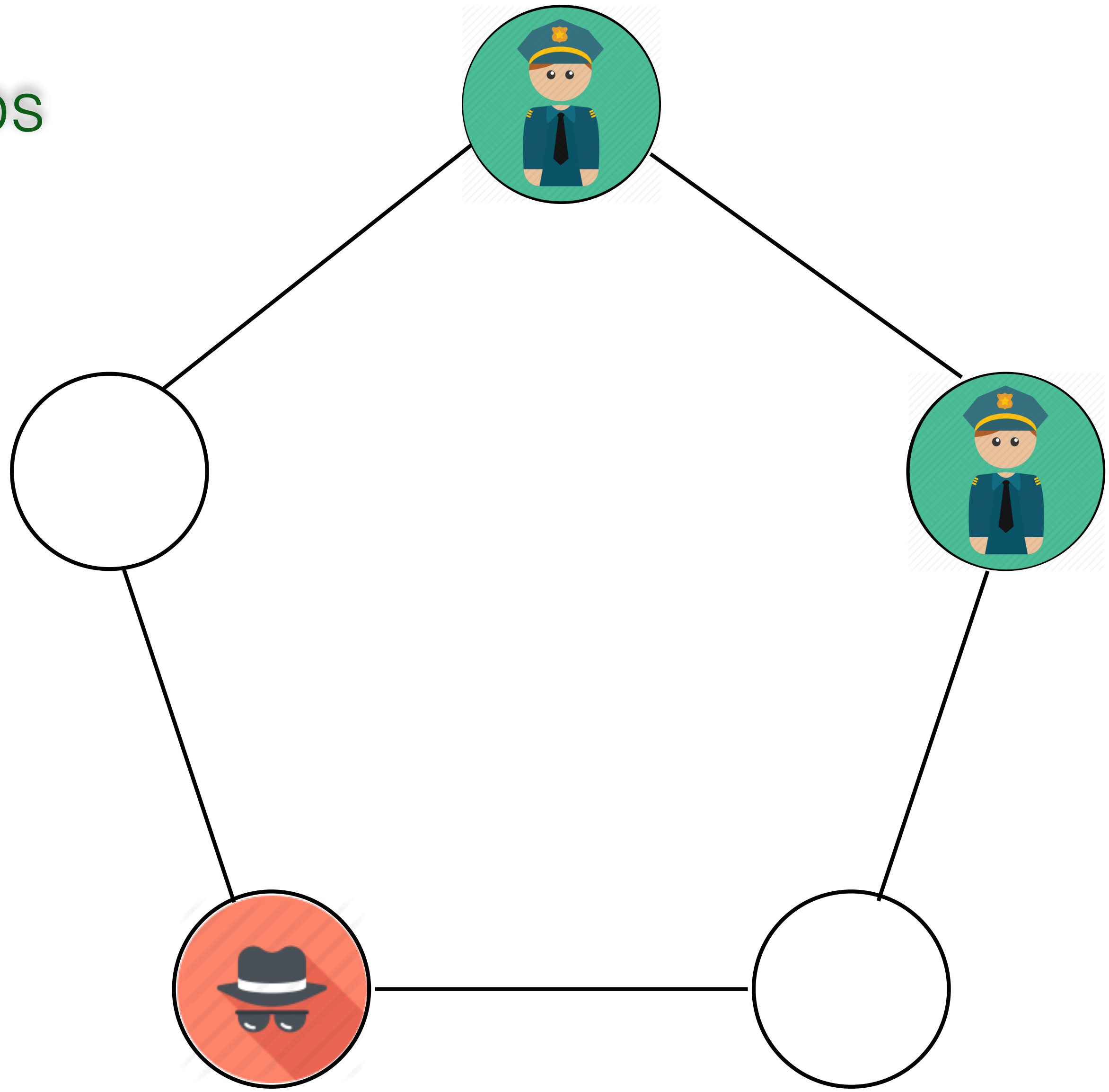







robber-win graph

2-cops



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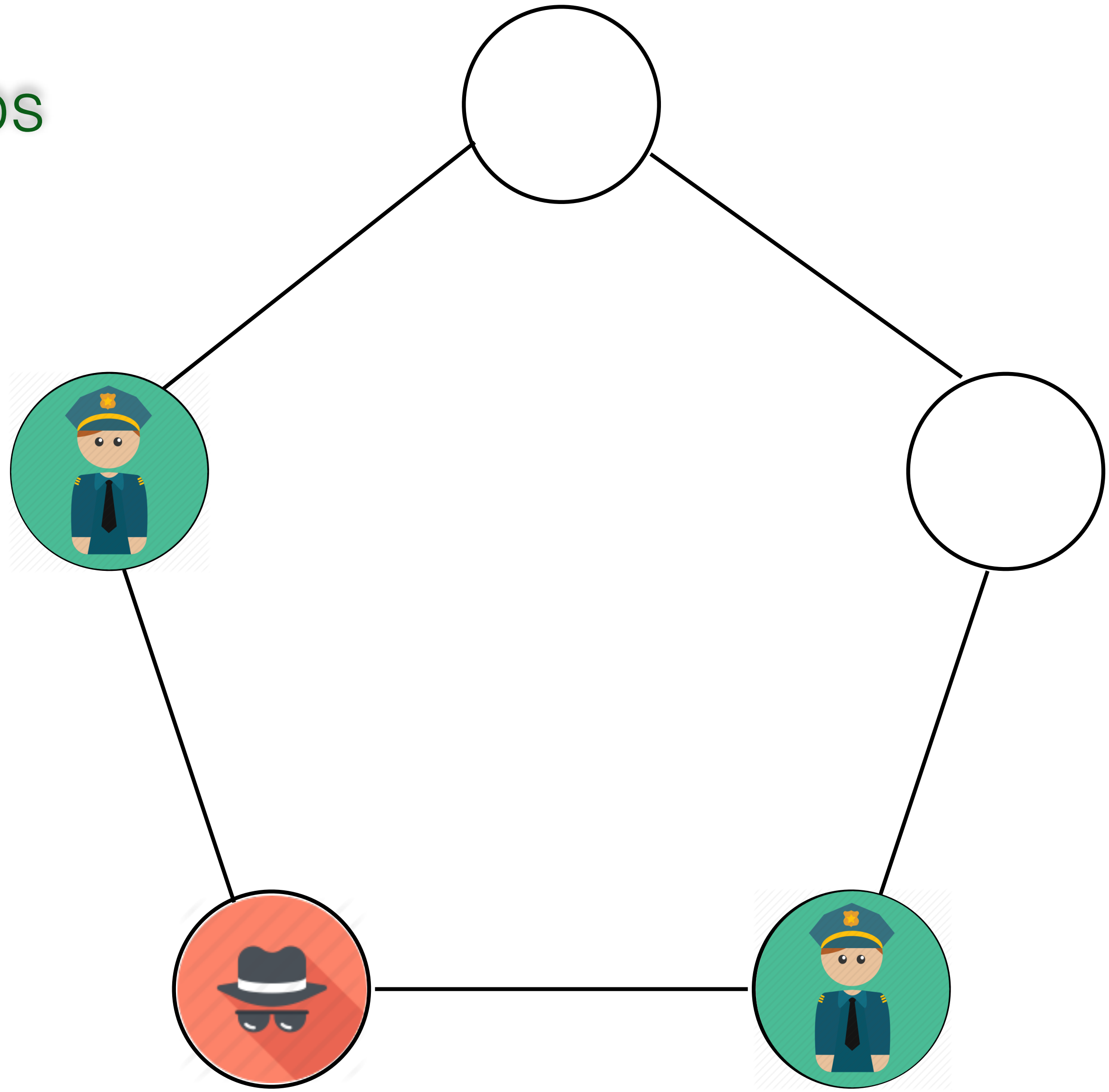

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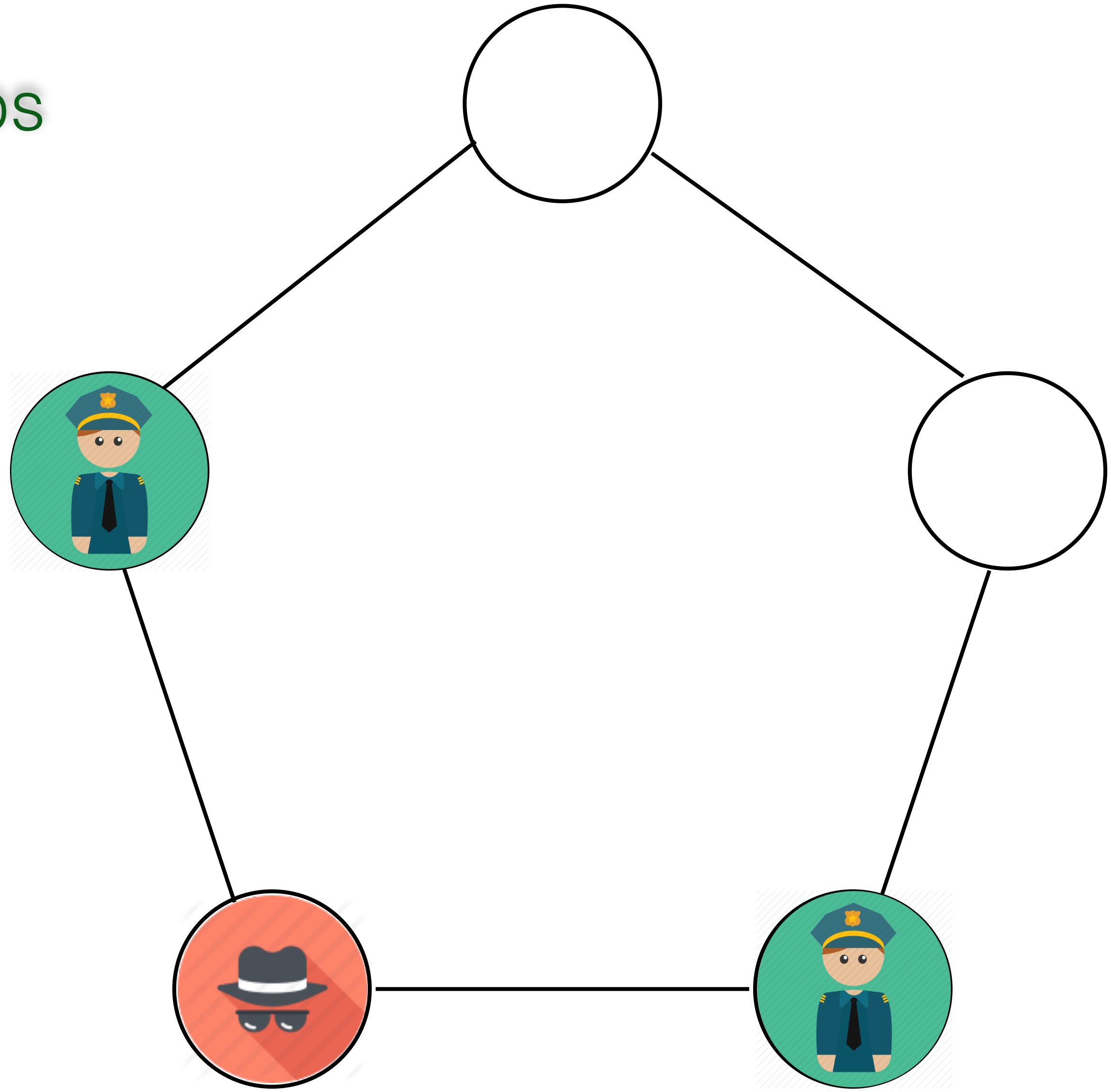
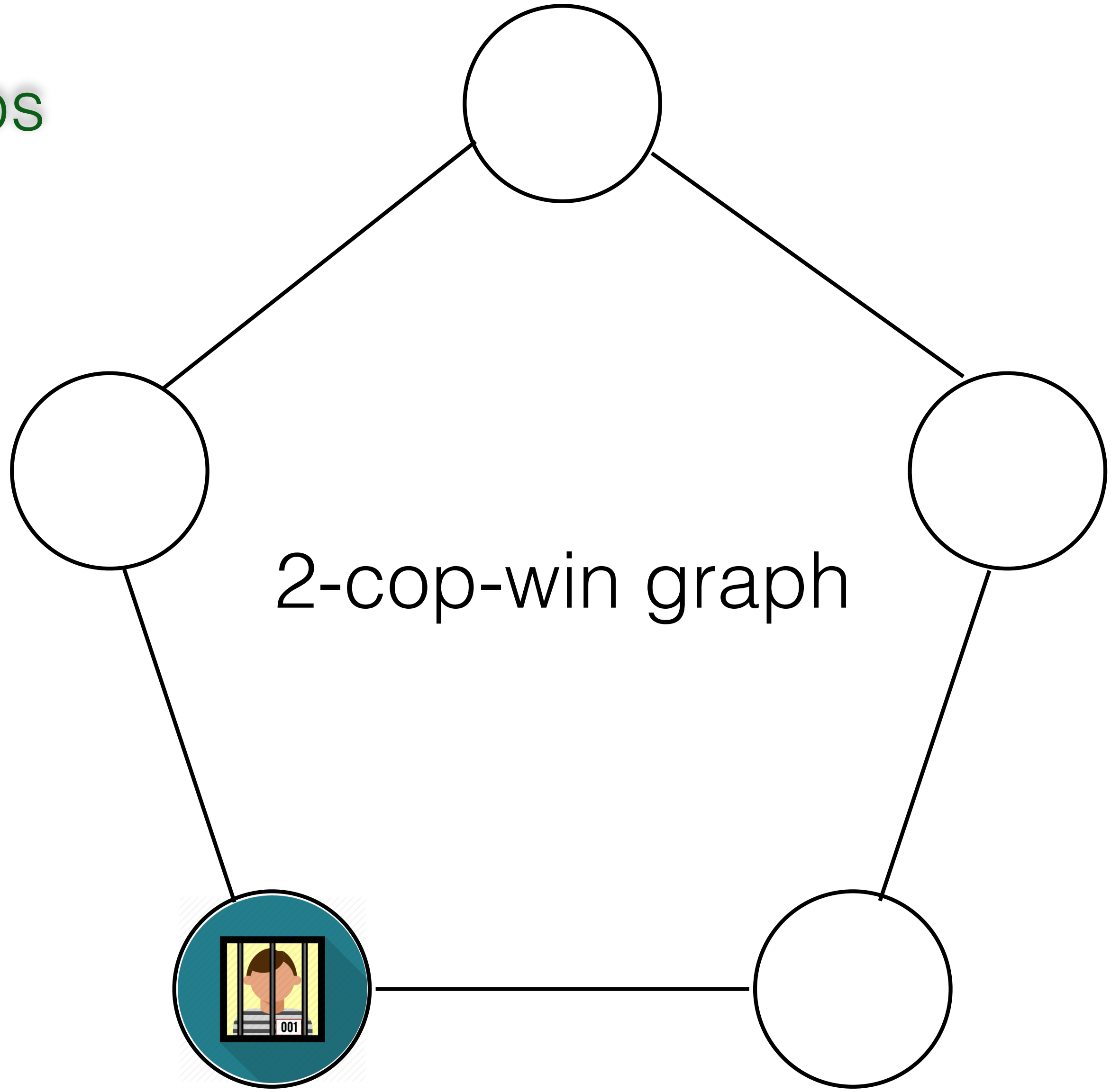


2-cops



How many cops are needed, at most?

- Every graph can catch a robber with *n* cops.
- Upper bound
    - on unconnected graphs: $c(G) = O(n)$
    - on connected graphs [Lu and Peng, 2011]:

$$c(G) \leq O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right)$$

How many cops are needed, at least?

- Lower bound
  - Meyniel's Conjecture: $c(n)=O(\sqrt{n})$
- Depends on the graph
  - For n≥4 0, $c(P_n) = c(W_n) = c(K_n) = 1$.
  - For C_n n≥4, $c(C_n) = 2$.
  - [Aigner and Fromme, 1984] **Planar graphs are 3-cop-win**
  - [Seymour and Thomas, 1993] A k-cop-win graph has tree width k-1.
  - [Chepoi, 1997] Bridged graphs are cop-win graphs.
- Other interesting bounds based on girth (length of minimum order cycle), minimum degree, maximum degree

References

(1) Anthony Bonato and Richard Nowakowski, The Game of Cops and Robbers on Graphs, Student Mathematical Library, Vol. 61, 2011.

(2) Richard Nowakowski and Peter Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math* 43, 235-239, 1983.

(3) A. Quilliot, "Homomorhismes, points fixes, rétractions et jeux de poursuite dans les graphes, les ensembles ordonnés et les espaces métriques," Thése d'État, Université de Paris VI, 1983.

(4) A game of cops & robbers. http://math.ucsd.edu/~fan/152/arch/coprob/. Mostafa Azizi and Fan Chung Graham, 2002.

Questions?