Collective Tree Spanners of Graphs

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Well-known **Tree *t* -Spanner Problem**

Given unweighted undirected graph *G=(V,E)* and integers *t,r.*
Does *G* admit a spanning tree *T =(V,E')* such that

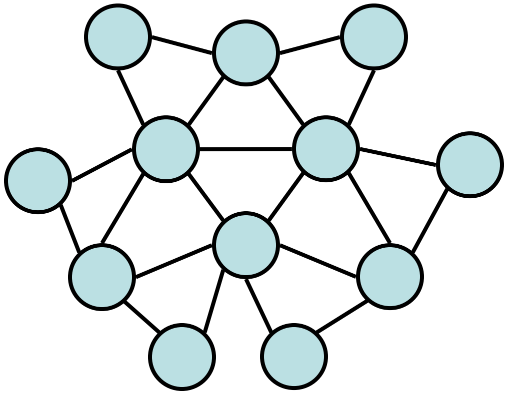
$$\forall u,v \in V, \quad dist_T(v,u) \leq t \times dist_G(v,u)$$

(a *multiplicative* tree *t-spanner* of *G)*

or

$$\forall u,v \in V, \quad dist_T(u,v) - dist_G(u,v) \leq r$$

(an *additive* tree *r-spanner* of *G)?*



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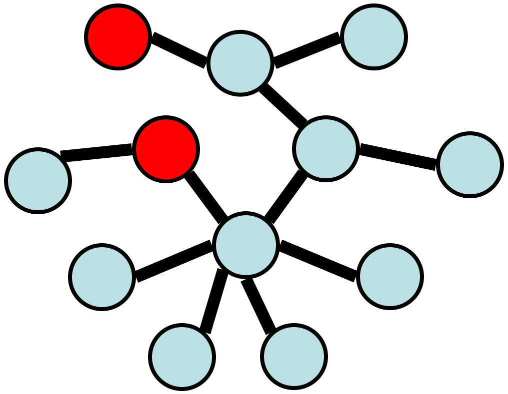


multiplicative tree 4- and additive tree 3-spanner of *G*

Well-known **Sparse *t* -Spanner Problem**

Given unweighted undirected graph *G=(V,E)* and integers *t, m,r.*
Does *G* admit a spanning graph *H =(V,E')* with $|E'| \leq m$ such that

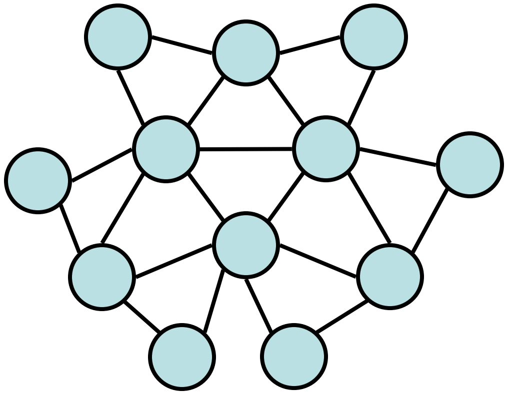
$$\forall u,v \in V,\quad dist_H(v,u) \leq t \times dist_G(v,u)$$

(a *multiplicative t-spanner* of *G)*

or

$$\forall u,v \in V,\quad dist_H(u,v) - dist_G(u,v) \leq r$$

(an *additive r-spanner* of *G)?*



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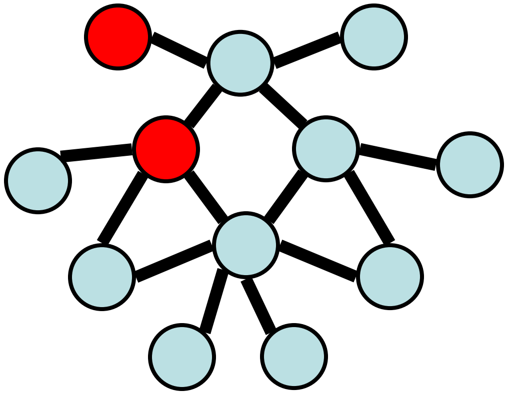


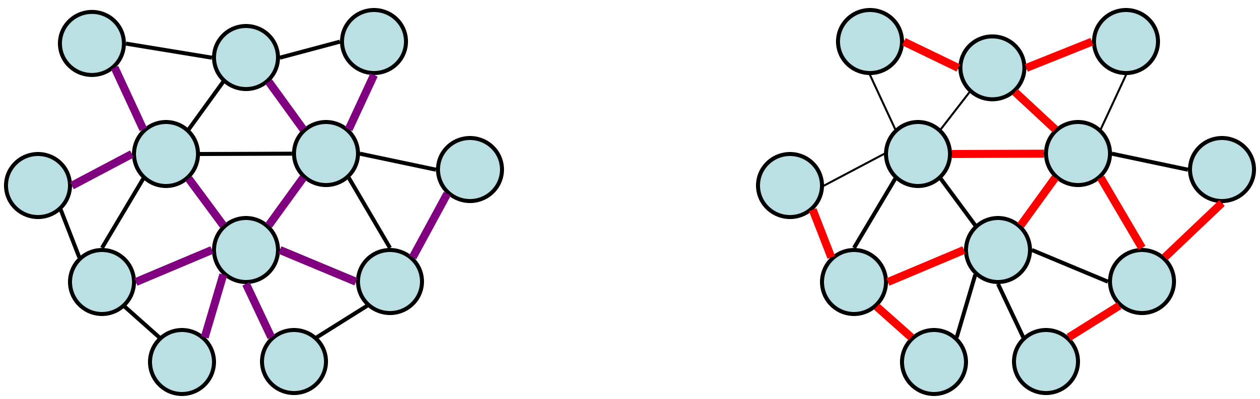
multiplicative 2- and additive 1-spanner of *G*

New Collective Additive Tree r-Spanners Problem

Given unweighted undirected graph $G=(V,E)$ and integers μ, r.
Does G admit a system of μ collective additive tree r-spanners $\{T_1, T_2, \ldots, T_\mu\}$ such that

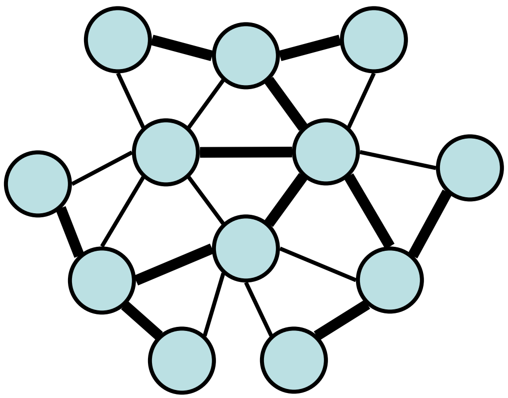
$$\forall\, u,v \in V \; and \;\; \exists 0 \le i \le \mu,\;\; dist_{T_i}(v,u) - dist_G(v,u) \le r$$

(a system of μ collective *additive tree r-spanners* of G)?



2 **collective additive tree *2*-spanners**

Applications of Collective Tree Spanners

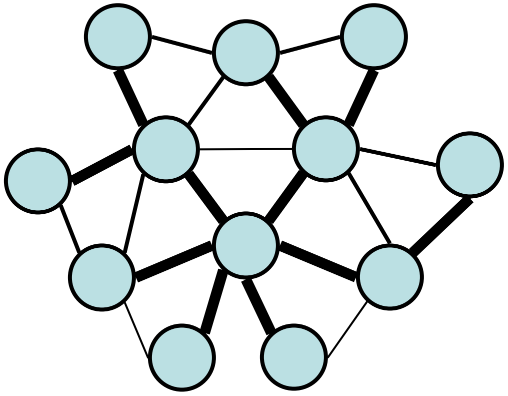


- message routing in networks

Efficient routing scheme is known for trees but very hard for graphs. For any two nodes, we can route the message between them in one of the trees which approximates the distance between them.

- solution for sparse *t*-spanner problem

If a graph admits a system of μ collective additive tree *r*-spanners, then the graph admits a sparse additive *r*-spanner with at most $\mu(n-1)$ edges, where *n* is the number of nodes.



2 collective tree 2-spanners for *G*

Some known results for the tree spanner problem

(mostly multiplicative case)

- general graphs [CC'95]
  - $t \geq 4$ is NP-complete. (t=3 is still open, $t \leq 2$ is P)
- approximation algorithm for general graphs [EP'04]
  - $O(\log n)$ approximation algorithm
- chordal graphs [BDLL'02]
  - $t \geq 4$ is NP-complete. (t=3 is still open.)
- planar graphs [FK'01]
  - $t \geq 4$ is NP-complete. (t=3 is polynomial time solvable.)

Some known results for sparse spanner problems

- general graphs [PS'89]
  - $t, m \geq 1$ is NP-complete
- n-vertex chordal graphs (multiplicative case) [PS'89]

  (G is chordal if it has no chordless cycles of length >3)
  - multiplicative 3-spanner with $O(n \log n)$ edges
  - multiplicative 5-spanner with $2n-2$ edges
- n-vertex c-chordal graphs (additive case) [CDY'03]

  (G is c-chordal if it has no chordless cycles of length >c)
  - additive (c+1)-spanner with $2n-2$ edges

➔ For chordal graphs: additive 4-spanner with $2n-2$ edges

Our results on the collective tree spanners problem

- (α, r)-decomposable graph
  - Sparse additive 2r -spanner with $(n-1)\log_{1/\alpha} n$ edges in polynomial time
  - $\log_{1/\alpha} n$ collective additive tree 2r - spanners in polynomial time
- c-chordal graphs
  - Sparse additive $2\lfloor c/2 \rfloor$-spanner with $O(n \log n)$ edges in polynomial time

    (extension & improvement of [PS'89] from chordal to c-chordal)
  - $\log n$ collective additive tree $2\lfloor c/2 \rfloor$-spanners in polynomial time
- chordal graphs
  - Sparse additive 2 -spanner with $O(n \log n)$ edges in polynomial time
  - $\log n$ collective additive tree 2-spanners in polynomial time

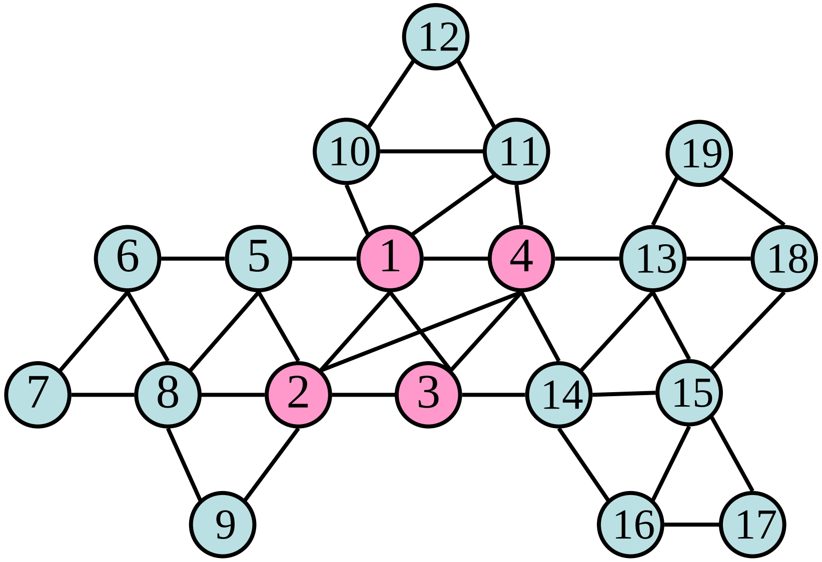
Our routing results

- Better routing scheme for *c*-chordal graphs

Graph class	Scheme construction time	Addresses and routing tables	Message initiation time	Routing decision time	Deviation
Chordal	$O(m \log n + n \log^2 n)$	$O(\log^3 n / \log\log n)$	$\log n$	$O(1)$	2
Chordal bipartite	$O(n\, m \log n)$	$O(\log^3 n / \log\log n)$	$\log n$	$O(1)$	2
Cocomparability	$O(m \log n + n \log^2 n)$	$O(\log^3 n / \log\log n)$	$\log n$	$O(1)$	2
c-Chordal	$O(n^3 \log n)$	$O(\log^3 n / \log\log n)$	$\log n$	$O(1)$	$2\lfloor c/2 \rfloor$

Constructing a Rooted Balanced Tree for (α, *r*)-decomposable graph

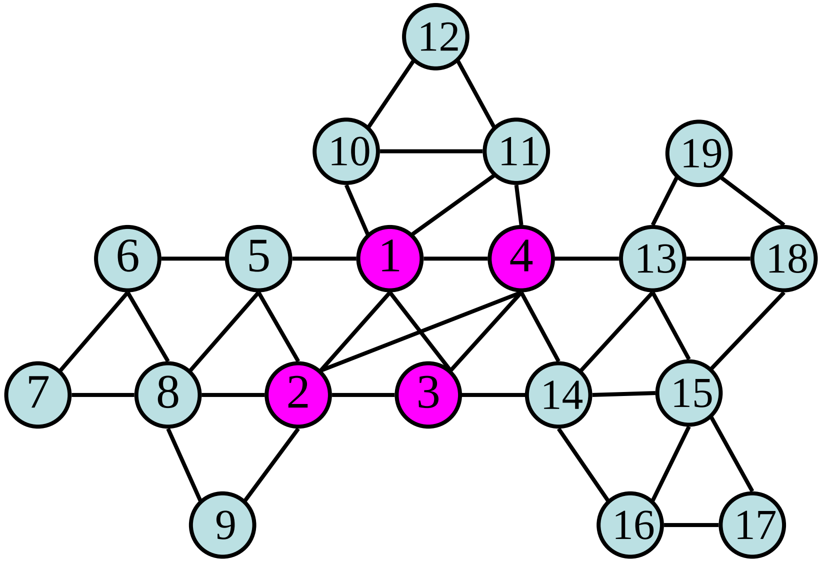
- An (α, *r*)-decomposable graph has
  - Balanced separator
  - Bounded separator radius
  - Hereditary family



(chordal graph)

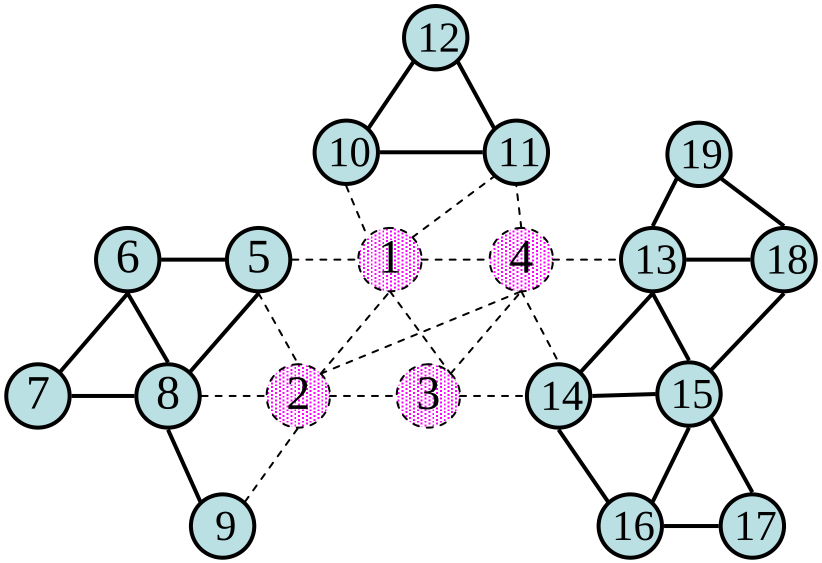
Decompose the Graph

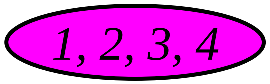
- Find the *balanced separator S* of *G*.



Decompose the Graph (*cont.*)

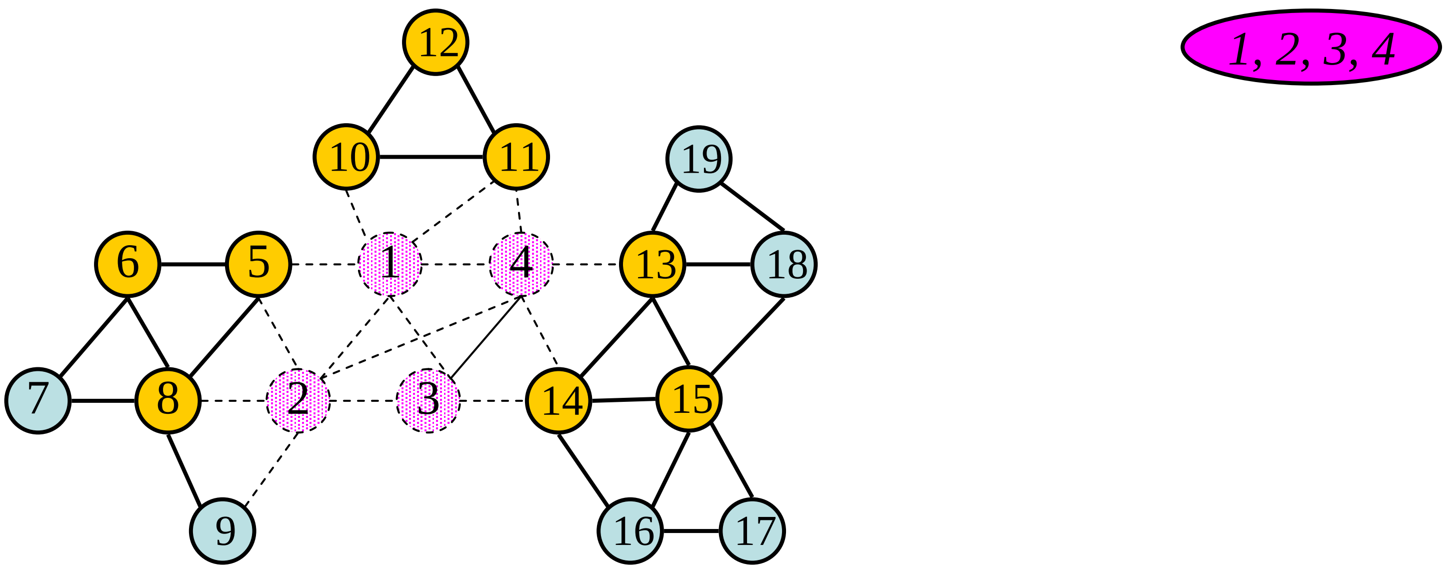
- Use *S* as the *root* of the *rooted balanced tree*.





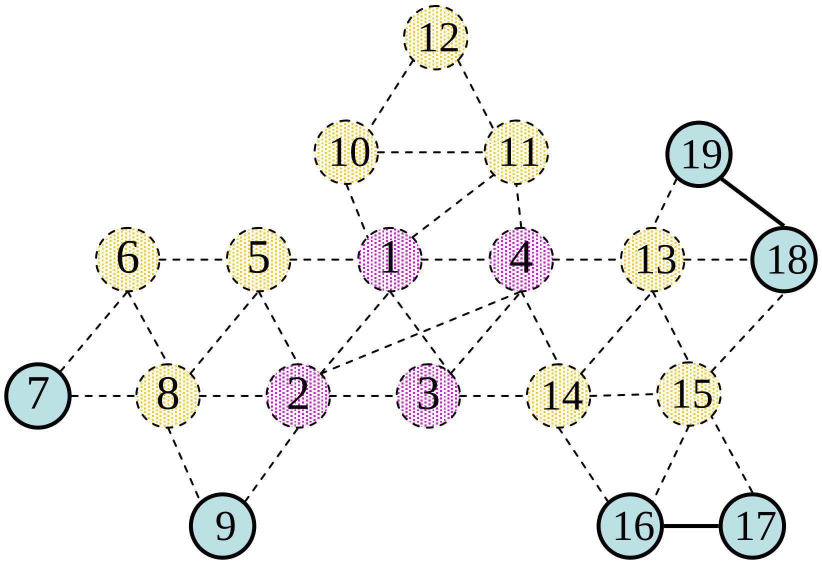
Decompose the Graph (*cont.*)

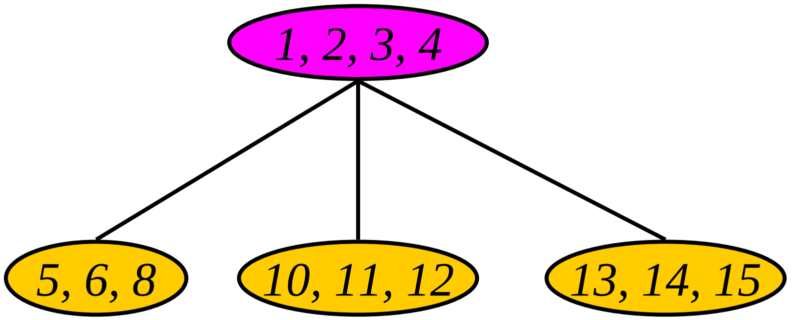
- For each connected component of *G\S*, find their *balanced separators*.



Decompose the Graph (*cont.*)

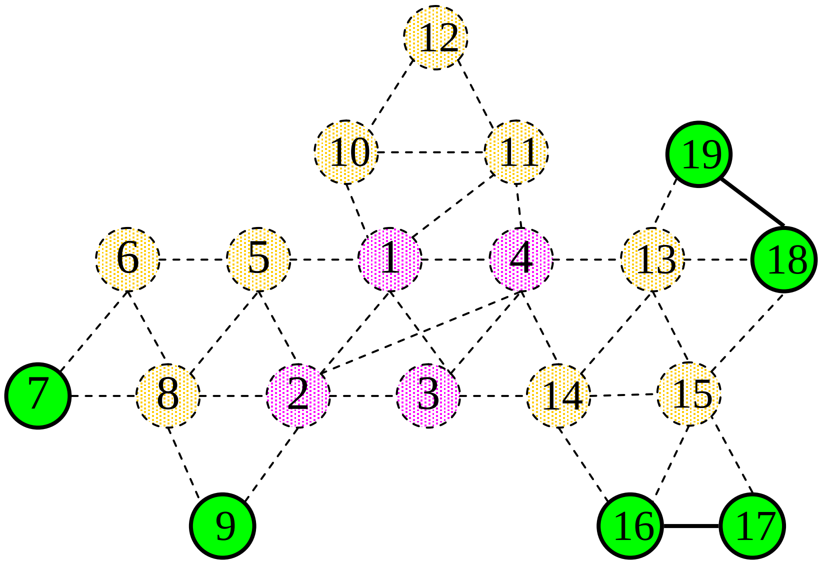
- Use the separators as nodes of the *rooted balanced tree* and let *S* be their father.

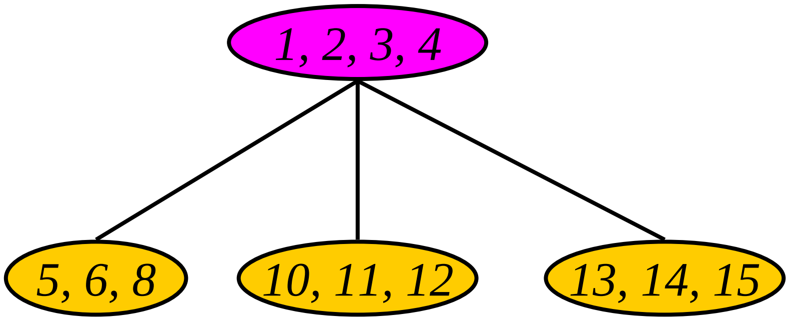




Decompose the Graph (*cont.*)

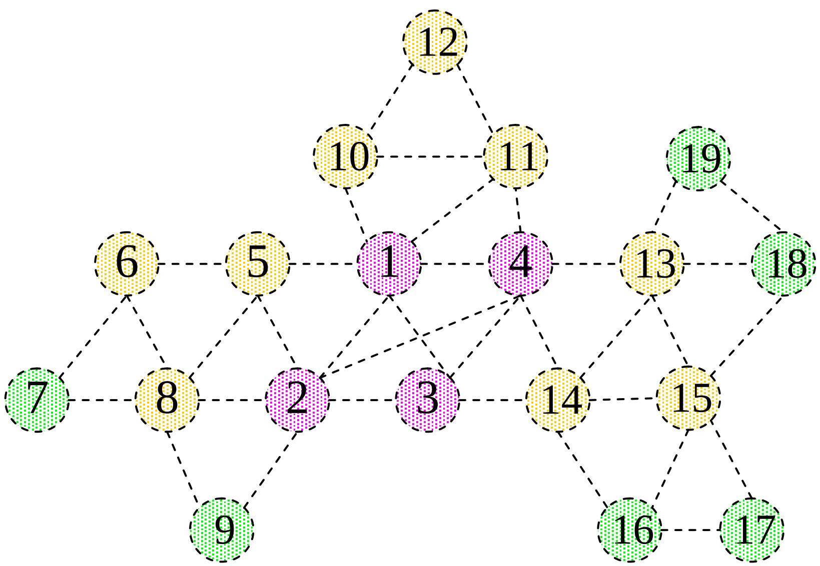
- Recursively repeat previous procedure until each connected component has radius less than or equal to *r* .

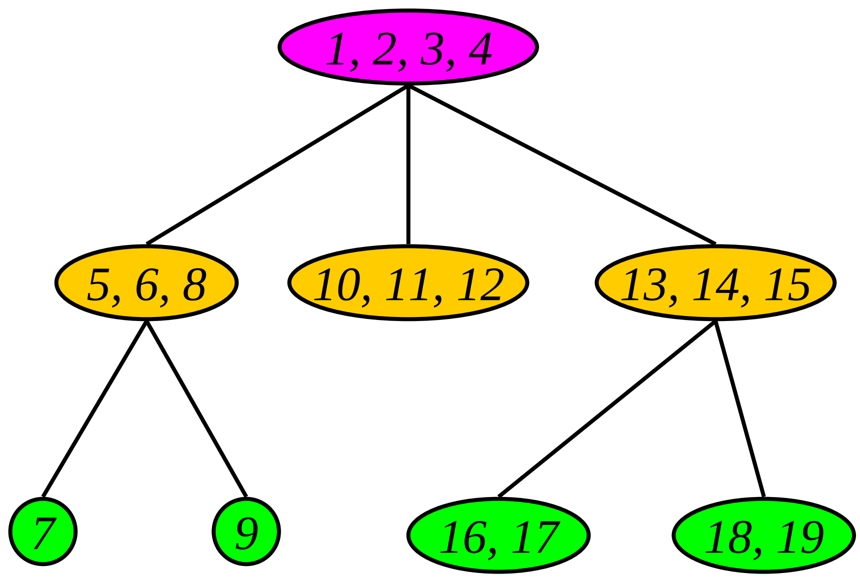




Decompose the Graph (*cont.*)

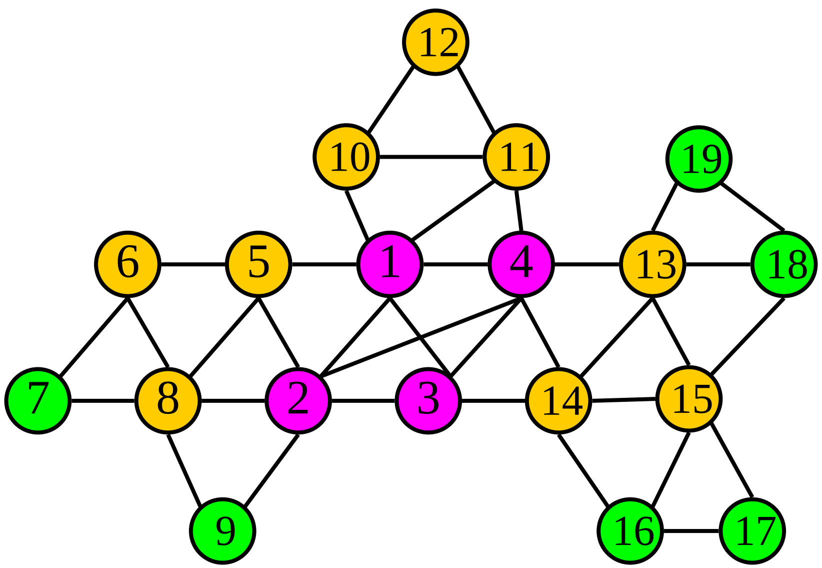
- Get the *rooted balanced tree*.

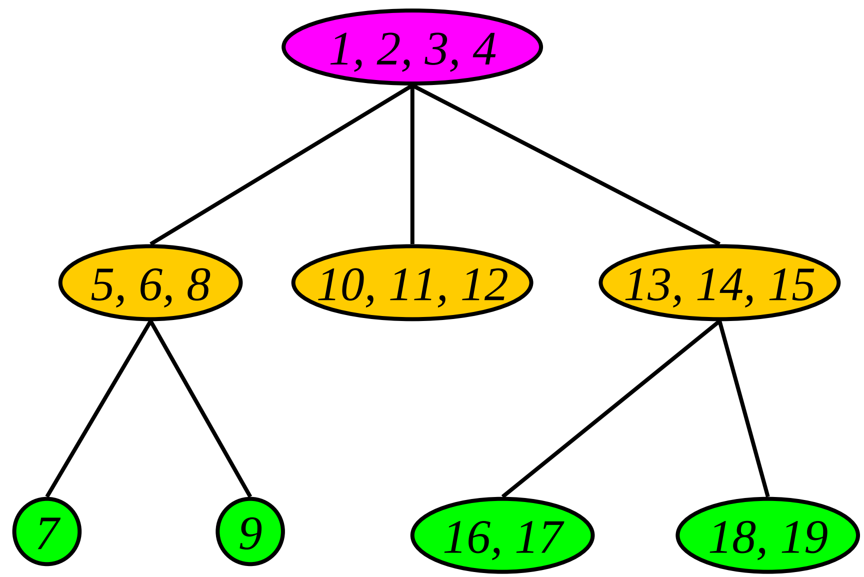




Rooted Balanced Tree

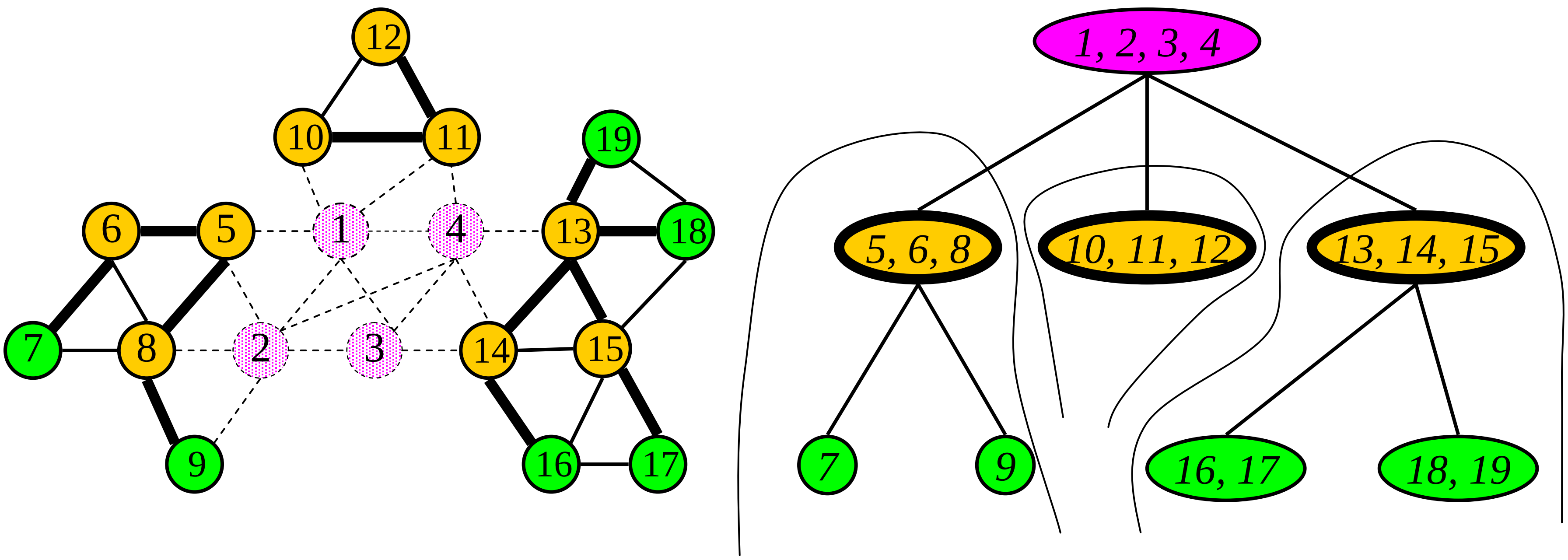
- Final *rooted balanced tree*.





Constructing Local Spanning Trees

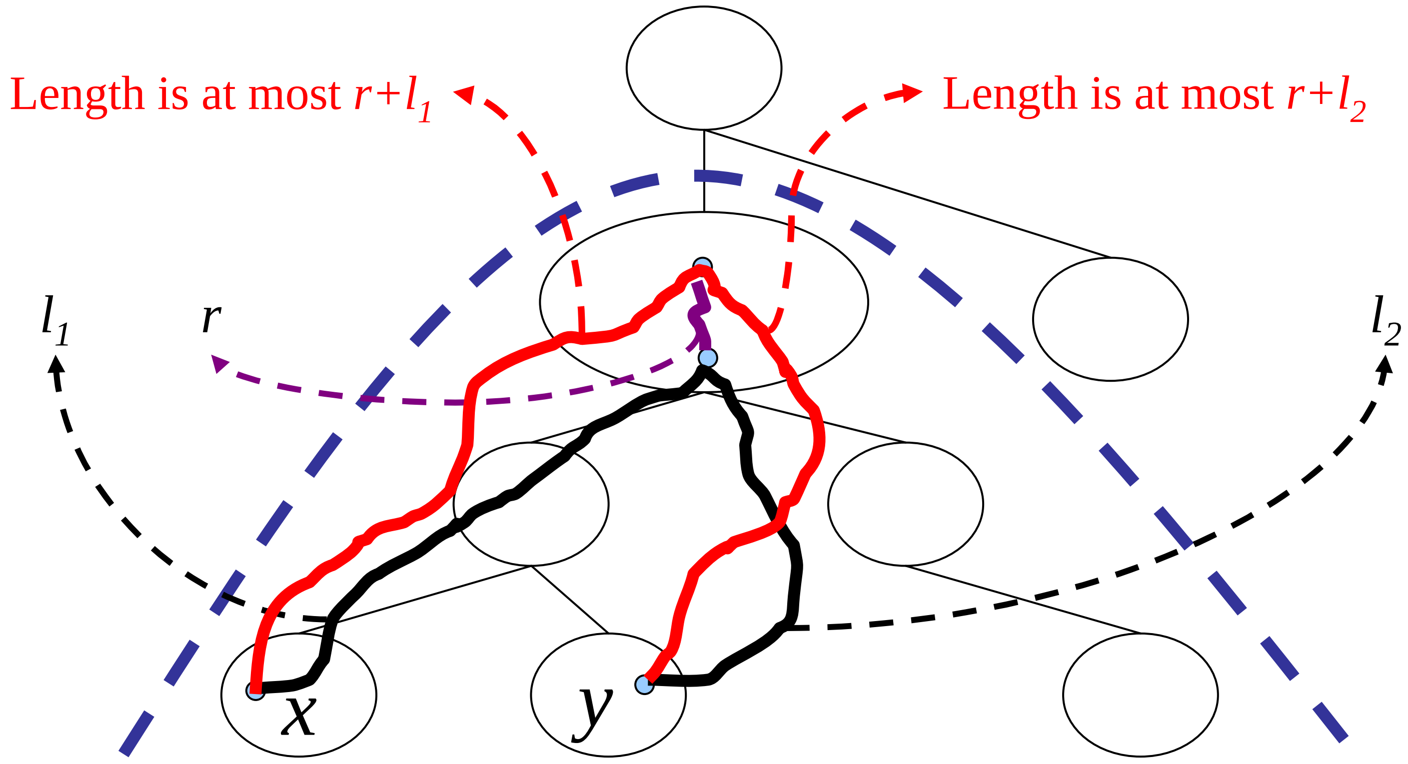
- Construction of *local spanning trees* of the 2nd layer.



- Construction of a spanning tree of the 2nd layer.

Main Result

- **Thm.** Given an (α,r)-decomposable graph $G=(V, E)$, a system of $\log_{1/\alpha} n$ collective additive tree $2r$-*spanners* of G can be constructed in polynomial time.



Further Results

➔ Any (α, r)-decomposable graph $G=(V, E)$ admits an additive $2r$-spanner with at most $n \log_{1/\alpha} n$ edges which can be constructed in polynomial time.

➔ Any (α, r)-decomposable graph $G=(V, E)$ admits a routing scheme of deviation $2r$ and with labels of size $O(\log_{1/\alpha} n \log^2 n / \log\log n)$ bits per vertex. Once computed by the sender in $\log_{1/\alpha} n$ time, headers never change, and the routing decision is made in constant time per vertex.

- The class of c-chordal graphs is $(1/2, \lfloor c/2 \rfloor)$-decomposable.

➔ log n trees with collective additive stretch factor $2\lfloor c/2 \rfloor$

Further Results

- The class of chordal graphs is (1/2, 1)-decomposable.

➔ log n trees with collective additive stretch factor 2

- The class of chordal bipartite graphs is (1/2, 1)-decomp.

➔ log n trees with collective additive stretch factor 2

(A bipartite graph $G=(X \cup Y, E)$ is *chordal bipartite* if it does not contain any induced cycles of length greater than 4.)

- There are chordal bipartite graphs on 2n vertices for which any system of collective additive tree 1-spanners will need to have at least $\Omega(n)$ spanning trees.

- There are chordal graphs on n vertices for which any system of collective additive tree 1-spanners will need to have at least $\Omega(\sqrt{n})$ spanning trees.

Open questions and future plans

- Find best possible trade-off between number of trees and additive stretch factor for planar graphs (currently: $\sqrt{n} \log n$ collective additive tree 0-spanners).

- Consider the collective additive tree spanners problem for other structured graph families.

- Complexity of the collective additive tree spanners problem for different μ and r on general graphs and special graph classes.

- More applications of …

Thank You