Spanners in sparse graphs *

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Abstract

A *t-spanner* of a graph G is a spanning subgraph S in which the distance between every pair of vertices is at most t times their distance in G. If S is required to be a tree then S is called a *tree t-spanner* of G. In 1998, Fekete and Kremer showed that on unweighted planar graphs the TREE t-SPANNER problem (the problem to decide whether G admits a tree t-spanner) is polynomial time solvable for $t \leq 3$ and is NP-complete as long as t is part of the input. They also left as an open problem whether the TREE t-SPANNER problem is polynomial time solvable for every fixed $t \geq 4$. In this work we resolve this open problem and extend the solution in several directions. We show that for every fixed t, it is possible in polynomial time not only to decide if a planar graph G has a tree t-spanner, but also to decide if G has a t-spanner of *bounded treewidth*. Moreover, for every fixed values of t and k, the problem, for a given planar graph G to decide if G has a t-spanner of treewidth at most k, is not only polynomial time solvable, but is *fixed parameter tractable* (with k and t being the parameters). In particular, the running time of our algorithm is linear with respect to the size of G. We extend this result from planar to a much more general class of sparse graphs containing graphs of bounded genus. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some vertices of G. We show that the problem of finding a t-spanner of treewidth k is fixed parameter tractable on graphs that do not contain some fixed apex graph as a minor, i.e. on *apex-minor-free graphs*. We prove that the tractability border of the t-spanner problem cannot be extended beyond the class of apex-minor-free graphs. In particular, for every $t \geq 4$, the problem of finding a tree t-spanner is NP-complete on K_6*-minor-free graphs*. Thus our results are tight, in a sense that the restriction of input graph being apex-minor-free cannot be replaced by H-minor-free for some non-apex fixed graph H. Graphs of bounded treewidth are sparse graphs and our technique can be used to settle the complexity of the parameterized version of the SPARSEST t-SPANNER problem, where for given t and m one asks if a given n-vertex graph has a t-spanner with at most $n - 1 + m$ edges. Our results imply that the SPARSEST t-SPANNER problem is fixed parameter tractable on apex-minor-free graphs with t and m being the parameters. Finally, we show that the optimization version of the SPARSEST t-SPANNER problem, which asks for a t-spanner with the minimum number of edges, admits PTAS for apex-minor-free graphs. This resolves an open question asked by Duckworth, Wormald, and Zito.

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1 Introduction

One of the basic questions in the design of routing schemes for communication networks is to construct a spanning network (a so-called spanner) which has two (often conflicting) properties: it should have simple structure and nicely approximate distances in the network. This problem fits in a larger framework of combinatorial and algorithmic problems that are concerned with distances in a finite metric space induced by a graph. An arbitrary metric space (in particular a finite metric defined by a graph) might not have enough structure to exploit algorithmically. A powerful technique that has been successfully used recently in this context is to embed the given metric space in a simpler metric space such that the distances are approximately preserved in the embedding. New and improved algorithms have resulted from this idea for several important problems (see, e.g., [4, 5, 11, 41]). Tree metrics are a very natural class of simple metric spaces since many algorithmic problems become tractable on them.

Peleg and Ullman [50] suggested the following parameter to measure the quality of a spanner. The spanner S of a graph G has the *stretch factor* t if the distance in S between any two vertices is at most t times the distance between these vertices in G. A *tree* t*-spanner* of a graph G is a spanning tree with a stretch factor t. If we approximate the graph by a tree t-spanner, we can solve the problem on the tree and the solution interpret on the original graph. Unfortunately, not many graph families admit good tree spanners. This motivates the study of sparse spanners, i.e. spanners with a small amount of edges. There are many applications of spanners in various areas; especially, in distributed systems and communication networks. In [50], close relationships were established between the quality of spanners (in terms of stretch factor and the number of spanner edges), and the time and communication complexities of any synchronizer for the network based on this spanner. Another example is the usage of tree t-spanners in the analysis of arrow distributed queuing protocols [21, 37]. Sparse spanners are very useful in message routing in communication networks; in order to maintain succinct routing tables, efficient routing schemes can use only the edges of a sparse spanner [51]. We refer to the survey paper of Peleg [47] for an overview on spanners.

In this work we study t-spanners of bounded treewidth (we postpone the definition of treewidth till the next section). Specifically,

PROBLEM: k-TREEWIDTH t-SPANNER
INSTANCE: A connected graph G and integers k and t.
QUESTION: Is there a t-spanner S of G of treewidth at most k?

Many algorithmic problems are tractable on graphs of bounded treewidth, and a spanner of small treewidth can be used to obtain an approximate solution to a problem on G. Since every connected graph with n vertices and at most $n-1+m$ edges is of treewidth at most $m+1$, we can see this problem as a generalization of the TREE t-SPANNER and the SPARSEST t-SPANNER problems. In this paper we also consider the optimization version of the SPARSEST t-SPANNER problem, which for a given graph G is to find a t-spanner with the minimum number of edges.

Related work. Substantial work has been done on the TREE t-SPANNER problem, also known as the minimum stretch spanning tree problem. Cai and Corneil [10] have shown that, for a given graph G, the problem to decide whether G has a tree t-spanner is NP-complete for any fixed $t \geq 4$ and is linear time solvable for $t = 1, 2$ (the status of the case $t = 3$ is open for general graphs). An $O(\log n)$-approximation algorithm for the minimum value of t for the TREE t-SPANNER problem is due to Emek and Peleg [30]. See the survey of Peleg [47] on more details on this problem and its variants.

The TREE t-SPANNER problem on planar graphs was studied intensively. Fekete and Kremer [33] proved that the TREE t-SPANNER problem on planar graphs is NP-complete (when t is part of the input). They also have shown that it can be decided in polynomial time whether a given planar graph has a tree 3-spanner and also have given a polynomial time algorithm that for any fixed t decides for planar graphs with bounded face length whether there is a tree t-spanner. For fixed $t \geq 4$, the complexity of the TREE t-SPANNER problem on planar graphs was left as an open problem [33].

There are several papers investigating the complexity of the problem on subclasses of planar graphs. Peleg and Tendler [49] showed that the problem can be solved in polynomial time on outerplanar graphs, and also in the special case of 1-face depth graphs in which no interior vertex has degree 2. Boksberger et al. [8] investigated the problem on grids and subgrids. They presented polynomial time algorithm on grids and $O(OPT^4)$-approximation for subgrids. Panda and Das [46] gave a polynomial time algorithm for tree 4-spanners in 2-trees.

Sparse t-spanners were introduced by Peleg, Schäffer and Ullman in [48, 50] and since that time were studied extensively. It was shown by Peleg and Schäfferin [48] that the problem of deciding whether a graph G has a t-spanner with at most m edges is NP-complete. Later, Kortsarz [38] showed that for every $t \geq 2$ there is a constant $c < 1$ such that it is NP-hard to approximate the sparsest t-spanner within the ratio $c \cdot \log n$, where n is the number of vertices in the graph. On the other hand, the problem admits a $O(\log n)$-ratio approximation for $t = 2$ [39, 38] and a $O(n^{2/(t+1)})$-ratio approximation for $t > 2$ [29]. For some other inapproximability and approximability results for the SPARSEST t-SPANNER problem on general graphs we refer the reader to [27, 28, 29] and papers cited therein.

On planar graphs the SPARSEST t-SPANNER problem was studied as well. Brandes and Handke have shown that the decision version of the problem remains NP-complete on planar graphs for every fixed $t \geq 5$ (the case $2 \leq t \leq 4$ is open) [9]. Duckworth, Wormald, and Zito [26] have shown that the problem of finding a sparsest 2-spanner of a 4-connected planar triangulation admits a polynomial time approximation scheme (PTAS). Their PTAS is based on a polynomial time reductions to the maximum edge star packing in planar cubic graphs and maximum induced matching in $2, 3$-regular bipartite planar graphs. The question raised by Duckworth, Wormald, and Zito in [26] concerns the existence of similar schemes for the problem of finding a sparsest t-spanner, $t > 2$, of a 4-connected planar triangulation or indeed, of a general planar graph.

Finally, let us briefly mention the relevant work on parameterized algorithms on planar graphs and more general classes of graphs (we refer to books [23, 34, 45] for more information on parameterized complexity and algorithms). Alber et al. [1] initiated the study of subexponential parameterized algorithms for the dominating set problem and its different variations. Demaine et al. [17, 19] gave a general framework called *bidimensionality* to design parameterized algorithms for many problems on planar graphs and showed how by making use of this framework to extend results from planar graphs to much more general graph classes including H-minor-free graphs. We refer to surveys [18, 22] for an overview of results and techniques related to bidimensionality. However, this framework cannot be used directly to solve the k-TREEWIDTH t-SPANNER problem because the theory of Demaine et al. is strongly based on the assumption that the parameterized problem should be minor or edge contraction closed, which is not the case for spanners. In particular, it is easy to construct an example when by contracting of an edge in a graph G with a t-spanner of treewidth k, one can obtain a graph which does not have such a spanner.

Our results and organization of the paper. In this paper we resolve the problems left open in [33] and [26] and extend solutions in several directions. Our general technique is combinatorial in nature and is based on the following observation. Let $\mathcal{G}$ be a class of graphs such that for every fixed t and every $G \in \mathcal{G}$, the treewidth of every t-spanner of G is $\Omega(treewidth(G))$. Then as an almost direct corollary of Bodlaender's Algorithm and Courcelle's Theorem (see Section 5 for details), we have that the k-Treewidth t-spanner problem is fixed parameter tractable on $\mathcal{G}$. Our main combinatorial result is the proof that the class of apex-minor-free graphs, which contains planar and bounded genus graphs, is in $\mathcal{G}$. We also answer the question explicitly mentioned in [9] and [26] about the approximability status of the Sparsest t-Spanner problem on planar graphs.

After preliminary Section 2, we start (Section 3) by proving the combinatorial properties of t-spanners in planar graphs. Our main result here is the proof that every t-spanner of a planar graph of treewidth k has treewidth $\Omega(k/t)$. The proof idea is based on a theorem due to Robertson, Seymour, Thomas [53] on planar graphs excluding a grid as a minor. A technical complication of a direct usage of this theorem is that non-existence of a k-treewidth t-spanner in a minor or a contraction of a graph G does not imply non-existence of a k-treewidth t-spanner in G. This is why we have to work here with walls and topological minors.

It is possible to extend the combinatorial result on t-spanners in planar graphs to apex-minor-free graphs (Section 4). This extension is quite technical and is based on a number of new insights on the structure of apex-minor-free graphs. The main tools here are the structural theorem of Robertson and Seymour characterizing graphs excluding a graph as a minor and the theorem of Demaine and Hajiaghayi on grid-minors in such graphs. We find the study of the k-treewidth t-spanner problem on apex-minor-free graphs interesting not only because this is a very general class of graphs, containing planar graphs and graphs of bounded genus. It appears that apex-minor-free graphs form a natural barrier for extension of many parameter/treewidth combinatorial bounds which hold for planar graphs [16]. However, for almost every such a parameter, the class of apex-minor-free graphs is not an algorithmic obstacle, in a sense, that very often it is possible to construct parameterized algorithms for H-minor-free graphs, where H is not necessary an apex graph, see, e.g. [17]. Surprisingly, this is not the case for the t-spanner problem. We show (Section 5) that the result on tractability of the problem on the class of apex-minor-free graphs is tight and cannot be extended further: the problem becomes untractable on H-minor-free graphs, when H is not an apex graph. In particular, for every $t \geq 4$, the problem of finding a tree t-spanner is NP-complete even on *K_6-minor-free graphs.*

In Section 6, we make a twist from the parameterized complexity to the approximability issue of Sparsest t-Spanner. We show that the problem admits a polynomial time approximation scheme (PTAS) on the class of apex-minor-free graphs for every $t \geq 1$. This not only answers the open question of Duckworth, Wormald, and Zito in [26] on the existence of a PTAS for finding a sparsest t-spanner, $t > 2$, of a 4-connected planar triangulation, but extends the answer to much more general classes of graphs. In a contrast to Duckworth, Wormald, and Zito in [26], we proceed here in a more common way by employing the technique of Baker [3], Demaine [15], Eppstein [31, 32], and Grohe [36].

In Section 7, we conclude with open problems and directions for further research.

2 Preliminaries

Let $G = (V, E)$ be an undirected unweighted graph with the vertex set V and edge set E. (We often will use notations $V(G) = V$ and $E(G) = E$.) The *distance* $\mathrm{dist}_G(u, v)$ between vertices u and v of a connected graph G is the length (the number of edges) of a shortest (u, v)-path in G.

Let t be a positive integer. A subgraph S of G, such that $V(S) = V(G)$, is called a *(multiplicative) t-spanner*, if $\mathrm{dist}_S(u, v) \leq t \cdot \mathrm{dist}_G(u, v)$ for every pair of vertices u and v. The parameter t is called the *stretch factor* of S. It is easy to see that the t-spanners can equivalently be defined as follows.

Proposition 1. *Let G be a connected graph, and t be a positive integer. A spanning subgraph S of G is a t-spanner of G if and only if for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \leq t$.*

Given an edge $e = (x, y)$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and replace all multiple edges by simple edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is a subgraph of a contraction of G. We say that a graph G is *H-minor-free* when it does not contain H as a minor. We also say that a graph class $\mathcal{G}$ is *H-minor-free* (or, excludes H as a minor) when all its members are H-minor-free. For example, the class of planar graphs is a K_5-minor-free graph class. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some vertices of G. A graph class $\mathcal{G}$ is *apex-minor-free* if $\mathcal{G}$ excludes a fixed apex graph H as a minor. If an edge of a graph G is replaced by the path between it's ends then it is said that this edge is *subdivided*. A graph H is a *topological minor* of a graph G, if G contains a subgraph which is isomorphic to a graph obtained from H by subdividing some of its edges.

The (r, s)-*grid* is the Cartesian product of two paths of lengths $r - 1$ and $s - 1$. The (r, s)-*wall* is a graph W_{rs} with the vertex set

$$\{(i, j)\colon 1 \leq i \leq r, 1 \leq j \leq s\}$$

such that two vertices (i, j) and (i', j') are adjacent if and only if either $i = i'$ and $j' \in \{j - 1, j + 1\}$, or $j = j'$ and $i' = i + (-1)^{i+j}$.

Let W_{rs} be a wall. By P_i^h we denote the shortest path connecting vertices $(i, 1)$ and (i, s), and by P_j^v is denoted the shortest path connecting vertices $(1, j)$ and (r, j) with assumption that, for $j > 1$, P_j^v contains only vertices (x, y) with $x = j - 1, j$. We call by the *southern part of* W_{rs} the path P_r^h, and by the *northern part of* W_{rs} the path P_1^h. Similar, the *eastern* and the *western* parts are the paths P_s^v and P_2^v, correspondingly. See Figure 1 for an illustration of these notions.

If W is obtained by subdivision of edges of W_{rs}, with slightly abusing the notation, we also will be using these terms for the paths obtained by subdivisions from the corresponding paths of W_{rs}.

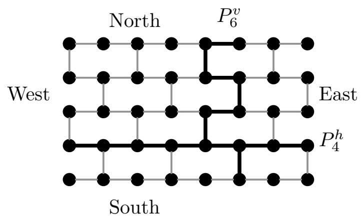

Figure 1: $(5,8)$-wall with parts and paths P_4^h, P_6^v

It is easy to check that if a graph G contains the (r,r)-grid as a minor, then it contains W_{rr} as a topological minor. Also if G contains W_{rr} as a topological minor, then it contains $(r, \lfloor r/2 \rfloor)$-grid as a minor.

A *tree decomposition* of a graph G is a pair (X, U) where U is a tree whose vertices we will call *nodes* and $X = (\{X_i \mid i \in V(U)\})$ is a collection of subsets of $V(G)$ such that

1. $\bigcup_{i \in V(U)} X_i = V(G)$,
2. for each edge $(v, w) \in E(G)$, there is an $i \in V(U)$ such that $v, w \in X_i$, and
3. for each $v \in V(G)$ the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of U.

The *width* of a tree decomposition $(\{X_i \mid i \in V(U)\}, U)$ equals $\max_{i \in V(U)} \{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G. We use notation $\mathbf{tw}(G)$ to denote the treewidth of a graph G. A tree decomposition with U being a path, is called a *path decomposition* and the pathwidth of G is the minimum width over all path decompositions of G.

We will need the following result which is due to Robertson, Seymour & Thomas [53].

Proposition 2 ([53])**.** *Every planar graph with no (r,r)-grid as a minor has treewidth $\leq 6r - 5$.*

A *surface* Σ is a compact 2-manifold without boundary (we always consider connected surfaces). A *line* in Σ is a subset homeomorphic to $[0, 1]$ and a (closed) *disc* $\Delta \subseteq \Sigma$ is a subset homeomorphic to $\{(x, y) : x^2 + y^2 \leq 1\}$. An O-arc is a subset of Σ homeomorphic to a circle. Whenever we refer to a Σ-*embedded graph* G we consider a 2-cell embedding of G in Σ. To simplify notations, we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider a graph G embedded in Σ as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$.

3 Planar graphs

In this section we prove that for every fixed t, a planar graph of large treewidth cannot have a t-spanner of small treewidth.

Theorem 1. *Let G be a planar graph of treewidth k and let S be a t-spanner of G. Then the treewidth of S is $\Omega(k/t)$.*

Proof. We need the following technical claim. Let G be a planar graph embedded in the plane and containing the wall W_{rs} as a topological minor. Let W be a subgraph of G isomorphic to a subdivision of W_{rs}. Let Δ be the disc in the plane which is bordered by the union of the southern, western, northern and eastern parts of W (with exclusion of pendant vertices) and containing W.

Claim 1. *For every $t \le \min\{s/4, r/2\} - 1$, every t-spanner S of G contains a path connecting the southern and the northern parts of W, and a path connecting the eastern and the western parts of W. Moreover, both these paths are in Δ.*

Proof. Let us prove the claim for the eastern and the western parts of W. Suppose that for some t-spanner S of G there is no path completely inside of Δ connecting the eastern and the western parts of W. Consider the path $P^h_{\lceil r/2 \rceil}$ in the wall. We find the first edge (x, y) in this path (starting from the western part) with the following property: there is a path in $S \cap \Delta$ connecting the eastern part of W and x but there are no such paths for y. Clearly, such an edge has to exist. Let P be a shortest path in S connecting x and y. By the choice of x and y, path P is not entirely in Δ. So it can be divided into three subpaths: the first path P_1 connects x with some vertex u on the border of Δ, the second part P_2 connects u with some vertex v, which also lies on the border of Δ, the third path P_3 connects v and y, and $P_1 \cup P_3 \subset \Delta$. Note that vertex u cannot belong to the eastern part, and vertex v cannot belong to the western part. The length of P is at least $\mathrm{dist}_S(x, u) + \mathrm{dist}_S(y, v) + 1$. If u is in the northern or the southern part, then $\mathrm{dist}_S(x, u) \ge r/2 - 1 \ge t$. If v is in the northern or the southern part. then $\mathrm{dist}_S(y, v) \ge r/2 - 1 \ge t$. If u is in the western part and v is in the eastern part, then $\mathrm{dist}_S(x, u) + \mathrm{dist}_S(y, v) \ge s/2 - 1 \ge t$. Hence, in all cases, the length of P is at least $t + 1$, and S is not a t-spanner.

The claim for the northern and southern parts is proved by similar arguments. We have only to consider path $P^v_{\lfloor s/2 \rfloor + 1}$ instead of $P^h_{\lceil r/2 \rceil}$. Note also that here we need the requirement $t \le s/4 - 1$. □

Set now $r = \lfloor \frac{k+4}{6} \rfloor$ and let S be a t-spanner of G. By Proposition 2, G has an (r, r)-grid as a minor. Thus G has an (r, r)-wall W_{rr} as a topological minor. Wall W_{rr} contains $\lfloor \frac{r}{4t+1} \rfloor$ disjoint $(4t + 1, r)$-walls. Let W be a subgraph of G isomorphic to a subdivision of W_{rr}. By applying Claim 1 to each $(4t + 1, r)$-wall, we have that there are $\lfloor \frac{r}{4t+1} \rfloor$ vertex disjoint paths in S connecting eastern and western parts of W. By similar arguments, S also contains $\lfloor \frac{r}{4t+1} \rfloor$ vertex disjoint paths connecting southern and northern parts of W. The union of these paths contains $(\lfloor \frac{r}{4t+1} \rfloor, \lfloor \frac{r}{4t+1} \rfloor)$-grid as a minor. So, S contains this grid as a minor, too, and the treewidth of S is at least $\lfloor \frac{r}{4t+1} \rfloor = \lfloor \frac{\lfloor (k+4)/6 \rfloor}{4t+1} \rfloor = \Omega(k/t)$. □

4 Apex-minor-free graphs

In this section, we extend Theorem 1 to graphs with bounded genus and to apex-minor-free graphs.

4.1 Bounded genus

The Euler genus $\mathbf{eg}(\Sigma)$ of a nonorientable surface Σ is equal to the nonorientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The Euler genus $\mathbf{eg}(\Sigma)$ of an orientable surface Σ is $2g(\Sigma)$, where

$g(\Sigma)$ is the orientable genus of Σ. The following extension of Proposition 2 on graphs of bounded genus is due to Demaine et al. [17].

Proposition 3 ([17])**.** *If G is a graph with treewidth more than $6r(\mathbf{eg}(G)+1)$ which is embeddable on a surface with Euler genus $\mathbf{eg}(G)$, then G has the (r,r)-grid as a minor.*

We also need a result roughly stating that if a graph G with a big wall as a topological minor is embedded on a surface Σ of small genus, then there is a disc in Σ containing a big part of the wall of G. This result is implicit in the work of Robertson and Seymour and there are simpler alternative proofs by Mohar and Thomassen [43, 55]. We use the following variant of this result from Geelen et al. [35].

Proposition 4 ([35])**.** *Let g, l, r be positive integers such that $r \geq g(l+1)$ and let G be an (r,r)-grid. If G is embedded in a surface Σ of Euler genus at most g^2-1, then some (l,l)-subgrid of G is embedded in a closed disc Δ in Σ such that the boundary cycle of the (l,l)-grid is the boundary of the disc.*

Recall that H is a *surface minor* of the embedded graph G, if H can be obtained from G by deleting edges and vertices, and performing contractions "on the surface" so that embedding is locally preserved (see [44]). If (r,r)-wall is embedded in a surface Σ, then $(r,\lfloor r/2 \rfloor)$-grid is a surface minor of the wall. Hence, by Proposition4 we have the following statement.

Corollary 1. *Let g, l, r be positive integers such that $r \geq 2g(l+1)$ and let G be an (r,r)-wall. If G is embedded in a surface Σ of Euler genus at most g^2-1, then some (l,l)-subwall of G is embedded in a closed disc Δ in Σ such that the west, east, northern and southern parts of the (l,l)-wall compose the boundary of the disc.*

Combining this result and Proposition 3, and using the same arguments as in the planar case, we have the following theorem.

Theorem 2. *Let G be a graph of treewidth k and Euler genus g, and let S be a t-spanner of G. Then the treewidth of S is $\Omega(\frac{k}{t \cdot g^{3/2}})$.*

Proof. Let G be a Σ-embedded graph. We put $g = \mathbf{eg}(\Sigma)$ and $r = \lfloor \frac{k}{6(g+1)} \rfloor$. By Proposition 3, G contains the (r,r)-grid as a minor, and thus, the (r,r)-wall as a topological minor. By Corollary 1, there is a subgraph $W \subseteq G$ which is isomorphic to a subdivision of the $(\lfloor \frac{r}{2\sqrt{g+1}} \rfloor - 1, \lfloor \frac{r}{2\sqrt{g+1}} \rfloor - 1)$-wall, such that the union of its eastern, western, southern and northern parts (with exclusion of pendant vertices) is the contractible O-line which borders the disk containing W (in Σ-embedded graph G). Let S be a t-spanner of G. By applying to W and S the same arguments as in the planar case, we have that S contains the $\Omega((\lfloor \frac{r}{2\sqrt{g+1}} \rfloor - 1)/t), \Omega((\lfloor \frac{r}{2\sqrt{g+1}} \rfloor - 1)/t)$-grid as a minor. Thus, the treewidth of S is $\Omega(\frac{k}{t \cdot g^{3/2}})$. □

4.2 Excluding apex as a minor

This extension of Theorems 1 and 2 to apex-minor-free graphs is based on a structural theorem of Robertson and Seymour [52]. Before describing this theorem we need some definitions.

Definition 1 (Clique-Sums)**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, and $k \geq 0$ an integer. For $i = 1, 2$, let $W_i \subset V_i$, form a clique of size h and let G_i' be the* *graph obtained from G_i by removing a set of edges (possibly empty) from the clique $G_i[W_i]$. Let $F : W_1 \to W_2$ be a bijection between W_1 and W_2. We define the h-*clique-sum *of G_1 and G_2, denoted by $G_1 \oplus_{h,F} G_2$, or simply $G_1 \oplus G_2$ if there is no confusion, as the graph obtained by taking the union of G'_1 and G'_2 by identifying $w \in W_1$ with $F(w) \in W_2$, and by removing all the multiple edges. The image of the vertices of W_1 and W_2 in $G_1 \oplus G_2$ is called the* join *of the sum.*

Note that some edges of G_1 and G_2 are not edges of G, because it is possible that they were added by clique-sum operation. Such edges are called *virtual*.

We remark that $\oplus$ is not well defined; different choices of G'_i and the bijection F could give different clique-sums. A sequence of h-clique-sums, not necessarily unique, which result in a graph G, is called a *clique-sum decomposition* of G.

Definition 2 (h-nearly embeddable graphs)**.** *Let Σ be a surface with boundary cycles $C_1, \ldots, C_h$, i.e. each cycle C_i is the border of a disc in Σ. A graph G is h-*nearly embeddable *in Σ, if G has a subset X of size at most h, called* apices*, such that there are (possibly empty) subgraphs $G_0, \ldots, G_h$ of $G \setminus X$ such that*

- $G \setminus X = G_0 \cup \cdots \cup G_h$,
- *G_0 is embeddable in Σ, we fix an embedding of G_0,*
- *graphs $G_1, \ldots, G_h$ (called* vortices*) are pairwise disjoint,*
- *for $1 \leq \cdots \leq h$, let $U_i := \{u_{i_1}, \ldots, u_{i_{m_i}}\} = V(G_0) \cap V(G_i)$, G_i has a path decomposition (B_{ij}), $1 \leq j \leq m_i$, of width at most h such that*
  - *for $1 \leq i \leq h$ and for $1 \leq j \leq m_i$ we have $u_j \in B_{ij}$*
  - *for $1 \leq i \leq h$, we have $V(G_0) \cap C_i = \{u_{i_1}, \ldots, u_{i_{m_i}}\}$ and the points $u_{i_1}, \ldots, u_{i_{m_i}}$ appear on C_i in this order (either if we walk clockwise or anti-clockwise).*

The following proposition is known as the Excluded Minor Theorem [52] and is the cornerstone of Robertson and Seymour's Graph Minors theory.

Proposition 5 ([52])**.** *For every graph H there exists an integer h, depending only on the size of H, such that every graph excluding H as a minor can be obtained by h-clique-sums from graphs that can be h-nearly embedded in a surface Σ in which H cannot be embedded.*

Moreover, it is known (see e.g. [19]) that there is a clique-sum decomposition with an additional property:

Proposition 6. *There is a clique decomposition of an H-minor-free graph G such that each clique-sum in the decomposition involves at most three vertices from each summand other than apices and vertices in vortices of that summand.*

Let us remark that by the result of Demaine et al. [20] such a clique-sum decomposition can be obtained in time $n^{O(1)}$ (the exponent in the running time depends only on H). However, we use Robertson and Seymour theorem only for the proof of the combinatorial bound, so we do not need to construct such a decomposition.

We also need the following results of Demaine and Hajiaghayi.

Proposition 7 ([19])**.** *If G is an H-minor-free graph with treewidth more than k, then G has the $(\Omega(k), \Omega(k))$-grid as a minor (the hidden constants in the Ω notation depend only on the size of H).*

Theorem 3. *Let H be a fixed apex graph. For every t-spanner S of an H-minor-free graph G, the treewidth of S is $\Omega(\mathbf{tw}(G))$. (The hidden constants in the Ω notation depend only on the size of H and t).*

Proof. Let G be an H-minor-free graph of treewidth k. It is well known, that for any pair of graphs G_1, G_2, $\mathbf{tw}(G_1 \oplus G_2) \le \max\{\mathbf{tw}(G_1), \mathbf{tw}(G_2)\}$. Thus, by decomposing G as a clique sum described in Propositions 5 and 6, we conclude that there is a summand G' in this clique sum such that a) G' h-almost embeddable in a surface Σ of genus h; b) the treewidth of G' is at least k.

Let us note that G' is not necessarily a subgraph of G because during h-clique-sums, when the join vertices are turned into clique, extra edges (virtual edges) can appear.

The further proof is performed in two steps. First we prove that Σ contains a closed disc $\Delta' \subset \Sigma$ such that *i)* $G' \cap \Delta'$ contains an $(\Omega(k), \Omega(k))$-wall as a topological minor and *ii)* no vertex of $G' \cap \Delta'$ is adjacent to an apex vertex and to a vertex from a vortex. In the second step, by extending the arguments used for planar graphs on the wall inside Δ', we prove that every t-spanner of G has a large grid as a minor, and thus has treewidth $\Omega(k)$.

Constructing Δ'. Let X be the set of apices of G' and $G_0, \ldots, G_h$ be subgraphs of $G' \setminus X$, where G_0 is Σ-embedded and $G_1, \ldots, G_h$ are vortices attached to boundary cycles $C_1, \ldots, C_h$ of G_0. It is known (see, for example, Lemmata 4.2 and 4.3 in [19]) that $\mathbf{tw}(G_0) = \Omega(\mathbf{tw}(G')) = \Omega(k)$.

By Proposition 7, G_0 contains the (ℓ, ℓ)-wall as a topological minor, where $\ell = \Omega(k)$. Then, by Proposition 4, there is a subgraph $W \subset G_0$ which is isomorphic to a subdivision of the $(\lfloor \frac{\ell}{2\sqrt{h}} \rfloor, \lfloor \frac{\ell}{2\sqrt{h}} \rfloor)$-wall, such that the union of its eastern, western, southern and northern parts is the contractible O-line which borders the disc $\Delta \supset W$ (in Σ-embedded graph G_0).

Now we want to prove that there is a subgraph W' of W which is isomorphic to a subdivision of an $(\Omega(\ell), \Omega(\ell))$-wall such that the union of the eastern, western, southern and northern parts of W' borders a disc $\Delta' \supset W'$ such that no vertex inside of Δ' is adjacent to a vertex neither from X, nor to a vertex from a vortex.

Let us prove at first that there is a disk in Δ such that no vertex inside of it is adjacent to a vertex from X. The intuition here is that if we fix an apex vertex x and partition the disc Δ into small discs like a chessboard, then making x adjacent to at least one vertex of every cell of the chessboard, will create a large apex-minor in G'. Then using these arguments recursively, we find the required disc. However, the technical difficulty here is that some edges of G' are virtual and the existence of an apex-minor in G' does not necessary contradict the assumption that G apex-minor-free. We need the following claim to overcome this obstacle.

Claim 2. *Let $x \in X$, and suppose that $Q \subseteq \Delta$ is a subdivision of a (q, q)-wall. Then there is a positive integer p (which depends only on H) such that if $\lfloor \frac{q}{p} \rfloor > 2$, then there is a $(\lfloor \frac{q}{p} \rfloor - 2, \lfloor \frac{q}{p} \rfloor - 2)$-subwall Q' of Q such that the disk formed by the union of the eastern, western, southern and northern parts of Q' does not have vertices adjacent to x.*

Proof. We denote by H_p the apex graph obtained from the (p, p)-grid by adding a universal vertex v adjacent to all vertices of the grid. It is known [53], that every planar graph F

with $2|V(F)| + 4|E(F)| \le p$ is contained in the (p,p)-grid as a minor. Therefore, for $p = 2|V(H)| + 4|E(H)|$, G does not contain as a minor H_p.

Assume that the disc Δ is bounded by the northern, southern, eastern and western parts of Q. The disc Δ can be partitioned into p^2 smaller discs such that each of these discs contains as a topological minor the $(\lfloor \frac{q}{p} \rfloor, \lfloor \frac{q}{p} \rfloor)$-wall. Such discs are obtained by cutting Δ along $p-1$ lines connecting the southern and the northern and $p-1$ lines connecting the eastern and the western parts of Q. For $i,j \in \{1,2,\dots,p\}$, we denote by Q_{ij} the subwall of Q captured inside these discs. For every Q_{ij}, we denote by Δ_{ij} the disc bounded by paths P^h_2, $P^h_{\lfloor q/p \rfloor - 1}$, P^v_2 and $P^v_{\lfloor q/p \rfloor - 1}$ in the wall Q_{ij} (see Figure 2) Since all these discs are bounded by northern, southern, western and eastern parts of some walls, we call by the northern, southern, western and eastern parts of a disc the corresponding part of the wall.

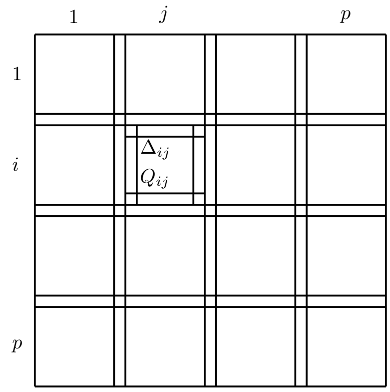

Figure 2: Wall Q_{ij} and disc Δ_{ij}.

If there are indices $i,j \in \{1,2,\dots,p\}$ such that disc Δ_{ij} does not contain a vertex adjacent to x, then Claim 2 holds. For sake of contradiction, suppose that every such disc contains a vertex v_{ij} which is adjacent to x.

Let (u,v) be a virtual edge of G'. It means that some graph F is connected by h-clique-sum to some clique U of G' and that $u,v \in U$. We always assume that the graph obtained from F by deleting vertices of U is connected. Otherwise we can decompose the clique-sum into clique-sums of the connected components of this graph. Thus there is an (u,v)-path P in F with all inner vertices in $V(F) \setminus U$. We say that P is a *cover* of the edge (u,v). If an edge (u,v) is not virtual (i.e. $(u,v) \in E(G)$), then (u,v) is its own cover. If P' is an (u,v)-path in G', then the union of covers of edges of P' is called the *cover of* P'. Note that the cover of a path is not necessary a path itself, however it contains an (u,v)-path. Also if an (u,v)-path P' lays completely in the disc Δ, i.e. $P' \subset \Delta$, then by Proposition 6, a cover of every edge contains at least one edge, which belongs to the (u,v)-path in the cover.

For $i,j \in \{1,2,\dots,p\}$, we denote by L_{ij} the path in G which is the cover of (x, v_{ij}). Let us note that by Proposition 6, these paths have no common vertices but x. Let L^h_{ij} be a shortest $(v_{i,j}, v_{i,j+1})$-path in G' such that it lays completely in Δ between paths $P^h_{(i-1)\lfloor q/p \rfloor + 2}$ and $P^h_{i\lfloor q/p \rfloor - 1}$ in Q, which form the northern and southern parts of $\Delta_{i,j}$ and $\Delta_{i,j+1}$, and also lays between the western part of $\Delta_{i,j}$ and the eastern part of $\Delta_{i,j+1}$, for $i \in \{1,2,\dots,p\}$ and $j \in \{1,2,\dots,p-1\}$. Let $\widetilde{L}^h_{ij}$ be the cover of L^h_{ij}. Correspondingly, let L^v_{ij} be a shortest $v_{i,j}, v_{i+1,j}$-path in G' such that it lays completely in Δ between paths $P^v_{(i-1)\lfloor q/p \rfloor + 2}$ and $P^v_{i\lfloor q/p \rfloor - 1}$ in

Q, which form the eastern and western parts of $\Delta_{i,j}$ and $\Delta_{i+1,j}$, and also lays between the northern part of $\Delta_{i,j}$ and the southern part of $\Delta_{i+1,j}$, for $i \in \{1, 2, \ldots, p-1\}$ and $j \in \{1, 2, \ldots, p\}$. Let $\widetilde{L}^v_{ij}$ be the cover of L^v_{ij}. By construction, all graphs $\widetilde{L}^h_{ij}$ and $\widetilde{L}^v_{ij}$ do not contain vertex x. Also, for every path L^h_{ij} (correspondingly, L^v_{ij}) we can chose an edge, which is crossed by line used to divide $Q_{i,j}$ and $Q_{i,j+1}$ (correspondingly, $Q_{i,j}$ and $Q_{i+1,j}$). All such edges do not have common incident vertices. By Proposition 6, the covers of these edges have no common vertices. Then the union of all graphs L_{ij}, $\widetilde{L}^h_{ij}$ and $\widetilde{L}^v_{ij}$, which is a subgraph of G, contains H_p as a minor. Thus G contains H as a minor, which is a contradiction. $\square$

By applying this claim inductively for each vertex of X, we conclude that there is subgraph W_X which is isomorphic to a subdivision of the $(\Omega(\frac{\ell}{p^h\sqrt{h}}), \Omega(\frac{\ell}{p^h\sqrt{h}}))$-wall such that the union of the eastern, western, southern and northern parts of W_X form the border of the disc $\Delta_X \supset W_X$ such that no vertex inside of Δ_X is adjacent to a vertex from X.

Every vortex boundary cycle C_i is either outside Δ_X, or is inside some face of W_X. Then for some $l' = \Omega(\frac{\ell}{hp^h\sqrt{h}})$, there is a subgraph W' containing as a topological minor the (l', l')-wall such that the union of the eastern, western, southern and northern parts of W' form the border of the disc $\Delta' \supset W'$.

Large wall in Δ' yields a spanner of large treewidth in G. As in the planar case, it is possible to show that the existence of the large wall in Δ' implies that every t-spanner in G' is of large treewidth. But again, one should be careful here because of possible existence of virtual edges in G'. Let F be a graph connected by h-clique-sum to G'. Let P be a path connecting two vertices u and v of the join in G' and such that all inner vertices of P are in $F \setminus G'$. Notice, that because u and v are in the join, there is edge (u, v) in G' (it is possible that this edge is virtual in G). We call (u, v) the *projection of P on G'*. For a path P in G' its projection on G' is P itself. Finally, the *projection* of an (u, v)-path P, where u and v are vertices of G' and P goes through some h-clique-sums $G' \oplus F_1 \oplus \cdots \oplus F_q$, is the union of the projections of parts of P.

Let X' be a subgraph of $G' \cap \Delta'$ isomorphic to a subdivision of the (r, s)-wall W_{rs}. Let $\Delta_{X'}$ be the disc in the plane which is bordered by the union of the southern, western, northern and eastern parts of X' (with exclusion of pendant vertices) and let $Y' = G' \cap \Delta_{X'}$. We also define $Y = G \cap \Delta_{X'}$. Thus, Y is a subgraph of G obtained from Y' by removing virtual edges and $Y \subseteq Y' \subseteq \Delta_{X'} \subseteq \Delta'$. (We assume that the drawing of Y in Σ is obtained from the drawing of Y' by erasing virtual edges.)

Since Y is inside of Δ', we have that no vertex of Y is adjacent to an apex vertex or to a vertex of a vortex. However, it is possible, that some subgraphs of G are attached to Y by h-clique-sums. Let $F_1, F_2, \ldots, F_q$ be the subgraphs of G attached to Y by h-clique-sums. Without loss of generality, we can assume that all these graphs are connected even after removal of the join set (see, e.g., [19]). With this assumption, for every virtual edge (u, v) of Y', there is an (u, v)-path in some of the summands F_i, and thus every path in Y' is a projection of some path in $Y \oplus F_1 \oplus \cdots \oplus F_q$.

Claim 3. *For every $t \leq \min\{s/4, r/2\} - 1$, every t-spanner S of G contains two paths in $Y \oplus F_1 \oplus \cdots \oplus F_q$ such that the projection of these paths on G' are in $\Delta_{X'}$ and one projection connects the southern and the northern parts of X', and the other projection connects the eastern and the western parts of X'.*

Proof. We prove the claim for the eastern and the western parts of X' (the proof for the southern and the northern parts is similar). For sake of contradiction, let us assume that for some t-spanner S of G there is no path in $Y \oplus F_1 \oplus \cdots \oplus F_q$ which projection connects the eastern and the western parts of X'. As in the planar case, we take the path $P^h_{\lceil r/2 \rceil}$ in the wall and find the first edge (x, y) in this path (starting from the western part) such that there is a path in Y' connecting the eastern part of X' and x but there is no such a path for y. (If no such an edge (x, y) exists, then $P^h_{\lceil r/2 \rceil}$ is the projection of a path satisfying the conditions of the claim.) Let P be a shortest path in S connecting x and y. The projection of P on G' is not entirely in $\Delta_{X'}$. As in the planar case, this implies that the length of the projection of P is at least $t + 1$. Since no vertex of Y' is adjacent to an apex, we conclude that the length of P is also at least $t + 1$. If (x, y) is an edge of G, then S is not a t-spanner in G. If (x, y) is a virtual edge, then x and y are in the join of some clique-sum, and thus there is an (x, y)-path P_F in some F_i. By the definition of (x, y), P_F is not entirely in S, thus there is an edge (u, v) of P_F not in S. But, since there is no path of length at most t in S projection of which is connecting x and y , we have that there is no path in S of length at most t connecting u and v. This is a contradiction and concludes the proof of Claim 3.

□

To finish the second step and the proof of the theorem, as in the planar case, we use Claim 3 to construct a big grid in a t-spanner of G. Let S be a t-spanner of G. The disc Δ' contains $\lfloor \frac{l'}{4t+1} \rfloor$ disjoint discs such that each of these small discs contains a subgraph of G' which has $(4t + 1, l')$-wall as a topological minor. By applying Claim 3 to each of these discs, we have that S has $\lfloor \frac{l'}{4t+1} \rfloor$ disjoint paths connecting the eastern and the western parts of W' and $\lfloor \frac{l'}{4t+1} \rfloor$ disjoint paths connecting the southern and the northern parts of W'. The projection in G' of every path from east to west intersects the projections of all paths from south to north. Because Y' is a plane drawing in Δ', these paths intersect only in vertices. Thus every east-west path in G intersect all south-west paths, and the union of these paths contains the $(\lfloor \frac{l'}{4t+1} \rfloor, \lfloor \frac{l'}{4t+1} \rfloor)$-grid as a minor. This yields that the treewidth of S is $\Omega(l') = \Omega(k)$. □

5 Algorithmic consequences

This section discusses algorithmic consequences of the combinatorial results obtained above. The proof of the following generic algorithmic observation is a combination of known results.

Lemma 1. *Let $\mathcal{G}$ be a class of graphs such that, for every $G \in \mathcal{G}$ and every t-spanner S of G, the treewidth of S is at least $\mathbf{tw}(G) \cdot f_{\mathcal{G}}(t)$, where $f_{\mathcal{G}}$ is the function only of t. Then for every fixed k and t, the existence of a t-spanner of treewidth at most k in $G \in \mathcal{G}$ can be decided in linear time.*

Proof. Let $G \in \mathcal{G}$ be a graph on n vertices and m edges. For given integers k and t, we use Bodlaender's Algorithm [6] to decide in time $O(n + m)$ if $\mathbf{tw}(G) \leq k/f_{\mathcal{G}}(t)$ (the hidden constants in the big-O depend only on k and $f_{\mathcal{G}}(t)$). If Bodlaender's Algorithm reports that $\mathbf{tw}(G) > k/f_{\mathcal{G}}(t)$, then we conclude that G does not have a t-spanner of treewidth at most k. Otherwise (when $\mathbf{tw}(G) \leq k/f_{\mathcal{G}}(t)$), Bodlaender's Algorithm computes a tree decomposition of G of width at most $k/f_{\mathcal{G}}(t)$.

Now we want to apply Courcelle's Theorem [12, 13], namely that every problem expressible in monadic second order logic (MSOL) can be solved in linear time on graphs of constant treewidth. To apply Courcelle's Theorem (and to finish the proof of our Theorem), we have to show that, for every fixed positive integers k and t, the property that a graph S is a t-spanner of treewidth at most k is expressible in MSOL. It is known that the property that a subgraph S has the treewidth at most k is expressible in MSOL for every fixed k (see, for example, [14]). Since any path is a sequence of adjacent edges, we have that the condition "for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \leq t$" can be written as an MSOL formula for every fixed t. By Proposition 1, this yields that "S is a t-spanner of treewidth at most k" is expressible in MSOL. $\square$

Theorem 3 and Lemma 1 imply the following result, which is the main algorithmic result of this paper. Let us note that for $k = 1$, Theorem 4 provides the answer to the question of Fekete and Kremer [33].

Theorem 4. *Let H be a fixed apex graph. For every fixed k and t, the existence of a t-spanner of treewidth at most k in an H-minor-free graph G can be decided in linear time.*

It is easy to see that the treewidth of a connected n-vertex graph with $n + m - 1$ edges is at most $m + 1$. Since for a fixed m, the property of that a S is a spanning subgraph of G with $n+m-1$ edges is in MSOL, we have (as in the proof of Lemma 1) that the combination of Theorem 3 with Bodlaender's Algorithm and Courcelle's Theorem implies the following corollary

Corollary 2. *Let H be a fixed apex graph. For every fixed m and t, the existence of a t-spanner with at most $n - 1 + m$ edges in an n-vertex H-minor-free graph G can be decided in linear time.*

It is easy to show that it is not possible to extend Theorem 3 to the class of H-minor free graphs, where H is not necessary an apex graph. For $i \geq 1$, let H_i be a graph obtained by adding to the (i, i)-grid a vertex v and making it adjacent to all vertices of the grid. Each of the graphs H_i, $i \geq 1$, does not contain the complete graph on six vertices K_6 as a minor. The treewidth of H_i is i, but it has a 2-spanner of treewidth one, which is the star with center in v. Thus, Lemma 1 cannot be used on graphs excluding a non-apex graph as a minor. Similar "apex-minor-free barrier" for using combinatorial bounds for parameterized algorithms was observed for other problems (e.g., parameterized dominating set [16]). However, for many of those problems, there are parameterized algorithms for H-minor-free graphs, which are based on dynamic programming over clique-sums of apex-minor-free graphs by making use of Robertson-Seymour structural theorem (Proposition 5), see, e.g. [17]. So, for many parameterized problems, combinatorial "apex-minor-free barrier" can be overcame. Surprisingly, this is not the case for the t-spanner problem. In particular, the Tree 4-Spanner problem is NP-complete on apex graphs, and since each apex graph is K_6-minor-free, it is NP-complete, for example, for K_6-minor-free graphs.

Note also that for apex graphs the claim of Theorem 3 is not correct. For $i \geq 1$, let H_i be a graph obtained by adding to the (i, i)-grid a vertex v and making it adjacent to all vertices of the grid. The graphs H_i, $i \geq 1$, do not contain the complete graph on six vertices K_6 as a minor. The treewidth of H_i is i, but it has a 2-spanner of treewidth one, which is the star with center in v.

Theorem 5. *For every fixed $t \geq 4$, deciding if an apex graph G has a tree t-spanner is NP-complete.*

Proof. The proof of this result is based on a modification of the reduction of Cai and Corneil [10] adapted to our purposes, and we start with the description of it.

The reduction of Cai and Corneil is from the well known NP-complete 3-Satisfiability problem. Let C be a boolean formula in conjunctive normal form with variables $u_1, u_2, \ldots, u_n$ and clauses $C_1, C_2, \ldots, C_m$, which contain three literals. Let $t \geq 4$ be an integer. For every variable u_i, a graph H_i is constructed by

1. taking five vertices $x_i, u_i, \overline{u}_i, y_i$, and z_i,
2. adding edges (x_i, y_i), (x_i, u_i), $(x_i, \overline{u}_i)$, (z_i, u_i), and $(z_i, \overline{u}_i)$,
3. joining y_i with z_i by a $(t-2)$-path, i.e., path of length $t-2$ (all internal vertices of the path are new), and joining endpoints of every edge of the path by two distinct t-paths,
4. joining u_i with $\overline{u}_i$ by a $(t-3)$-path (all internal vertices of this path are also new), and joining endpoints of every edge of the path by two distinct t-paths.

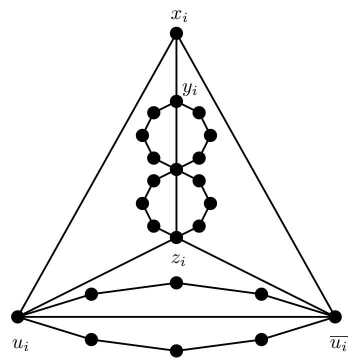

Figure 3: Graph H_i for $t = 4$.

Figure 3 shows such a graph for $t = 4$. Note that graph H_i is planar, and that it can be embedded such that vertices u_i and $\overline{u}_i$ are on the boundary of one (exterior) face. In the next stage, vertices $C_1, C_2, \ldots, C_m$ are introduced, and for every C_j edges between C_j and its literals are added. Then vertices $x_1, x_2, \ldots, x_n$ are merged into a single vertex x. Let $G(C)$ be the resulting graph.

The following statement is due to Cai and Corneil (Theorem 4.10 in [10]).

Proposition 8 ([10])**.** *Graph $G(C)$ has a t-tree spanner if and only if the formula C can be satisfied.*

Now we proceed with the proof of Theorem 5.

Instead of 3-Satisfiability, we use the following variant of the Planar 3-Satisfiability problem. Let C be a boolean formula in conjunctive normal form with variables $u_1, u_2, \ldots, u_n$ and clauses $C_1, C_2, \ldots, C_m$. For the formula C, let $G'(C)$ be the graph with the vertex set $\{u_1, u_2, \ldots, u_n\} \cup \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n\} \cup \{C_1, C_2, \ldots, C_m\}$ such that vertices u_i and $\overline{u}_i$ are adjacent for every $i \in \{1, 2, \ldots, n\}$, and for every $j \in \{1, 2, \ldots, m\}$, C_j and $z \in \{u_1, u_2, \ldots, u_n\} \cup \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n\}$ are adjacent if and only if clause C_j contains literal z. We use the fact that the Satisfiability problem remains NP-complete even if each clause contains no more than three literals, and graph $G'(C)$ is planar [40].

We assume (without loss of generality) that each clause contains at least two literals, and construct graph $G(C)$. It can be easily seen, that $G(C)$ is an apex graph with apex vertex x, since for the construction of the planar embedding of $G(C)-x$ we have only to replace edges $(u_i, \overline{u}_i)$ in the embedding of $G'(C)$ by graphs $H_i - x_i$. To complete the proof of the theorem, we note that the proof of the Proposition 8 holds even if every clause contains not exactly three literals, but two or three literals. □

6 A PTAS for the Sparsest Spanners problem on apex-minor-free Graphs

In this section we provide a PTAS for the Sparsest t-Spanner problem on apex-minor-free graphs. The property of apex-minor-free graphs used here is that for every fixed apex graph H, the class of H-minor-free graphs is the minor closed class of graphs of bounded local treewidth.

6.1 Graphs of bounded local treewidth and partial spanners

The notion of local treewidth was introduced by Eppstein [31, 32] under the name diameter-treewidth property. We say that a graph class $\mathcal{G}$ has bounded local treewidth if there is a function $f(r)$ (which depends only on r) such that for any graph G in $\mathcal{G}$ and any $v \in V(G)$, the treewidth of the subgraph of G induced by the set of vertices at distance at most r from v, is at most $f(r)$. A graph class $\mathcal{G}$ has *linear local treewidth* if $f(r) = O(r)$. For example, it is known [7, 1] that for every planar graph G, $f(r) \leq 3r - 1$. Moreover, for a planar graph of radius r, a corresponding tree decomposition of width at most $3r-1$ can be found in time $O(rn)$.

Eppstein [31, 32] characterized all minor-closed graph classes that have bounded local treewidth. It was proved that they are exactly apex-minor-free graphs. These results were improved by Demaine and Hajiaghayi [15] who proved that all apex-minor-free graphs have linear local treewidth.

For an edge subset $A \subseteq E(G)$ of a graph G, a *partial t-spanner for A* is a subgraph S of Gsuch that for every edge $(x, y) \in A$, $\mathrm{dist}_S(x, y) \leq t$. Thus for $A = E(G)$ a partial t-spanner for A is also a t-spanner for G.

The Sparsest Partial t-Spanner problem is to find a partial t-spanner with the minimum number of edges for a given graph G, an integer t and a set $A \subset E(G)$.

Using the fact that optimization problems which are expressible in MSOL can be solved in linear time for graphs with bounded treewidth [2], and applying the same arguments as in the proof of Lemma 1, we immediately obtain the following lemma.

Lemma 2. *Let k and t be positive integers. Let also G be a graph of treewidth at most k, and let $A \subseteq E(G)$. The* Sparsest Partial t-Spanner *problem can be solved by a linear-time algorithm (the constant which is used in the bound of the running time depends only on k and t) if a corresponding tree decomposition of G is given.*

Let us remark that dynamic-programming algorithm for the case $A = E(G)$ was given by Makowsky and Rotics [42]. The algorithm of Makowsky and Rotics can be easily adapted to solve the problem for arbitrarily choice of A.

Let u be a vertex of a graph G. For $i \geq 0$ we denote by L_i the i-th level of breadth first search, i.e. the set of vertices at distance i from u. We refer to the partition of the vertex set $V(G)$,

$$\mathcal{L}(G,u) = \{L_0, L_1, \dots, L_r\}$$

as the *breadth first search (BFS) decomposition* of G. We assume for convenience that for BFS decomposition $\mathcal{L}(G,u)$, $L_i = \emptyset$ for $i < 0$ or $i > r$. It is well known that the BFS decomposition can be constructed by the breadth first search in linear time.

Let G be a graph with BFS decomposition $\mathcal{L}(G,u) = (L_0, L_1, \dots, L_r)$, and let t be a positive integer. For integers $i \leq j$, we define

$$G_{ij} = G[\bigcup_{k=i}^{j} L_k].$$

Graph G_{ij} is shown in Figure 4.

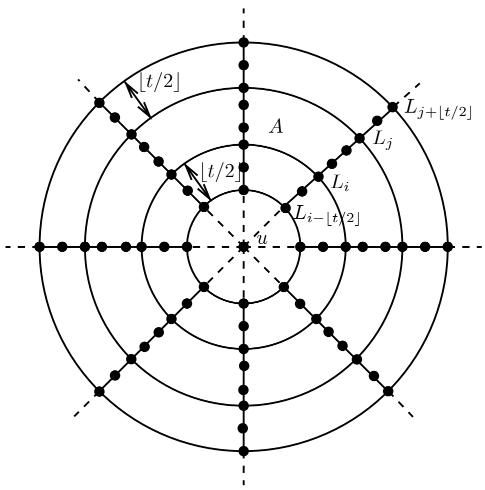

Figure 4: Graphs G_{ij} and G'_{ij}

We need the following result of Demaine and Hajiaghayi [15].

Lemma 3 ([15]). *Let G be an apex-minor-free graph. Then* $\mathbf{tw}(G_{ij}) = O(j - i)$.

Let $G'_{ij} = G_{i-\lfloor t/2 \rfloor, j+\lfloor t/2 \rfloor}$ and let $A = E(G_{ij})$ (see Fig. 4). Let S be a t-spanner of G and S' be the subgraph of S induced by $V(G'_{ij})$. We need the following claim.

Lemma 4. *Graph S' is a partial t-spanner for A in G'_{ij}.*

Proof. Let $(x,y) \in A$. Note that $x, y \in V(G_{ij})$. Since S is a t-spanner for G, we have that there is a x,y-path P in S of length at most t. Suppose that some vertex v of this path does not belong to G'. Then $v \in L_l$, for some $l < i - \lfloor t/2 \rfloor$ or $l > j + \lfloor t/2 \rfloor$. By the definition of the BFS decomposition, $\mathrm{dist}_G(x,v) > \lfloor t/2 \rfloor$ and $\mathrm{dist}_G(y,v) > \lfloor t/2 \rfloor$. But then P is of length at least $\mathrm{dist}_G(x,v) + \mathrm{dist}_G(v,y) \geq 2\lfloor t/2 \rfloor + 2 > t$. So, all vertices of P are vertices of G'_{ij}, and this path is a path in S'. $\square$

6.2 Description of the algorithm

Now we are ready to describe our algorithm. Let t, k be positive integers, $t < k$. For a given apex-minor-free graph G the BFS decomposition $\mathcal{L}(G, u) = (L_0, L_1, \ldots, L_r)$ is constructed for some vertex u.

If $r \le k$ then a t-spanner S of G is constructed directly. We use the fact that $\mathbf{tw}(G) = O(k)$ and use Bodlaender's Algorithm [6] to construct in linear time a suitable tree decomposition of G. Then, by Lemma 2, a sparsest t-spanner of G can be found in linear time.

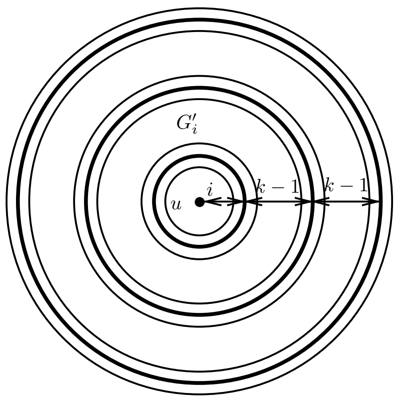

Figure 5: Graphs G'_j

Suppose now that $r > k$. We consequently construct t-spanners S_i of G for $i = 1, 2, \ldots, k-1$ as follows. Let

$$J_i = \{j \in \{2-k, 3-k, \ldots, r-1\} : j \equiv i \pmod{k-1}\}.$$

For every $j \in J_i$ we consider graph $G'_j = G_{j-\lfloor t/2 \rfloor, j+k+\lfloor t/2 \rfloor -1}$ and set of edges $A_j = E(G_{j,j+k-1})$. In other words, we "cover" graph G by graphs $G'_{i-(k-1)}, G'_i, G'_{i+(k-1)}, \ldots$, such that two consecutive graphs "overlap" by $2\lfloor t/2 \rfloor + 1$ levels in the BFS decomposition (see Fig. 5). The union of all sets A_j is the set $E(G)$. By Lemma 3, $\mathbf{tw}(G'_j) = O(k+t)$. For every graph G'_j, we construct a sparsest partial t-spanner S_{ij} for A_j in G'_j by making use of Lemma 2. We define

$$S_i = \bigcup_{j \in J_i} S_{ij}.$$

Finally, we choose among graphs $S_1, S_2, \ldots, S_{k-1}$ a graph with the minimum number of edges and denote it by S.

The following lemma describes the properties of the graph S.

Lemma 5. *Let S be a subgraph of an apex-minor-free graph G, obtained by the algorithm described above. Then the following holds*

1. *S is a t-spanner of G.*

2. *For every t and $k > t$, S can be constructed by a linear-time algorithm.*

3. *S has at most $(1 + \frac{t+1}{k-1})\mathrm{OPT}(G)$ edges, where $\mathrm{OPT}(G)$ is the number of edges in the solution of the* Sparsest t-Spanner *problem on G.*

Proof. 1. Every S_i is a t-spanner of G. Indeed, for every $(x,y) \in E(G)$, there is $j \in J_i$ such that $(x,y) \in A_j$, and $\mathrm{dist}_{S_i}(x,y) \leq \mathrm{dist}_{S_{ij}}(x,y) \leq t$.

2. The second claim follows from Lemmata 2 and 3.

3. If $k \geq r$, then the claim is trivial. Let $k < r$ and let T be a t-spanner of G with the minimum number of edges $m = |E(T)| = \mathrm{OPT}(G)$. Assume that $i \in \{1, 2, \ldots, k-1\}$ and $j \in J_i$. Let $T_j = T[V(G'_j)]$. By Lemma 4, T_j is a partial t-spanner for the set A_j in T_j. Then

$$|E(T_j)| \geq |E(S_{ij})|,$$

and

$$\begin{aligned} |E(S_i)| &\leq \sum_{j \in J_i} |E(T_j)| \\ &= m + \sum_{j \in J_i} |E(T) \cap E(G_{j-\lfloor t/2 \rfloor, j+\lfloor t/2 \rfloor})|. \end{aligned}$$

We have only to note that

$$\begin{aligned} |E(S)| &= \min_{1 \leq i \leq k-1} |E(S_i)| \\ &\leq m + \min_{1 \leq i \leq k-1} \sum_{j \in J_i} |E(T) \cap E(G_{j-\lfloor t/2 \rfloor, j+\lfloor t/2 \rfloor})| \\ &\leq m + \min_{1 \leq i \leq k-1} \sum_{j \in J_i} |E(G_{j-\lfloor t/2 \rfloor, j+\lfloor t/2 \rfloor})| \\ &\leq (1 + \frac{t+1}{k-1})m. \end{aligned}$$

□

Finally, we have the following corollary.

Theorem 6. *For every $t \geq 1$, the* Sparsest t-Spanner *problem admits a PTAS with linear running time for the class of apex-minor-free graphs (and, hence, for planar graphs and for graphs of bounded genus).*

7 Conclusions and open problems

We have shown that for fixed k and t, one can decide in linear time if an apex-minor-free graph G has a t-spanner of treewidth at most k. The results we used in our proof, Bodlaender's Algorithm and Courcelle's Theorem, have huge hidden constants in the running time, and thus Theorem 4 is of theoretical interest mainly. Because the class of apex-minor-free graphs is a very general class of graphs and since for K_6-minor-free graphs and $t = 4$ the problem is NP complete, we doubt that it is possible to design fast practical algorithms solving t-spanner problem on apex-minor-free graphs. However, on planar graphs and for small values of t, our ideas can be used to design practical algorithms. First of all, instead of using

Bodlaender's algorithm, one can use Ratcatcher algorithm of Seymour-Thomas [54] to find exact branchwidth of a planar graph. The running time of the algorithm is cubic, but there is no hidden constants. The second bottleneck of our approach for practical applications is the usage of Courcelle's Theorem. Instead of that, for small values of t, it is more reasonable to construct dynamic programming algorithms that use the properties of planarity and of the problem. Also it would be interesting to find a more efficient solution to the SPARSEST t-SPANNER problem at least for the class of planar graphs by utilizing the dynamic programming technique, the planarity of the graph and the specifics of the problem. For example, for planar graphs, the initial problem on G can be reduced to a subproblem of constructing a sparsest partial t-spanner for a subgraph of G with bounded outerplanarity.

Parameterized algorithm and PTAS in this paper are for unweighted graphs. It is easy to generalize our results to graphs with weighted edges if all weights do not exceed some constant. However, with unbounded weights our combinatorial arguments do not hold and one needs different ideas here. We leave the parameterized complexity of TREE t-SPANNER on weighted planar graphs as an open problem. Another left open problem is the existence of a PTAS for the SPARSEST t-SPANNER problem on weighted planar graphs.

We have shown that SPARSEST t-SPANNER admits a PTAS on apex-minor-free graphs. The approximability of the problem on graphs excluding some non-apex graph as a minor is an interesting open question.

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