CHAPTER 1
Regular Languages

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Regular expressions: definition

• An algebraic equivalent to finite automata.
• We can build complex languages from simple languages using operations on languages.
• Let \( \Sigma = \{a_1, \ldots, a_n\} \) be an alphabet. The simple languages over \( \Sigma \) are
  • the empty language \( \emptyset \), which contains no word.
  • for every symbol \( a \in \Sigma \), the language \( \{a\} \), which contains only the one-letter word “a”.
• The regular operations on languages are \( \cup \) (union), \( \circ \) (concatenation), and \( \ast \) (iteration).
• An expression that applies regular operations to simple languages is called a regular expression (and the resulting language is a regular language; we will see later why…).
• \( L(E) \) is the language defined by the regular expression \( E \).

Formally, \( R \) is a regular expression if \( R \) is
1. \( a \) for some \( a \) in the alphabet \( \Sigma \) (stands for a language \( \{a\} \)),
2. \( \epsilon \), standing for a language \( \{\epsilon\} \),
3. \( \emptyset \), standing for the empty language,
4. \( (R_1 \cup R_2) \), where \( R_1, R_2 \) are regular expressions,
5. \( (R_1 \circ R_2) \), where \( R_1, R_2 \) are regular expressions,
6. \( (R_1^\ast) \), where \( R_1 \) is a regular expression.
Notations

• When writing regular expressions, we use the following conventions:
  • For simple languages of the form \{a\}, we write a (omitting braces).
  • Parentheses are omitted according to the rule that iteration binds stronger
    than concatenation, which binds stronger than union.
  • The concatenation symbol \cdot is often omitted.
  • We write \Sigma for a_1 \cup \ldots \cup a_n.
  • We write \epsilon for \emptyset (which is the language that contains only the empty word).
• For example, 01^*\epsilon stands for the expression \{(0\cdot 0^*)\cup (0^*)\}.

Examples of expressions

\Sigma^*000\Sigma^* … the language of all words that contain the substring 000
(\Sigma^*)^* … the language of all words with an even number of letters
(0^*10^*)^*0^* … the language of all words that contain an even number of 1’s

Note that concatenating the empty set to any set yields the empty set: 1^*\emptyset = \emptyset

Equivalence with Finite Automata

• Regular expressions and finite automata are equivalent in their descriptive power.
• Any regular expression can be converted into a finite automaton
  that recognizes the language it describes, and vice versa.
• We will prove the following result

Theorem. A language is recognizable by a FA if and only if some
regular expression describes it.

• This theorem has two directions. We state each direction as a separate lemma.

Lemma 1. If a language is described by a regular expression, then it
is recognizable by a FA.

• We have a regular expression R describing some language A.
• We show how to convert R into an NFA recognizing A.
• We proved before that if an NFA recognizes A then a DFA recognizes A.
• To convert R into an NFA N, we consider the six cases in the formal
  definition of regular expression.
Proof of Lemma 1 (6 cases)

1. \( R = a \) for some \( a \) in \( \Sigma \). Then \( L(R) = \{a\} \), hence

\[
N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})
\]

\[
\delta(q_1, a) = \{q_2\}
\]

\[
\delta(r, b) = \emptyset \quad \text{for } r \neq q_1 \text{ or } b \neq a.
\]

2. \( R = \varepsilon \). Then \( L(R) = \{\varepsilon\} \), hence

\[
N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})
\]

\[
\delta(r, b) = \emptyset \quad \text{for any } r \text{ and } b.
\]

3. \( R = \emptyset \). Then \( L(R) = \emptyset \), hence

\[
N = (\{q\}, \Sigma, \delta, q, \emptyset)
\]

\[
\delta(r, b) = \emptyset \quad \text{for any } r \text{ and } b.
\]

4. \( R = R_1 \cup R_2 \).
5. \( R = R_1 \circ R_2 \).
6. \( R = R_1^* \).

- in these cases we use the constructions given in the proofs that the class of regular languages is closed under the regular operations.
- We construct the NFA for \( R \) from NFAs for \( R_1, R_2 \) and the appropriate closure construction.

Example 1

\[
\begin{align*}
a & \quad \text{a} \\
b & \quad \text{b} \\
ab & \quad \text{ab} \\
ab \cup a & \quad (ab \cup a)^
\end{align*}
\]

Building an NFA from the regular expression \((ab \cup a)^*\)
Example 2

\[ a \cup b \]

\[ (a \cup b)^{\ast} \]

ABA

\[ a \cup b \]

\[ (a \cup b)^{\ast} \]

\[ (a \cup b)^{\ast}aba \]

Building an NFA from the regular expression \((a \cup b)^{\ast}aba\)

Equivalence with Finite Automata

- We are working on the proof of the following result

**Theorem.** A language is regular if and only if some regular expression describes it.

- We have proved

**Lemma 1.** If a language is described by a regular expression, then it is regular.
  - For given regular expression \(R\), describing some language \(A\), we have shown how to convert \(R\) into an NFA recognizing \(A\).
  - Now we will prove the other direction

**Lemma 2.** If a language is regular then it is described by a regular expression.
  - For a given regular language \(A\), we need to write a regular expression \(R\), describing \(A\).
  - Since \(A\) is regular, it is accepted by a DFA.
  - We will describe a procedure for converting DFAs into equivalent regular expressions.
  - We will define a new type of finite automaton, generalized NFA (GNFA).
  - and show how to convert DFAs into GNFAs and then GNFAs into regular expression.
Generalized Non-deterministic Finite Automata

- **Generalized non-deterministic finite automata** are simply NFAs wherein the transition arrows may have any regular expressions as labels, instead of only members of the alphabet or ε.

For convenience we require that GNFA always have a form that meets the following conditions.

- the start state has arrows going to every other state but no incoming arrows.
- there is only one accepting state. It has incoming arrows from every other state but no outgoing arrows.
- moreover, the start state is not the same as the accept state.
- except for the start and accept states, one arrow goes from every state to every other state and also from each state to itself.

Formal definition of GNFA

A GNFA is a 5-tuple \((Q, \Sigma, \delta, q_{\text{start}}, q_{\text{accept}})\), where

1. \(Q\) is the finite set of states,
2. \(\Sigma\) is the input alphabet,
3. \(\delta: (Q - \{q_{\text{accept}}\}) \times (Q - \{q_{\text{start}}\}) \to R\) is the transition function,
4. \(q_{\text{start}}\) is the start state, and
5. \(q_{\text{accept}}\) is the accept state.

A GNFA accepts a string \(w\) in \(\Sigma^*\) if \(w = w_1, w_2, ..., w_n\), where each \(w_i\) is in \(\Sigma^*\) and a sequence of states \(r_0, r_1, r_2, ..., r_n\) exists such that

1. \(r_0 = q_{\text{start}}, r_n = q_{\text{accept}}\)
2. For each \(i\), we have \(w_i \in L(R_i)\), where \(R_i = \delta(r_{i-1}, r_i)\); in other words, \(R_i\) is the expression on the arrow from \(r_{i-1}\) to \(r_i\).

From DFAs to GNFA:

- add a new state with an ε arrow to the old start state, a new accept state with ε arrows from the old accept states.
- if any arrows have multiple labels (or if there are multiple arrows going between the same two states in the same direction) replace each with a single arrow whose label is the union of the previous labels.
- add arrows labeled ∅ between states that had no arrows.
From GNFAs to Regular Expressions.

Convert(G)
1. Let k be the number of states of GNFA G.
2. If k=2, then G must consist of a start state, an accept state, and a single arrow connecting them and labeled with a regular expression R. Return the expression R.
3. If k>2, select any state \(q_r \in Q\) different from start and accept states and let \(G'\) be the GNFA \((Q', \Sigma, \delta', q_{start}, q_{accept})\), where
   \[Q' = Q - \{q_r\}\]
   And for any \(q_i \in Q' - \{q_{accept}\}\) and any \(q_j \in Q' - \{q_{start}\}\) let
   \[\delta'(q_i, q_j) = (R_i)(R_j)^*(R_i) \cup (R_j),\]
   for \(R_i = \delta(q_i, q_r)\), \(R_2 = \delta(q_r, q_j)\), \(R_3 = \delta(q_i, q_j)\), \(R_4 = \delta(q_j, q_r)\).
4. Compute Convert(G') and return this value.

Claim. For any GNFA G, G' is equivalent to G.

Proof of Claim.

Claim. For any GNFA G, G' is equivalent to G.
- We show that G and G' recognize the same language
- Suppose G accepts an input \(w\)
  - then there exists a sequence of states s.t.
    \[q_{start} \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow ... \rightarrow q_{accept},\]
    \(w \in L(R_i)\), \(w = w_1w_2...w_k\)
  - if none of them is \(q_r\), then G' accepts \(w\) since each of the new regular expressions labeling arrows of G' contains the old reg. expression as a part of union
  - if \(q_r\) does appear, removing each sequence of consecutive \(q_r\) states forms an accepting path in G'.
    - the states \(q_r\) and \(q_j\) bracketing a sequence have a new regular expression on the arrow between them that describes all strings taking \(q_j\) to \(q_j\) via \(q_r\) on G.'
  - So, G' accepts \(w\).
- Suppose G' accepts \(w\)
  - as each arrow between any states \(q_i\) and \(q_j\) in G' describes the collection of strings taking \(q_i\) to \(q_j\) in G, either directly or via \(q_r\), G must also accept \(w\).
Example

(a(aa ∪ b)* ab ∪ b) ((ba ∪ a)(aa ∪ b)* ab ∪ bb)*((ba ∪ a)(aa ∪ b)* ε) ∪ a(aa ∪ b)*

Theory of Computation, Feodor F. Dragan, Kent State University