CHAPTER 1
Regular Languages

Contents

- Finite Automata (FA or DFA)
    - definitions, examples, designing, regular operations
- Non-deterministic Finite Automata (NFA)
    - definitions, equivalence of NFAs and DFAs, closure under regular operations
- **Regular expressions**
    - **definitions, equivalence with finite automata**
- Non-regular Languages
    - the pumping lemma for regular languages

Regular expressions: definition

- An algebraic equivalent to finite automata.
- We can build complex languages from simple languages using operations on languages.
- Let $\Sigma = \{a_1, ..., a_n\}$ be an alphabet. The simple languages over Σ are
  - the empty language $\varnothing$, which contains no word.
  - for every symbol $a \in \Sigma$, the language *{a},* which contains only the one-letter word "*a*".
- The regular operations on languages are $\bigcup$ (union), $\circ$ (concatenation), and * (iteration).
- An expression that applies regular operations to simple languages is called a ***regular expression*** (and the resulting language is a regular language; we will see later why…).
- *L(E)* is the language defined by the regular expression *E.*

Formally, *R* is a ***regular expression*** if *R* is

1. *a* for some *a* in the alphabet Σ (stands for a language *{a}*),
2. ε, standing for a language $\{\varepsilon\}$,
3. $\varnothing$, standing for the empty language,
4. $(R_1 \bigcup R_2)$, where R_1, R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1, R_2 are regular expressions,
6. (R_1^*), where R_1 is a regular expression.

Notations

- When writing regular expressions, we use the following conventions:
  - For simple languages of the form *{a},* we write *a* (omitting braces).
  - Parentheses are omitted according to the rule that iteration binds stronger than concatenation, which binds stronger than union.
  - The concatenation symbol $\circ$ is often omitted.
  - We write Σ for $a_1 \cup \ldots \cup a_n$.
  - We write ε for $\varnothing^*$(which is the language that contains only the empty word).
- For example, $01^* \cup \varepsilon$ stands for the expression $(\{0\} \circ (\{1\}^*)) \cup (\varnothing^*)$.

Examples of expressions

$\Sigma^* 000 \Sigma^*$ … the language of all words that contain the substring 000

$(\Sigma\Sigma)^*$ … the language of all words with an even number of letters

$(0^*10^*1)^*0^*$ … the language of all words that contain an even number of 1's

Note that concatenating the empty set to any set yields the empty set; $1^*\varnothing = \varnothing$

Equivalence with Finite Automata

• Regular expressions and finite automata are equivalent in their descriptive power.

• Any regular expression can be converted into a finite automaton that recognizes the language it describes, and vice versa.

• We will prove the following result

Theorem. A language is recognizable by a FA if and only if some regular expression describes it.

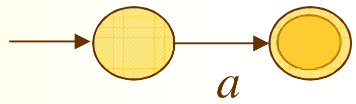
• This theorem has two directions. We state each direction as a separate lemma.

Lemma 1. If a language is described by a regular expression, then it is recognizable by a FA.

- We have a regular expression R describing some language A.
- We show how to convert R into an NFA recognizing A.
- We proved before that if an NFA recognizes A then a DFA recognizes A.
- To convert R into an NFA N, we consider the six cases in the formal definition of regular expression.

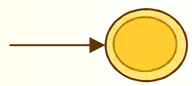
Proof of Lemma 1 (6 cases)

1. $R = a$ *for some* a *in* Σ. *Then* $L(R) = \{a\}$, *hence*



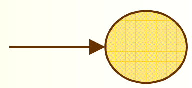
$$N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$$
$$\delta(q_1, a) = \{q_2\}$$
$$\delta(r, b) = \varnothing \quad \text{for } r \neq q_1 \text{ or } b \neq a.$$

2. $R = \varepsilon$. *Then* $L(R) = \{\varepsilon\}$, *hence*



$$N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$$
$$\delta(r, b) = \varnothing \quad \text{for any } r \text{ and } b.$$

3. $R = \varnothing$. *Then* $L(R) = \varnothing$, *hence*



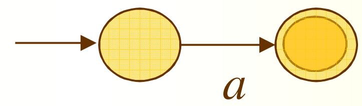
$$N = (\{q\}, \Sigma, \delta, q, \varnothing)$$
$$\delta(r, b) = \varnothing \quad \text{for any } r \text{ and } b.$$

4. $R = R_1 \cup R_2$.
5. $R = R_1 \circ R_2$.
6. $R = R_1^*$.

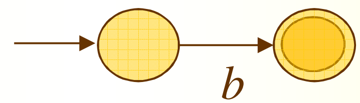
- in these cases we use the constructions given in the proofs that the class of regular languages is closed under the regular operations.
- We construct the NFA for ***R*** from NFAs for R_1, R_2 and the appropriate closure construction.

Example 1

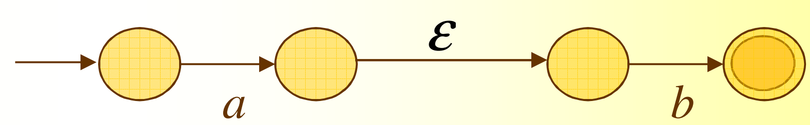
a



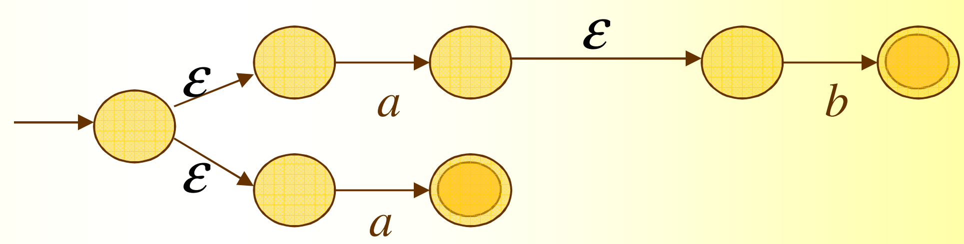
b



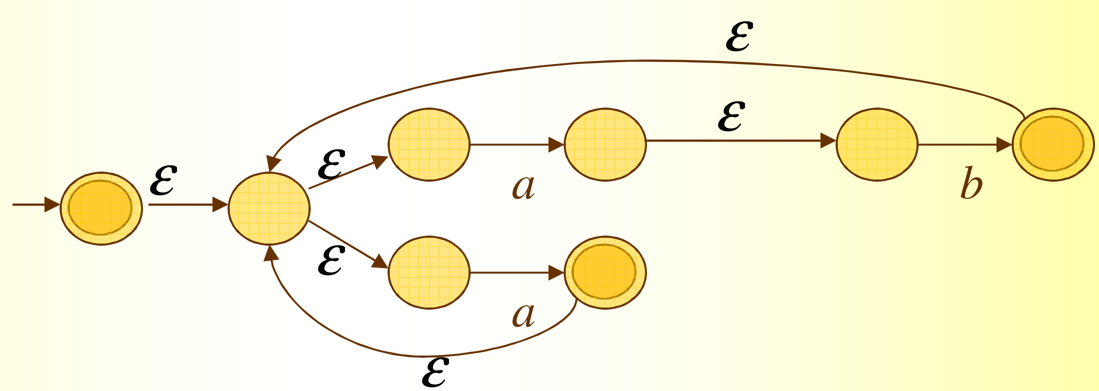
ab



$ab \cup a$

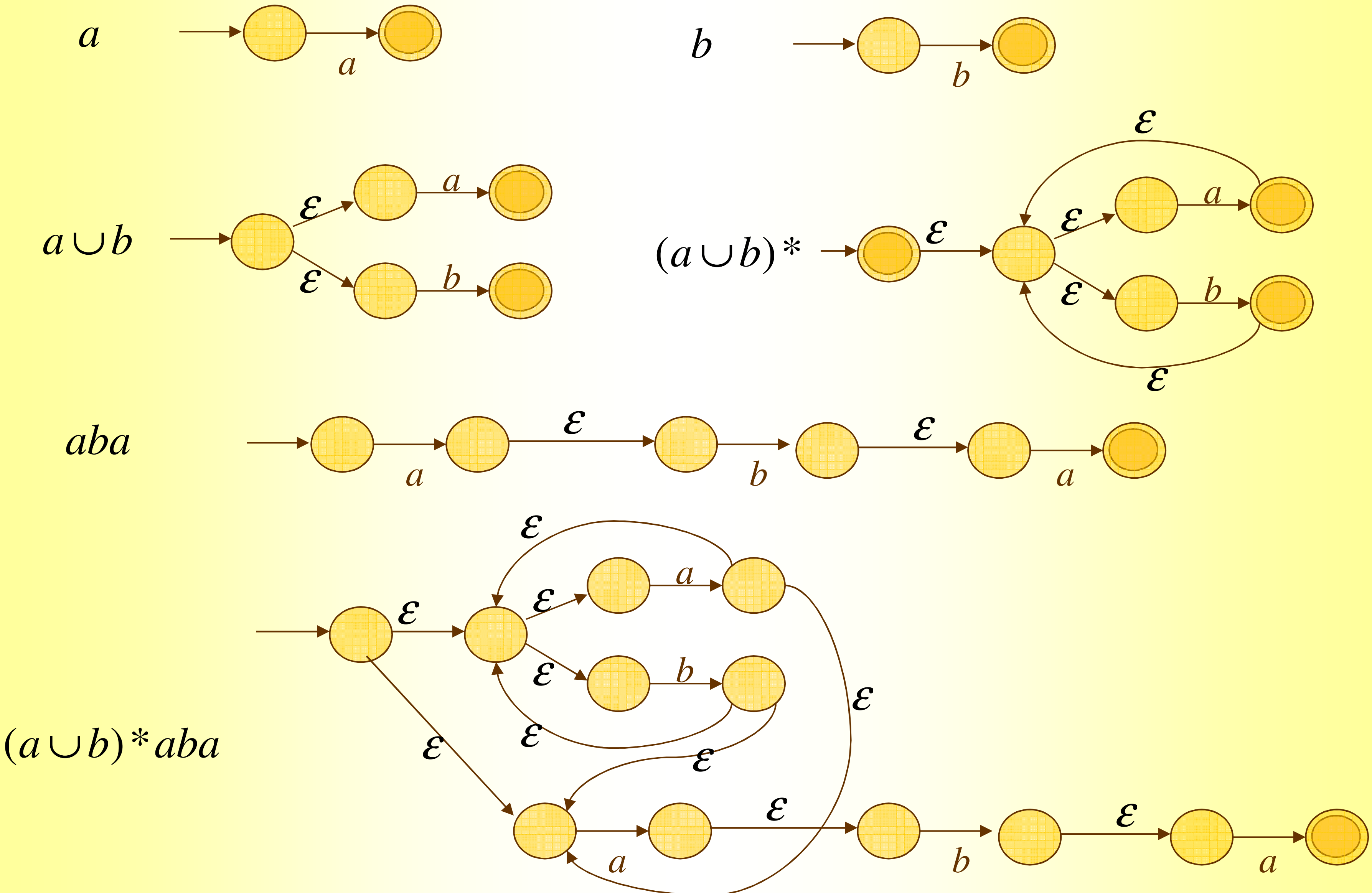


$(ab \cup a)^*$



Building an NFA from the regular expression $(ab \cup a)^*$

Example 2



Building an NFA from the regular expression $(a \cup b)^* aba$

Equivalence with Finite Automata

- We are working on the proof of the following result

Theorem. A language is regular if and only if some regular expression describes it.

- We have proved

Lemma 1. If a language is described by a regular expression, then it is regular.

  - For given regular expression *R,* describing some language *A, w*e have shown how to convert *R* into an NFA recognizing *A*.

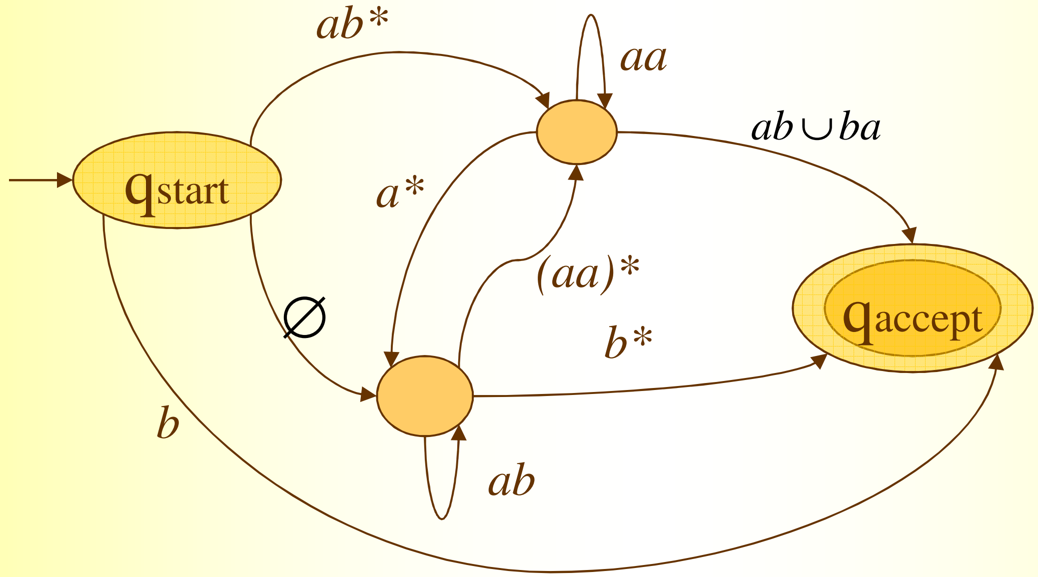
- Now we will prove the other direction

Lemma 2. If a language is regular then it is described by a regular expression.

  - For a given regular language *A*, we need to write a regular expression *R,* describing *A*.
  - Since *A* is regular, it is accepted by a DFA.
  - We will describe a procedure for converting DFAs into equivalent regular expressions.
  - We will define a new type of finite automaton, *generalized* NFA (GNFA).
  - and show how to convert DFAs into GNFAs and then GNFAs into regular expression.

Generalized Non-deterministic Finite Automata

- ***Generalized non-deterministic finite automata*** are simply NFAs wherein the transition arrows may have any regular expressions as labels, instead of only members of the alphabet or ε.



•For convenience we require that GNFAs always have a form that meets the following conditions.

- the start state has arrows going to every other state but no ingoing arrows.
- there is only one accepting state. It has ingoing arrows from every other state but no outgoing arrows.
- moreover, the start state is not the same as the accept state.
- except for the start and accept states, one arrow goes from every state to every other state and also from each state to itself.

Formal definition of GNFAs

• A GNFA is a 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where

1. Q is the finite set of states,
2. Σ is the input alphabet,
3. $\delta : (Q - \{q_{accept}\}) \times (Q - \{q_{start}\}) \to \mathrm{R}$ is the transition function,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.

• A GNFA accepts a string w in Σ^* if $w = w_1, w_2, \ldots, w_n$, where each w_i is in Σ^* and a sequence of states $r_0, r_1, r_2, \ldots, r_n$ exists such that

1. $r_0 = q_{start}, r_n = q_{accept}$
2. For each i, we have $w_i \in L(R_i)$, where $R_i = \delta(r_{i-1}, r_i)$; in other words, R_i is the expression on the arrow from r_{i-1} to r_i .

From DFAs to GNFAs

• add a new state with an ε arrow to the old start state, a new accept state with ε arrows from the old accept states.

• if any arrows have multiple labels (or if there are multiple arrows going between the same two states in the same direction) replace each with a single arrow whose label is the union of the previous labels.

• add arrows labeled $\varnothing$ between states that had no arrows.

From GNFAs to Regular Expressions.

Convert(G)

1. Let k be the number of states of GNFA G.
2. If $k=2$, then G must consist of a start state, an accept state, and a single arrow connecting them and labeled with a regular expression R. Return the expression R.
3. If $k>2$, select any state $q_r \in Q$ different from start and accept states and let G' be the GNFA $(Q', \Sigma, \delta', q_{start}, q_{accept})$, where

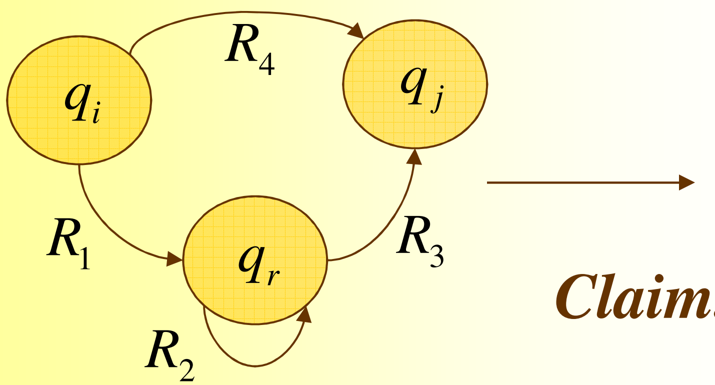
$$Q' = Q - \{q_r\},$$

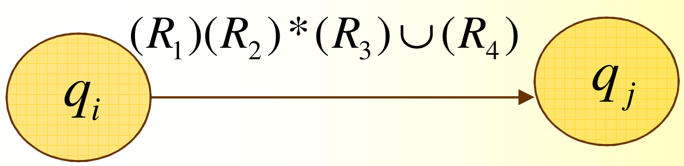
And for any $q_i \in Q' - \{q_{accept}\}$ and any $q_j \in Q' - \{q_{start}\}$ let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$

for $R_1 = \delta(q_i, q_r), R_2 = \delta(q_r, q_r), R_3 = \delta(q_r, q_j), R_4 = \delta(q_i, q_j).$

4. Compute ***Covert(G')*** and return this value.





Claim. For any GNFA G, G' is equivalent to G.

Proof of Claim.

Claim. For any GNFA *G, G'* is equivalent to *G*.

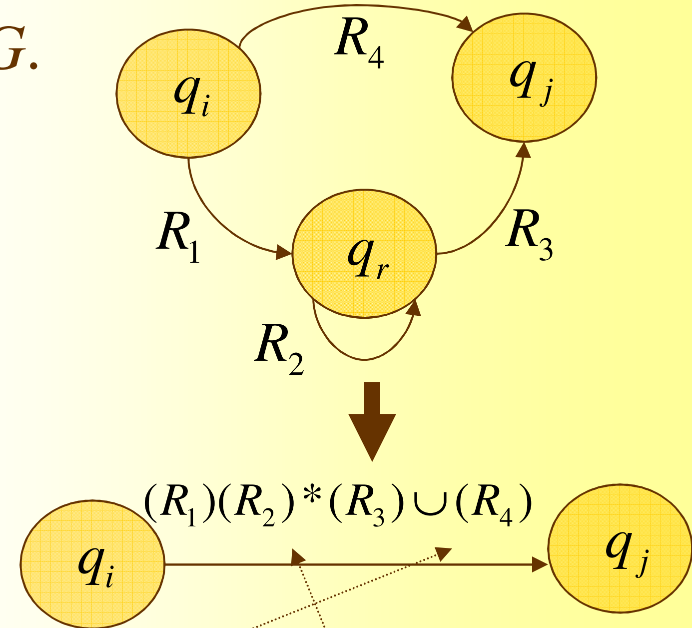
- We show that *G* and *G'* recognize the same language
- Suppose *G* accepts an input *w*
  - then there exists a sequence of states s.t.

$$q_{start} \xrightarrow{R_1} q_1 \xrightarrow{R_2} q_2 \xrightarrow{R_3} q_3 \xrightarrow{R_4} \ldots \xrightarrow{R_k} q_{accept},$$

$$w_i \in L(R_i),\ \ w = w_1\, w_2 \ldots w_k$$

  - if none of them is q_r, then *G'* accepts *w*

    since each of the new regular expressions labeling arrows of *G'* contains the old reg. expression as a part of union

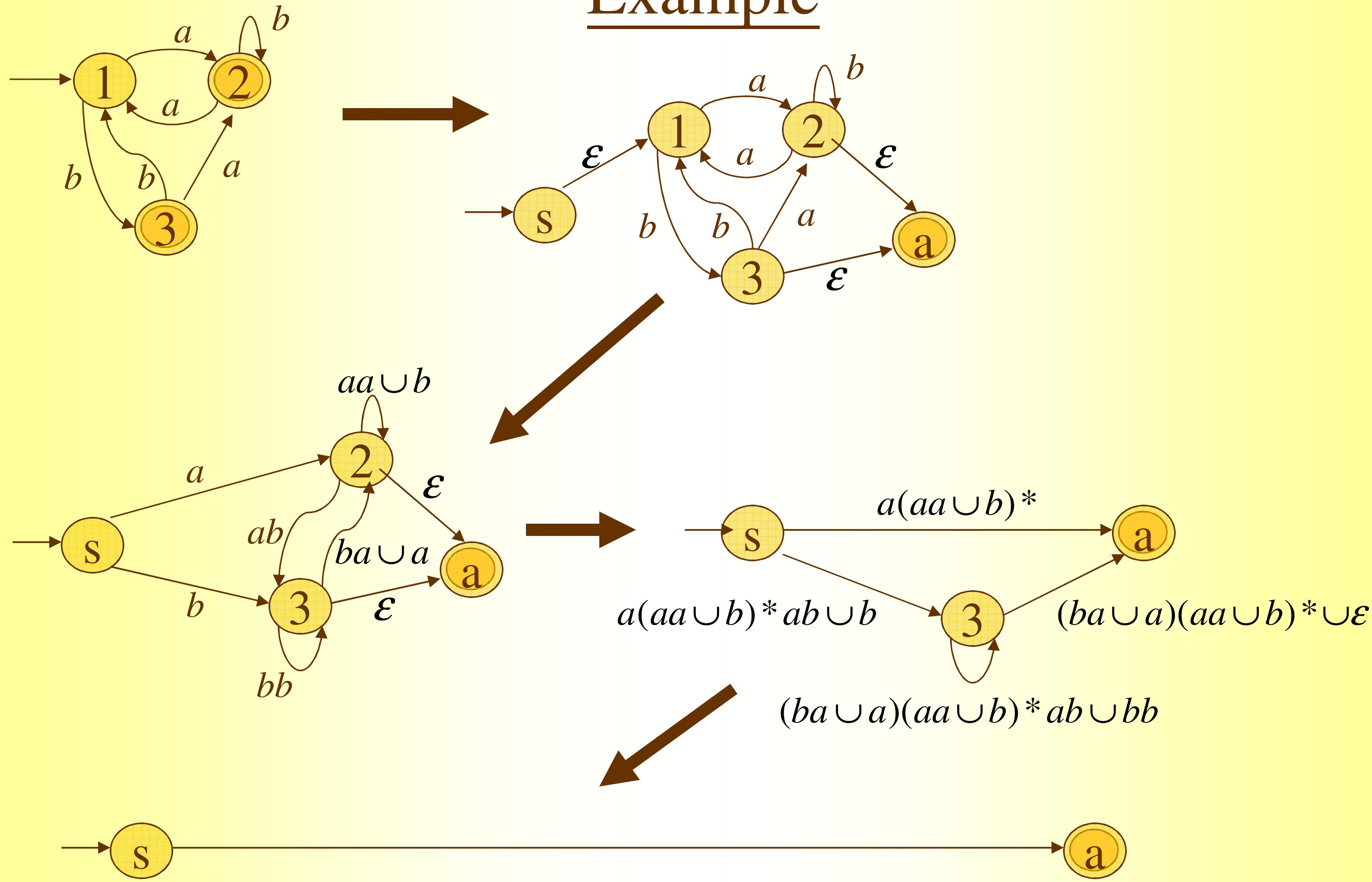


  - if q_r does appear, removing each sequence of consecutive q_r states forms an accepting path in G'.

    the states q_i and q_j bracketing a sequence have a new regular expression on the arrow between them that describes all strings taking q_i to q_j via q_r on G

  - So, *G'* accepts *w*.
- Suppose *G'* accepts *w*
  - as each arrow between any states q_i and q_j in *G'* describes the collection of strings taking q_i to q_j in *G*, either directly or via q_r, *G* must also accept *w.*

Example



$$(a(aa\cup b)^*ab\cup b)\,((ba\cup a)(aa\cup b)^*ab\cup bb)^*((ba\cup a)(aa\cup b)^*\cup \varepsilon)\cup a(aa\cup b)^*$$