CHAPTER 2
Context-Free Languages

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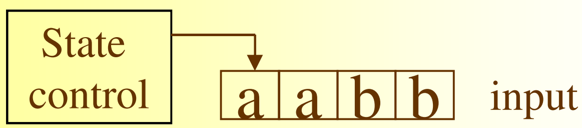
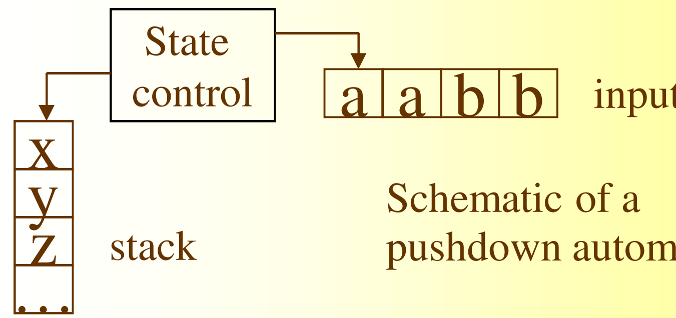
- Context-Free Grammars
  - definitions, examples, designing, ambiguity, Chomsky normal form
- **Pushdown Automata**
  - **definitions, examples, equivalence with context-free grammars**
- Non-Context-Free Languages
  - the pumping lemma for context-free languages

Pushdown Automata (PDAs)

- A new type of computational model.
- It is like a NFA but has an extra component called ***stack.***
- The stack provides additional memory beyond the finite amount available in the control.
- The stack allows pushdown automata to recognize some non-regular languages.
- Pushdown automata are equivalent in power to context-free grammars.



Schematic of a finite automaton



Schematic of a pushdown automaton

- A PDA can write symbols on stack and read them back later
- Writing a symbol is **pushing,** removing a symbol is **popping**
- Access to the stack, for both reading and writing, may be done only at the **top** (last in, first out)
- A stack is valuable because it can hold an unlimited amount of information.

Example

- Consider the language $\{0^n 1^n : n \geq 0\}$.
- Finite automaton is unable to recognize this language.
- A PDA is able to do this.

Informal description how the PDA for this language works.

- Read symbols from the input.
- As each *0* is read, push it into the stack.
- As soon as *1*s are seen, pop a *0* off the stack for each *1* read.
- If reading the input is finished exactly when the stack becomes empty of *0*s, accept the input.
- If the stack becomes empty while *1*s remain or

  if the *1*s are finished while the stack still contains *0*s or

  if any *0*s appear in the input following *1*s,

  reject the input.

Formal Definition of PDAs

- A ***pushdown automaton*** (PDA) is specified by a 6-tuple
  $(Q, \Sigma, \Gamma \delta, q_0, F)$, where

Non-deterministic

Is a collection of all subsets

Q	is a finite set of states,
Σ	is a finite input alphabet,
Γ	is a finite stack alphabet,
$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \boxed{P(Q \times \Gamma_\varepsilon)}$	is the transition function,
$q_0 \in Q$	is the initial state,
$F \subseteq Q$	is the set of final states.

$\Sigma_\varepsilon = \Sigma \bigcup \{\varepsilon\}$

$\Gamma_\varepsilon = \Gamma \bigcup \{\varepsilon\}$

- It computes as follows: it ***accepts*** input *w* if *w* can be written as $w = w_1, w_2, \ldots, w_n$, where each $w_i \in \Sigma_\varepsilon$ and a sequence of states $r_0, r_1, r_2, \ldots, r_n \in Q$ and strings $s_0, s_1, s_2, \ldots, s_n \in \Gamma^*$ exist that satisfy the next three conditions (the strings s_i represent the sequence of stack contents that PDA has on the accepting branch of the computation.

1. $r_0 = q_0, \; s_0 = \varepsilon$
2. $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a), \;\; i = 0, \ldots, n-1,$
   $\quad where \;\; s_i = at, \; s_{i+1} = bt \;\; for \; some \; a, b \in \Gamma_\varepsilon \; and \; t \in \Gamma^*.$
3. $r_n \in F$

Example

- Consider the language $\{0^n 1^n : n \geq 0\}$.

$M = (Q, \Sigma, \Gamma \delta, q1, F)$

$Q = \{q1, q2, q3, q4\}$

$\Sigma = \{0,1\}$

$\Gamma = \{0,\$\}$

$F = \{q1, q4\}$

δ :

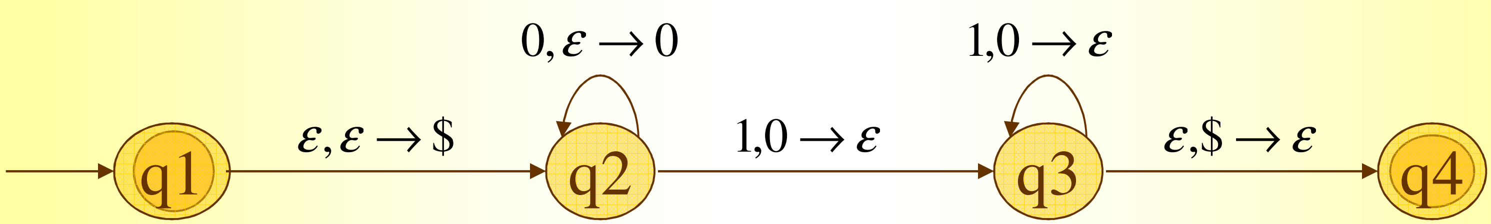
Input:	*0*			*1*			ε		
Stack:	*0*	*\$*	ε	*0*	*\$*	ε	*0*	*\$*	ε
q1	$\emptyset$								*{(q2,\$)}*
q2	$\emptyset$		*{(q2,0)}*	*{(q3, ε)}*					
q3	$\emptyset$			*{(q3, ε)}*				*{(q4, ε)}*	
q4	$\emptyset$								$\emptyset$

$a, b \rightarrow c$: when the machine is reading an ***a*** from the input it may replace the symbol ***b*** on the top of stack with a ***c***. Any of ***a,b,*** and ***c*** may be ε .

a is ε , the machine may take this transition without reading any input symbol.

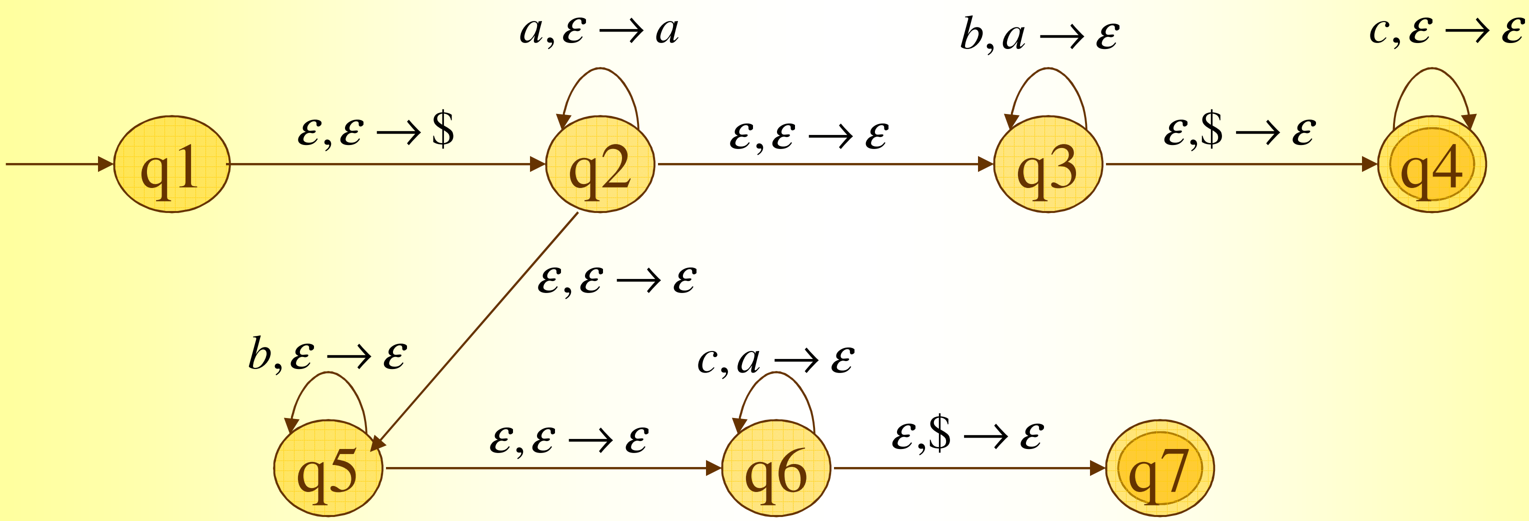
b is ε , the machine may take this transition without reading and popping any stack symbol.

c is ε , the machine does not write any symbol on the stack when going along this transition.

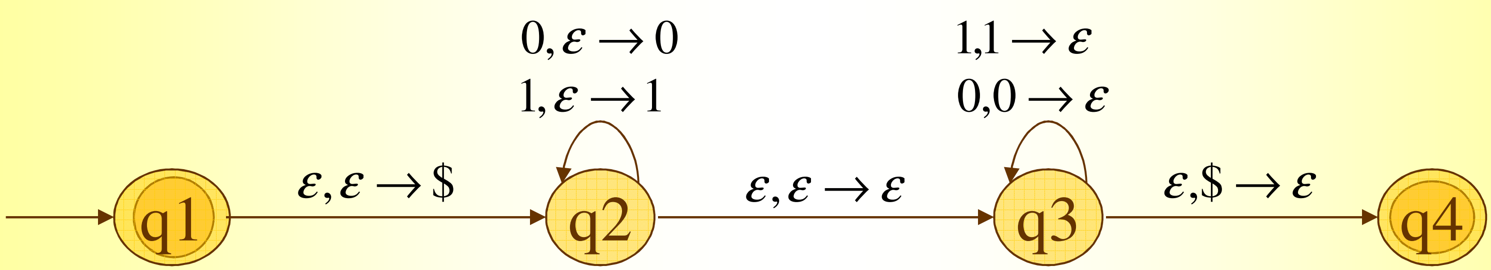


More Examples

- Language $\{a^i b^j c^k : i, j, k \geq 0 \ and \ i = j \ or \ i = k\}$.



- Language $\{ww^R : w \in \{0,1\}^*\}$.



Equivalence with Context-free Grammars

- Context-free grammars and pushdown automata are equivalent in their descriptive power. Both describe the class of context-free languages.

- Any context-free grammar can be converted into a pushdown automaton that recognizes the same language, and vice versa.

- We will prove the following result

Theorem. A language is context-free if and only if some pushdown automaton recognizes it.

- This theorem has two directions. We state each direction as a separate lemma.

Lemma 1. If a language is context-free, then some pushdown automaton recognizes it.

  - We have a context-free grammar G describing the context-free language L.
  - We show how to convert G into an equivalent PDA P.
  - The PDA P will accept string w iff G generates w, i.e., if there is a leftmost derivation for w.
  - Recall that a derivation is simply the sequence of substitution made as a grammar generates a string.

$$S \rightarrow AS \mid \varepsilon$$
$$A \rightarrow 0A1 \mid A1 \mid 01$$

- $S \underset{lm}{\Rightarrow} AS \underset{lm}{\Rightarrow} A1S \underset{lm}{\Rightarrow} 011S \underset{lm}{\Rightarrow} 011AS \underset{lm}{\Rightarrow} 0110A1S \underset{lm}{\Rightarrow} 0110011S \underset{lm}{\Rightarrow} 0110011$

How do we check that G generates 0110011 ?

$$S \rightarrow AS \mid \varepsilon$$
$$A \rightarrow 0A1 \mid A1 \mid 01$$

ε

$0A11S \Rightarrow 0A111S$

- $S \underset{lm}{\Rightarrow} AS \underset{lm}{\Rightarrow} A1S \underset{lm}{\Rightarrow} 011S \underset{lm}{\Rightarrow} 011AS \underset{lm}{\Rightarrow} 0110A1S \underset{lm}{\Rightarrow} 0110011S \underset{lm}{\Rightarrow} 0110011$

$A11S$

$0A1S \Rightarrow 0A11S$

$00A11S$

$0011S$

$01S \Rightarrow 011$

$01AS \Rightarrow \ldots$

Promising variants

Idea

- We can use stack to store an intermediate string of variables and terminals.
- It is better to keep only part (suffix) of the intermediate string, the symbols starting with the first variable.
- Any terminal symbols appearing before the first variable are matched immediately with symbols in the input string.
- Use non-determinism, make copies.

Informal description of PDA *P*

1. Place the marker symbol *$* and the start symbol on the stack.
2. Repeat the following steps forever.

   a) If top of stack is a variable symbol *A*, non-deterministically select one of the rules for A and substitute A by the string on the right-hand side of the rule.

   b) If the top of stack is terminal symbol *a*, read the next symbol from the input and compare it to *a*. If they match, pop *a* and repeat. If they do not match, reject on this branch of the non-determinism.

   c) If the top of the stack is the symbol *$*, enter the accept state. Doing so accepts the input if it has all been read.

$$\varepsilon$$

$$0A11S \Rightarrow 0A111S$$

- $S \underset{lm}{\Rightarrow} AS \underset{lm}{\Rightarrow} A1S \underset{lm}{\Rightarrow} 011S \underset{lm}{\Rightarrow} 011AS \underset{lm}{\Rightarrow} 0110A1S \underset{lm}{\Rightarrow} 0110011S \underset{lm}{\Rightarrow} 0110011$

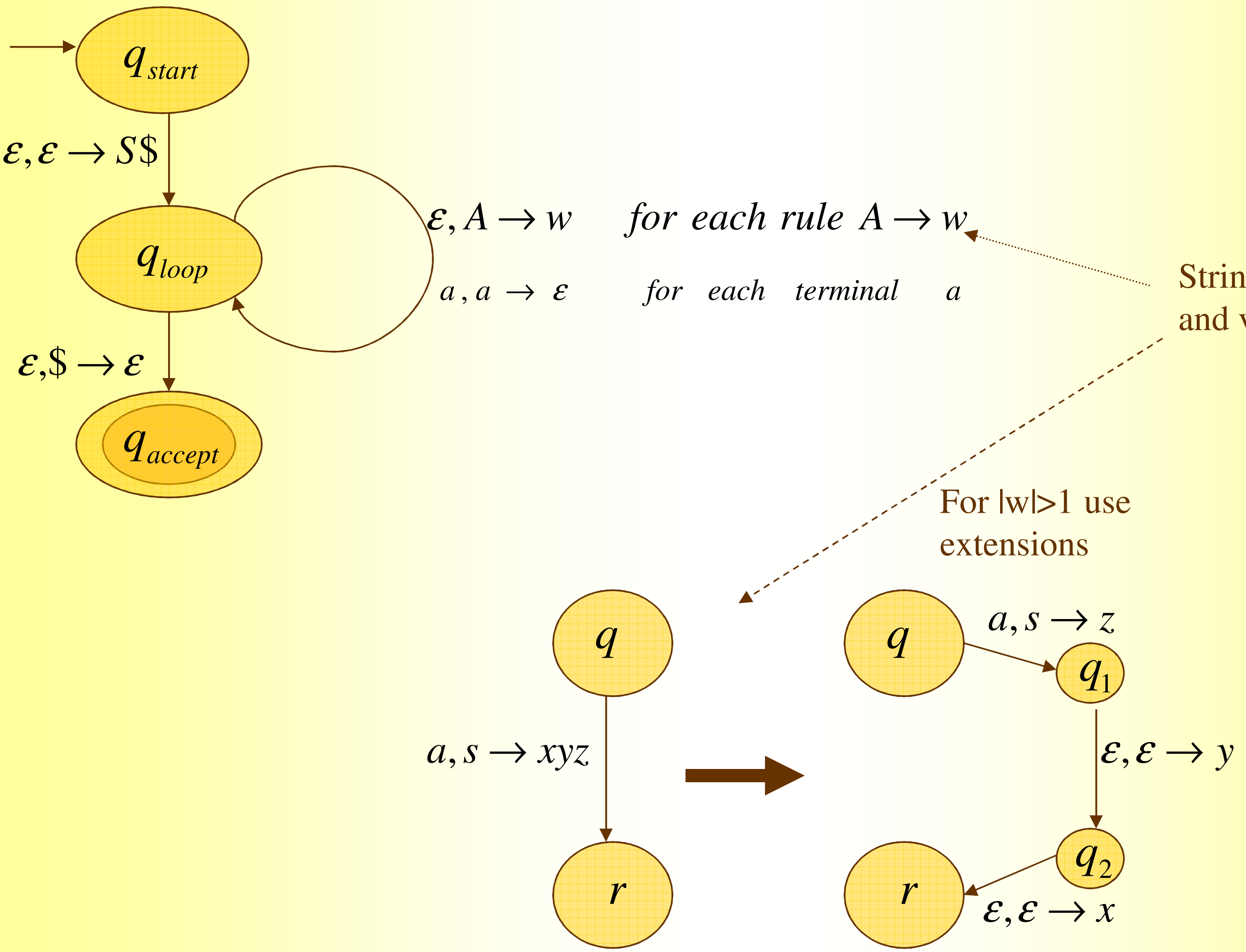
$$A11S$$

$$0A1S \Rightarrow 0A11S$$

$$00A11S$$

$$0011S$$

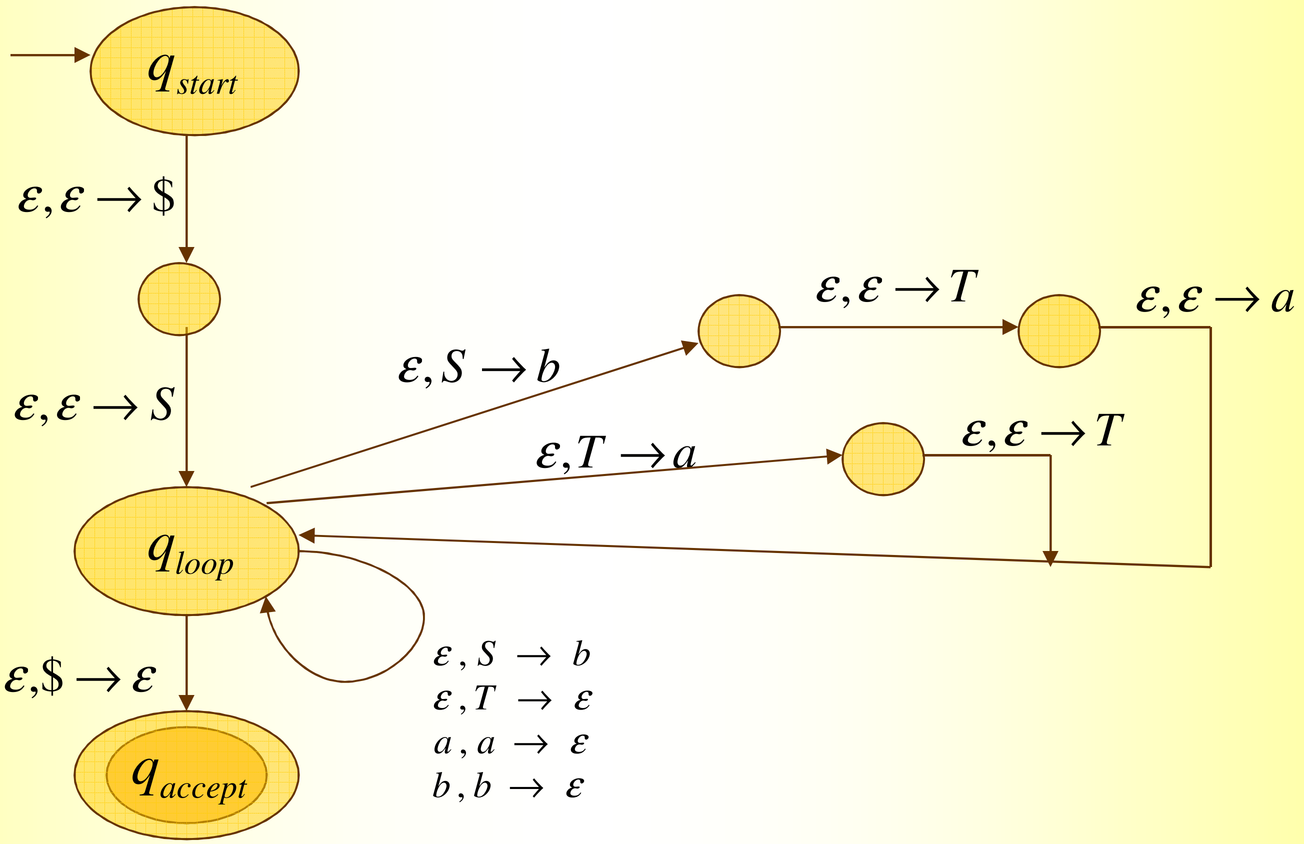
Construction of PDA P



Example

$S \to aTb \mid b$

$T \to Ta \mid \varepsilon$



Equivalence with Context-free Grammars

• We are working on the proof of the following result

Theorem. A language is context-free if and only if some pushdown automaton recognizes it.

• We have proved

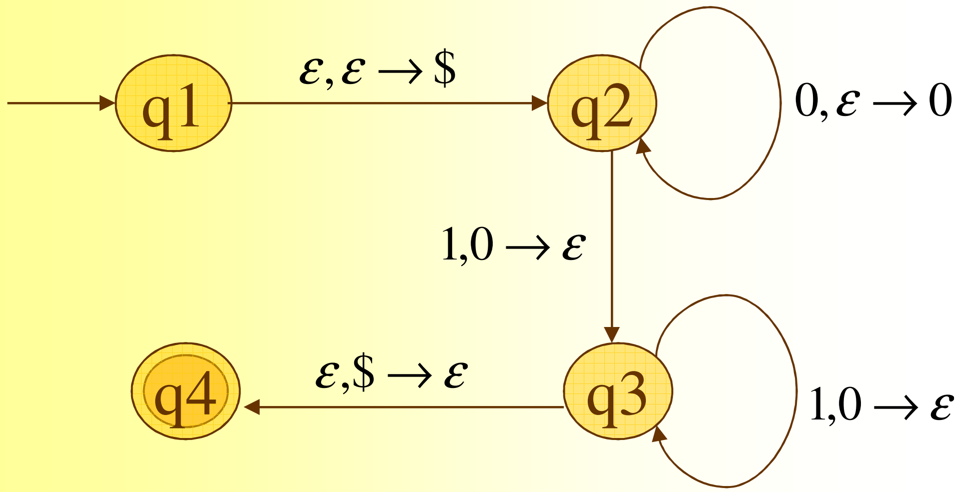
Lemma 1. If a language is context-free, then some pushdown automaton recognizes it.

- We have shown how to convert a given CFG *G* into an equivalent PDA *P*.
- Now we will consider the other direction

Lemma 2. If a pushdown automaton recognizes some language, then it is context-free.

- We have a PDA *P*, and want to create a CFG *G* that generates all strings that *P* accepts.
- That is *G* should generate a string if that string causes the PDA to go from its start state to an accept state (*takes* P from start state to an accept state).

Example



- string 000111 takes P from start state to a finite state;
- string 00011 does not.

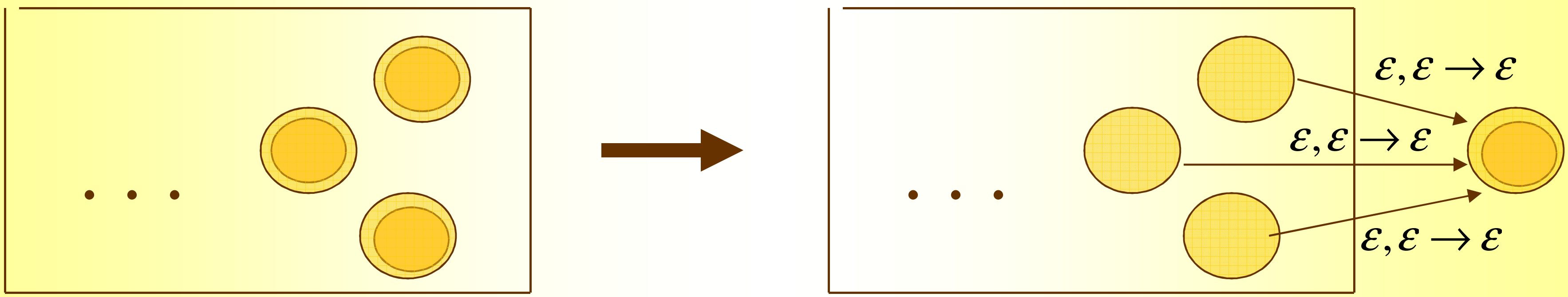
Design a Grammar

- Let P be an arbitrary PDA.
- For each pair of states p and q in P the grammar will contain a variable A_{pq}
- This variable will generate all strings that can take P from state p with empty stack to q with an empty stack
- Clearly, all those strings can also take P from p to q, regardless of the stack contents at p, leaving the stack at q in the same condition as it was at p.

Design a Grammar (cont.)

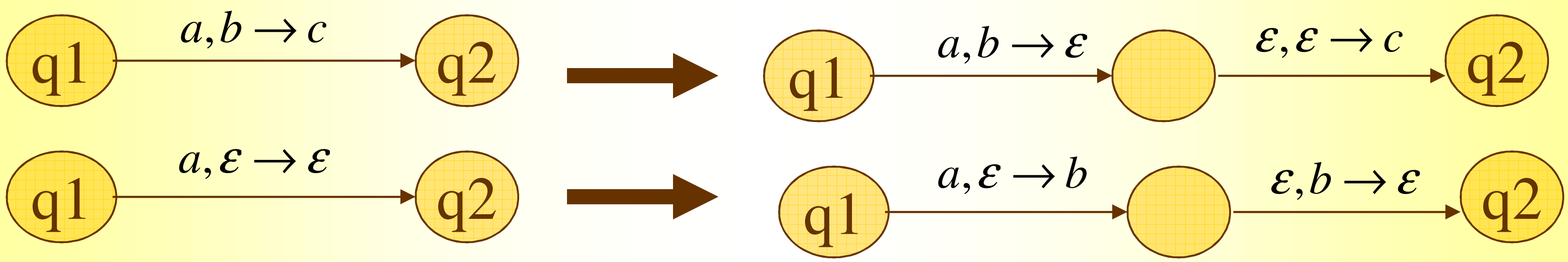
First we modify *P* slightly to give it the following three features.

1. It has a single accept state, q_{accept} .



2. It empties its stack before accepting.

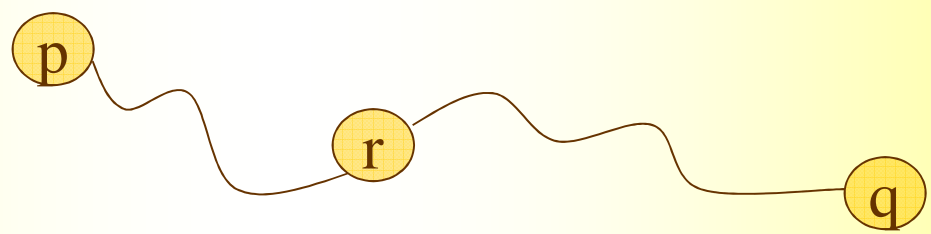
3. Each transition either pushes a symbol onto stack (a *push* move) or pops one off the stack (a *pop* move), but does not do both at the same time.



Design a Grammar (ideas)

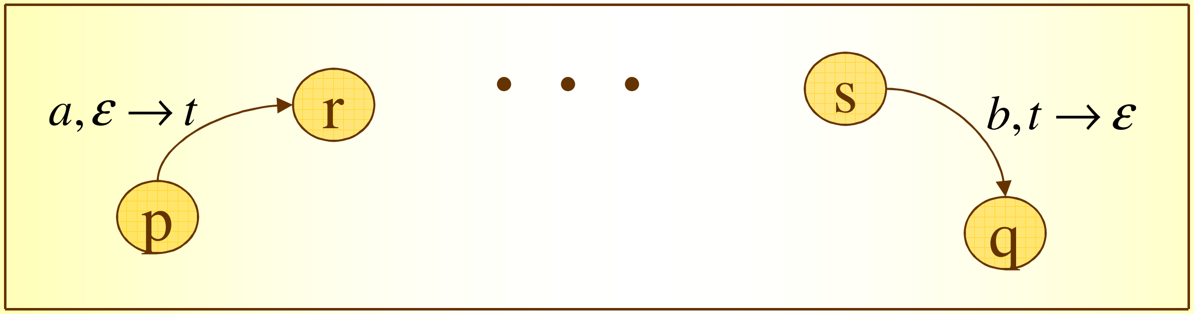
- For any string x that take P from p to q, starting and ending with an empty stack, P's first move on x must be a push; the last move on x must be a pop. (Why?)
- **If** the symbol pushed at the beginning is the symbol popped at the end, the stack is empty only at the beginning and the end of P's computation on x.
  - We simulate this by the rule $A_{pq} \to aA_{rs}b$, where a is the input symbol read at the first move, b is the symbol read at the last move, r is the state following p, and s the state preceding q.

- **Else**, the initially pushed symbol must get popped at some point before the end of x, and thus the stack becomes empty at this point.
  - We simulate this by the rule $A_{pq} \to A_{pr}A_{rq}$, r is the state when the stack becomes empty.



Formal Design

- Let $P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{accept}\})$.
- We construct G as follows.
  - The variables are $\{A_{pq} : p, q \in Q\}$
  - The start variable is $A_{q_0 q_{accept}}$
  - Rules:
    - For each *p,q,r,s* from *Q*,, $t \in \Gamma,\ a, b \in \Sigma_\varepsilon$ if we have



put the rule $A_{pq} \to aA_{rs}b$ in *G*.

- For each *p,q,r* from *Q*,, **put the rule** $A_{pq} \to A_{pr}A_{rq}$ in *G*.
- For each *p* from *Q*,, **put the rule** $A_{pp} \to \varepsilon$ in *G*.