CHAPTER 3

The Church-Turing Thesis

Contents

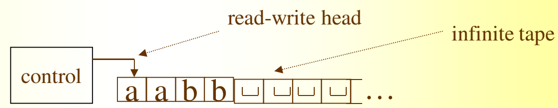
Turing Machines (intro)

- So far in our development of theory of computation we have presented several models for computing devices
- **Finite automata** are good models for devices that have a small amount of memory.
- **Pushdown automata** are good models for devices that have an unlimited memory that is usable only in the last in, first out manner of a stack.
- We have shown that some very simple tasks are beyond the capabilities of these models.
- Now we will consider a much more powerful model, first proposed by Alan Turing in 1936, called the **Turing Machine (TM)**.
- It is similar to a finite automaton but with an *unlimited* and *unrestricted memory*.
- TM is much more accurate model of a general purpose computer.
- It can do everything that a real computer can do.
- But a TM also cannot solve certain problems.
- There are problems that are beyond the theoretical limits of computation.

Turing Machines (informal)

- The Turing machine model uses an *infinite tape* as its unlimited memory.
- It has a *head* that can read and write symbols and move around on the tape.
- Initially the tape contains only the input string and is blank everywhere else.
- If TM needs to store information, it may write this info on the tape.
- To read the information that it has written, TM can move its head back over it.
- The machine continues computing until it produces an output.
- The output *accept* and *reject* are obtained by entering designated accepting and rejecting states.
- If it does not enter an accepting or a rejecting state, it will go on forever, never halting.

Schematic of a Turing Machine:



- The differences between finite automata and Turing machines.
  - A TM can both write on the tape and read from it.
  - The read-write head can move both to the left and to the right.
  - The tape is infinite.
  - The special states for rejecting and accepting take immediate effect.

Example

- We want to design a TM *M1* which accepts if its input is a member of *B*

$$B = \{w\#w : w \in \{0,1\}^*\}.$$

Informal description how the TM works on input string *s*.

- Scan the input to be sure that it contains a single # symbol. If not, *reject.*
- Zig-zag across the tape to corresponding positions on either side of the # symbol to check on whether these positions contain the same symbol. If they do not, *reject.* Cross off symbols as they are checked to keep track of which symbols correspond.
- When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #. If any symbols remain, *reject*; otherwise, *accept.*

M1 on input 011000#011000

$011000\#011000\_\_ \ldots$

$x11000\#011000\_\_ \ldots$

$x11000\#x11000\_\_ \ldots$

$x11000\#x11000\_\_ \ldots$

$xx1000\#x11000\_\_ \ldots$

$xxxxxx\#xxxxxx\_\_ \ldots$ accept

Formal Definition of TMs

- A ***Turing machine*** (TM) is specified by a 7-tuple

$(Q, \Sigma, \Gamma \delta, q_0, q_{accept}, q_{reject})$, where

Q	is a finite set of states,
Σ	is a finite input alphabet not containing $\sqcup$,
Γ	is a finite tape alphabet, such that $\sqcup \in \Gamma,\ \Sigma \subset \Gamma$,
$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$	is the transition function,
$q_0 \in Q$	is the start state,
$q_{accept} \in Q$	is the accept state, and
$q_{reject} \in Q$	is the reject state, where $q_{accept} \neq q_{reject}$.

- The heart of the definition of a TM is the transition function because it tells us how the machine gets from one step to the next.

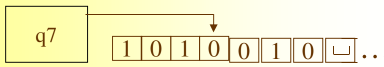
$\delta(q, a) = (r, b, L)$ means that when the machine is in a certain state q and head is over a tape square containing a symbol a, the machine writes the symbol b replacing the a, and goes to state r. The third component is either L or R and indicates whether the head moves to the left or right after writing.

How does a TM compute?

- Initially TM receives its input $w = w_1 w_2 ... w_n \in \Sigma^*$ on the leftmost n squares of the tape, and the rest of the tape is blank.
- The head starts on the leftmost square of the tape.
- Note that Σ does not contain the blank symbol, so the first blank symbol appearing on the tape marks the end of the input.
- Once TM starts, the computation proceeds according to the rules described by the transition function.
- If TM ever tries to move its head to the left off the left-hand end of the tape, the head stays in the same place for that move, even though the transition function indicates L.
- The computation continues until it enters either accept state or reject state at which point it halts.
- If neither occurs, TM goes on forever.

Acceptance of Strings and the Language of TM

A **configuration** C of the TM.



For a state q and two strings u and v over the tape alphabet Γ we write uqv for the configuration where the current state is q, the current tape contents is uv, and the current head location is the first symbol of v.

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Let $a,b,c \in \Gamma, \;\; u,v \in \Gamma^*, \;\; q_i, q_j \in Q.$

We say that configuration

$$ua\,q_i\,bv \;\; yields \;\; \begin{cases} u\,q_j\,acv & if \;\; \delta(q_i,b)=(q_j,c,L) \\ u\,ac\,q_j\,v & if \;\; \delta(q_i,b)=(q_j,c,R) \end{cases}$$

Note that

$$q_i\,bv \;\; yields \;\; \begin{cases} q_j\,cv & if \;\; \delta(q_i,b)=(q_j,c,L) \\ c\,q_j\,v & if \;\; \delta(q_i,b)=(q_j,c,R) \end{cases}$$

and that $ua\,q_i$ *is equivalent to* $ua\,q_i␣$ and we can handle this as before.

Acceptance of Strings and the Language of TM (cont.)

- The ***start configuration*** of TM on input w is q_0w.
- In an ***accepting configuration*** the state is q_{accept}.
- In an ***rejecting configuration*** the state is q_{reject}.

} **Halting configurations**

- A Turing machine TM ***accepts input*** w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists where
  - C_1 is the start configuration of TM on input w,
  - each C_i yields C_{i+1} and
  - C_k is an accepting configuration.
- If L is a set of strings that TM accepts, we say that L is the ***language of* TM** and write *L=L(TM).*
- We say TM ***recognizes*** *L* or TM ***accepts*** *L.*
- A language is ***Turing-recognizable*** if some TM recognizes it.

- For a TM three outcomes are possible on an input: it may *accept*, *reject* or *loop.*
- ***Deciders*** are TMs that always make a decision to accept or reject the input.
- A language is ***Turing-decidable*** or simply ***decidable*** if it is accepted by a decider.

Example 1.

• A TM *M2* which decides the language $A = \{0^{2^n} : n \geq 0\}$.

Higher-level description.

M2="On input string *w:*

1. Sweep left to right across the tape, crossing off every other *0*.
2. If in stage 1 the tape contained a single *0*, *accept*.
3. If in stage 1 the tape contained more than a single *0* and the number of *0*'s was odd, *reject*.
4. Return the head to the left-hand end of the tape.
5. Go to stage 1."

Formal description.

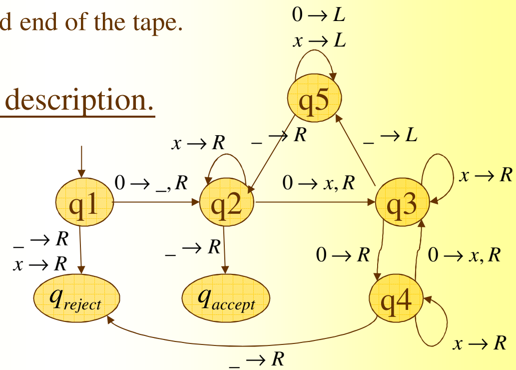
$M2 = (Q, \Sigma, \Gamma, \delta, q1, q_{accept}, q_{reject})$

$Q = \{q1, q2, q3, q4, q5, q_{accept}, q_{reject}\}$

$\Sigma = \{0\}$

$\Gamma = \{0, x, \sqcup\}$

Start state



Run *M2* on input 0000 and 000

Example 2.

• A TM *M1* which decides the language $B = \{w\#w : w \in \{0,1\}^*\}$.

For higher-level description see slide #4.

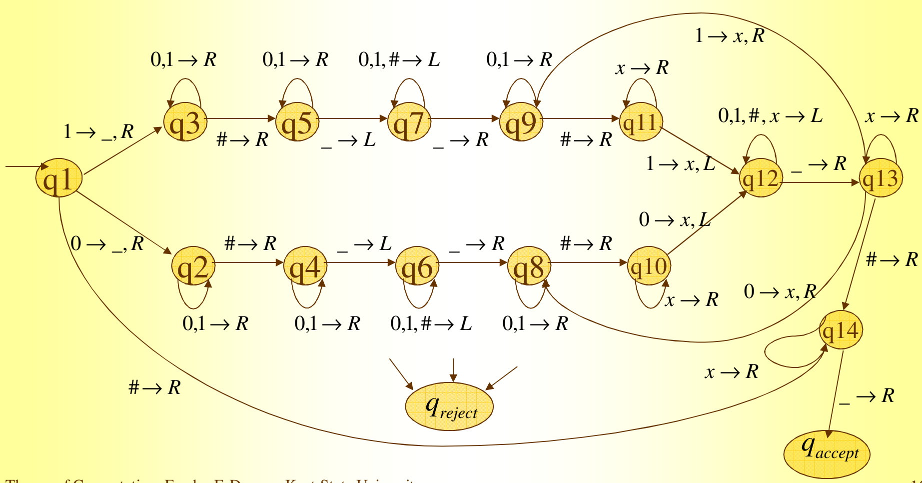
Formal description.

$M1 = (Q, \Sigma, \Gamma, \delta, q1, q_{accept}, q_{reject})$

$Q = \{q1, \ldots, q14, q_{accept}, q_{reject}\}$

$\Sigma = \{0, 1, \#\}$

$\Gamma = \{0, 1, \#, x, \sqcup\}$



• Example 3: A TM solving the *element uniqueness problem*. It is given a list of strings over {0,1} separated by #s and its job is to accept if all strings are different. The language is

$$E = \{\#x_1\#x_2\#...\#x_l : x_i \in \{0,1\}^* \ \forall i \in \{1,...,l\},\ x_i \neq x_j \ \ for\ i \neq j\}.$$

• TM M3 works by comparing x_1 with x_2 through x_l, then by comparing x_2 with x_3 through x_l, and so on.

Higher-level description: M3="On input *w:*

1. Place a mark on top of the leftmost tape symbol. If that symbol was blank, *accept.* If it was a #, continue with the next stage. Otherwise, *reject.*
2. Scan right to the next # and place a second mark on top of it. If no # is encountered before a blank symbol, only x_1 was present , so *accept.*
3. By zig-zagging, compare the two strings to the right of the marked #s. If they are equal, *reject*.
4. Move the rightmost of the two marks to the next # symbol to the right. If no # symbol is encountered before a blank symbol, move the leftmost mark to the next # to its right and the rightmost mark to the # after that. If no # is available for the rightmost mark, all the strings have been compared, so *accept.*
5. Go to stage 3."

(In the actual implementation, the machine has two different symbols, # and $\dot{\#}$, in its tape alphabet).