Reducibility

- Now we examine several additional unsolvable problems.
- In doing so we introduce the primary method for proving that problems are computationally unsolvable.
- It is called ***reducibility***.
- A reduction is a way of converting one problem into another problem in such a way that a solution to the second problem can be used to solve the first problem.
- When *A* is reducible to *B*, solving *A* cannot be harder that solving *B* because a solution to *B* gives a solution to *A*.
- In terms of computability theory, if *A* is reducible to *B* and *B* is decidable then *A* also is decidable.
- Equivalently, if *A* is undecidable and reducible to *B*, *B* is undecidable.
- This is the key to proving that various problems are undecidable.
- Our method for proving that a problem is undecidable will be: show that some other problem already known to be undecidable reduces to it.
- We will consider the following problems (~ as membership in languages):

$$HALT_{TM} = \{<M, w>: M \text{ is a TM that halts on input string } w\},$$
$$E_{TM} = \{<M>: M \text{ is a TM such that } L(M) = \emptyset\},$$
$$EQ_{TM} = \{<M_1, M_2>: M_1, M_2 \text{ are TMs with } L(M_1) = L(M_2)\}.$$

The Halting Problem for TMs.

- We have seen that the acceptance problem for TMs is undecidable

$$A_{TM} = \{<M, w>: M \text{ is a TM that accepts input string } w\}.$$

Theorem: A_{TM} is undecidable.

- Consider the problem determining whether a Turing machine halts (by accepting or rejecting) on a given input.

$$HALT_{TM} = \{<M, w>: M \text{ is a TM that halts on input string } w\}.$$

Theorem 1: $HALT_{TM}$ is undecidable.

- We use undecidability of A_{TM} to prove the undecidability of $HALT_{TM}$ by reducing A_{TM} to $HALT_{TM}$.
- Let assume that TM *R* decides $HALT_{TM}$. We construct a TM *S* to decide A_{TM}.

S = "on input <*M,w*>, where *M* is a TM and *w* is a string:

1. Run TM *R* on input <*M,w*>.
2. If *R* rejects, *reject.*
3. If *R* accepts, simulate *M* on *w* until it halts.
4. If *M* has accepted, *accept*; if *M* rejected, *reject*."

Clearly, if *R* decides $HALT_{TM}$, then *S* decides A_{TM}. Because A_{TM} is undecidable, $HALT_{TM}$ is undecidable too.

The Emptiness Problem for the Language of a TM.

$$E_{TM} = \{<M>: M \text{ is a TM such that } L(M) = \varnothing\}.$$

Theorem 2: E_{TM} is undecidable.

• Let assume that TM *R* decides E_{TM}. We construct a TM *S* to decide A_{TM}.

• Idea is for *S* to run *R* on input <*M*> and see whether it accepts. If it does then *L(M)* is empty and hence *M* does not accept *w*. But if *M* rejects …(???) we still do not know whether *M* accepts *w.*

• Instead of running *R* on <*M*> we run *R* on a modification of <*M*> *(*<*M1*>). The only string *M1* accepts is *w*, so its language is nonempty if and only if it accepts *w*.

S = "on input <*M,w*>*,* an encoding of a TM *M* and a string *w*:

1. Use the description of *M* and *w* to construct the following TM *M1.*
   M1 = "on input *x*:
   1. If $x \neq w$, *reject.*
   2. If *x* = *w*, run *M* on input *w* and *accept* if *M* does."
2. Run *R* on input <*M1*>.
3. If *R* accepts, *reject*; if *R* rejects, *accept*."

• The test whether *x* = *w* is obvious; scan the input and compare it character by character with *w* to determine whether they are the same.

• Note that *S* must be able to compute a description of *M1* from a description of *M* and *w*. It is able because it needs only add extra states to *M* that perform the *x* = *w* test.

• If *R* were a decider for E_{TM}, *S* would be a decider for A_{TM} which is impossible.

The Equivalence Problem for TMs.

$$EQ_{TM} = \{<M_1, M_2>: M_1, M_2 \text{ are TMs with } L(M_1) = L(M_2)\}.$$

Theorem 3: EQ_{TM} is undecidable.

• We could prove it by a reduction from A_{TM}, but we use this opportunity to give an example of an undecidability proof by reduction from E_{TM}.

• Let TM *R* decides EQ_{TM} and construct TM *S* to decide E_{TM} as follows.

S = "on input <*M*>*,* an encoding of a TM *M*:

1. Run *R* on input <*M,M1*>*,* where *M1* is a TM that rejects all inputs.
2. If *R* accepts, *accept*; if *R* rejects, *reject*."

• The E_{TM} problem is a special case of the EQ_{TM} problem wherein one of the machines is fixed to recognize the empty language.

• This idea makes giving the reduction easy.

• So, If *R* were a decider for EQ_{TM}, *S* would be a decider for E_{TM}, which is impossible.

• One can also show that EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

In the textbook, a simple problem called ***Post Correspondence Problem*** *is shown to be unsolvable by algorithms.*