# Time complexity

- Here we will consider ***elements of computational complexity theory*** – an investigation of the time (or other resources) required for solving computational problems.
- We introduce a way of measuring the time used to solve a problem. Then we will classify problems according to the amount of time required.
- We will see that certain decidable problems require enormous amounts of time and how to determine when you are faced with such a problem.
- Let consider an example of a TM *M1* which decides the language $A = \{0^k 1^k : k \geq 0\}$.

*M1* = "on input *w*:
1. Scan across the tape and *reject* if a *0* is found to the right of a *1*.
2. Repeat the following if both *0*s and *1*s remain on the tape.
3. Scan across the tape, crossing off a single *0* and a single *1*.
4. If *0*s still remain after all the *1*s have been crossed off, or if *1*s remain after all the *0*s have been crossed off, *reject*. Otherwise, if neither *0*s nor *1*s remain on the tape, *accept*."

- How much time does a single type TM need to decide *A*?
- We count the number of steps that algorithm uses on a particular input as a function of the length of the string representing the input.
- We consider ***worst case analysis,*** i.e., the longest running time of all inputs of a particular length.



# Asymptotic notation : big-*O* and small-*o*

- ***Def. 1:*** Let *M* be a TM that halts on all inputs. The ***running time*** or ***time complexity*** of *M* is the function *f: N→N*, where *f(n)* is the maximum number of steps that *M* uses on any input of length *n*. We say *M* runs in time *f(n)* and *M* is an *f(n)* time Turing machine.
- ***Def. 2:*** Let *f* and *g* be two functions $f, g : N \rightarrow R^+$. Say that *f(n)=O(g(n))* if positive integers *c* and *n'* exist so that for every $n \geq n',\ f(n) \leq c\, g(n)$ . We say that *g(n)* is an ***upper bound*** for *f(n)* (or ***asymptotic upper bound***).
- Intuitively, this means that *f* is less than or equal to *g* for sufficient large *n* if we disregard differences up to a constant factor. *O* represents that constant; constant is hidden under *O*.

$$\textit{If}\ \ f(n) = 5n^3 + 2n^2 + 22n + 6,\ \textit{then}\ f(n) = O(n^3)\ \textit{or}\ f(n) = O(n^4),\ \textit{but}\ f(n) \neq O(n^2).$$
$$\textit{If}\ \ f(n) = 3n\log_2 n + 5n\log_2\log_2 n + 2,\ \textit{then}\ f(n) = O(n\log n).$$
$$\textit{If}\ \ f(n) = O(n^2) + O(n),\ \textit{then}\ f(n) = O(n^2).$$

- Other examples of run-time: $2^{O(n)},\ O(1),\ n^{O(1)}\ (= 2^{O(\log n)})$. Bounds of the form $n^c$ for *c >0* are called ***polynomial bounds.*** Bounds of the form $2^{(n^\delta)}\ (\delta > 0)$ are called ***exponential bounds.***
- ***Def. 3:*** Let *f* and *g* be two functions $f, g : N \rightarrow R^+$. Say that *f(n)=o(g(n))* if for any real *c>0*, a number *n'* exists so that for every $n \geq n',\ f(n) < c\, g(n)$ i.e., $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$
- Examples: $\sqrt{n} = o(n)$, $n = o(n\log\log n)$, $n\log\log n = o(n\log n)$, $n\log n = o(n^2)$, $n^2 = o(n^3)$, $\textit{but}\ f(n) \neq o(f(n))$.

# Analyzing Algorithms

Let's analyze the algorithm we gave for the language $A = \{0^k 1^k : k \geq 0\}$.

*M1* = "on input *w*:

1. Scan across the tape and *reject* if a *0* is found to the right of a *1*.
2. Repeat the following if both *0*s and *1*s remain on the tape.
3. Scan across the tape, crossing off a single *0* and a single *1*.
4. If *0*s still remain after all the *1*s have been crossed off, or if *1*s remain after all the *0*s have been crossed off, *reject*. Otherwise, if neither *0*s nor *1*s remain on the tape, *accept*."

- ***Stage 1:*** verifies that input is of form 0*1* in 2n steps. Hence *O(n)* steps.
- ***Stages 2,3:*** each scan uses *O(n)* steps, at most *n/2* scans. Hence $O(n^2)$ steps.
- ***Stage 4:*** at most *O(n)* steps.
- Hence, the total time of *M1* on input of length *n* is $O(n) + O(n^2) + O(n) = O(n^2)$.

• ***Def. 4:*** Let $t : N \to N$ be a function. Define the time complexity class, *TIME(t(n)),* to be

$$TIME(t(n)) = \{L : L \textit{ is a language decided by an } O(t(n)) \textit{ time Turing machine}\}.$$

• We have $A \in TIME(n^2)$. Is there a machine that decides *A* asymptotically more quickly?



# $A = \{0^k 1^k : k \geq 0\} \in TIME(n \log n)$

*M2* = "on input *w*:

1. Scan across the tape and *reject* if a *0* is found to the right of a *1*.
2. Repeat the following if some *0*s and some *1*s remain on the tape.
3. Scan across the tape, checking whether the total number of 0s and 1s remaining is even or odd. If odd, *reject.*
4. Scan again across the tape, crossing off every other *0* starting with the first *0*, and then crossing off every other *1* starting with the first *1*.
5. If no *0*s and no *1*s remain on the tape, *accept*. Otherwise, *reject*."

- ***Why does M2 decide A?***
  - on every scan performed in stage 4, the total number of *0*s (of *1*s) remaining is cut in half and any remainder is discarded.
  - in stage 3 we check whether the parities of # of *0*s and # of *1*s are the same.
- **Running Time:**
  - ***All Stages*** take *O(n)* steps.
  - ***Stages 1 and 5*** are executed once.
  - ***Stages 2,3,4*** are executed at most (1+log n) time.
- Hence, the total time of *M2* on input of length *n* is
  $O(n) + (1 + \log n)(O(n) + O(n) + O(n)) + O(n) = O(n \log n)$.
- So, $A \in TIME(n \log n)$. This result cannot be further improved on a single tape TM.

# Linear time two-tape Turing machine for *A*.

*M3* = "on input *w*:

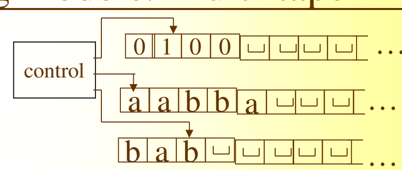
1. Scan across the tape and *reject* if a *0* is found to the right of a *1*.
2. Scan across the *0*s on tape 1 until the first *1*. At the same time, copy the *0*s onto tape 2.
3. Scan across the *1*s on tape 1 until the end of the input. For each *1* read on tape 1, cross off a *0* on tape 2. If all *0*s are crossed off before all the *1*s are read, *reject*.
4. If all the *0*s have now been crossed off, *accept*. If any *0*s remain, *reject*."

- Clearly, this is a decider for *A*. Running time is clearly *O(n)*.
- ***Summary:***
  - We presented a single tape TM *M2* that decides *A* in *O(n* log *n)* time.
  - We mentioned (w/o proof) that no single tape TM can do it more quickly.
  - Then we presented a two-tape TM *M3* that decides *A* in linear time.
  - Hence, the complexity of *A* depends on the model of computation selected.
- This shows an important difference between *complexity theory* and *computability theory.*
- In *computability theory*, The Church-Turing thesis implies that all reasonable models of computation are equivalent, i.e., they decide the same class of languages. In *complexity theory*, the choice of model affects the time complexity of languages.



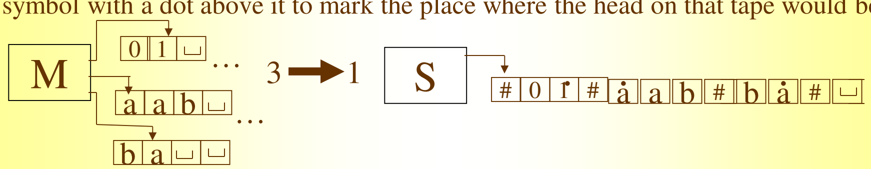
# Complexity relations among models: Multi-tape TM

$$\delta: Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k,$$
$$\delta(q, a_1, \ldots, a_k) = (r, b_1, \ldots, b_k, L, R, \ldots, L)$$




***Theorem 1:*** Let *t(n)* be a function, where $t(n) \geq n$. Then every *t(n)* time multi-tape TM has an equivalent $O(t^2(n))$ time single-tape TM.

- We have seen before how to convert a multi-tape TM *M* to an equivalent single tape TM *S*, that simulates it.
- Let *M* be a *k*-tape TM that runs in *t(n)* time. We will show that simulating each step of the multi-tape TM uses at most *O(t(n))* steps of the single-tape TM. Hence the total time used is $O(t^2(n))$.
- *S* simulates the effect of *k* tapes by storing their information on its single tape.
- It uses new symbol # as a delimiter to separate the contents of the different tapes.
- *S* must also keep track of the locations of the heads. It does so by writing a tape symbol with a dot above it to mark the place where the head on that tape would be.



## Multi-tape TM vs. Single-tape TM

S="On input $w = w_1 w_2 ... w_n$ :

1. First *S* puts its tape into the format that represents all *k* tapes of *M*. The formatted tape contains $\# \dot{w}_1 w_2 ... w_n \# \dot{\sqcup} \# \dot{\sqcup} \# ... \#$
2. To simulate a single move, *S* scans its tape from the first #, which marks the left-hand end, to the *(k+1)*st #, which marks the right-hand end, in order to determine the symbols under the virtual heads. Then *S* makes a second pass to update the tapes according to the way that *M*'s transition function dictates.
3. If at any point *S* moves one of the virtual heads to the right onto a #, this action signifies that *M* has moved the corresponding head onto the previously unread blank portion of that tape. So *S* writes a blank symbol on this tape cell and shifts the tape contents, from this sell until the rightmost #, one unit to the right. Then it continues the simulation as before.

**Running Time:**

- ***Stage 1*** takes *O(n)* steps and is executed once.
- ***Stages 2,3:*** *S* simulates each of the *t(n)* steps of *M*, using *O(t(n))* steps.

- The length of the active portion of *S*'s tape determines how long *S* takes to scan it.
- A scan of the active portion of *S*'s tape uses *O(t(n))* steps. (Why???)

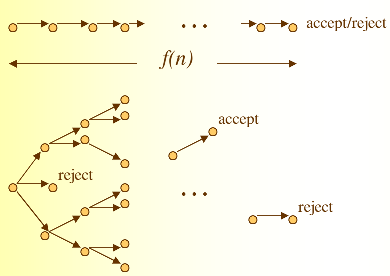
- Hence, the total time of *S* on input of length *n* is

$$O(n) + t(n) \times O(t(n)) = O(t^2(n)).$$



## Complexity relations among models: Non-deterministic TM

- A non-deterministic TM is a decider if all its computation branches halt on all inputs.
- ***Def 5:*** Let *N* be a non-deterministic TM that is a decider. The ***running time*** of *N* is the function *f: N→N*, where *f(n)* is the maximum number of steps that *N* uses on any branch of its computation on any input of length *n*.




Deterministic

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

Non-deterministic

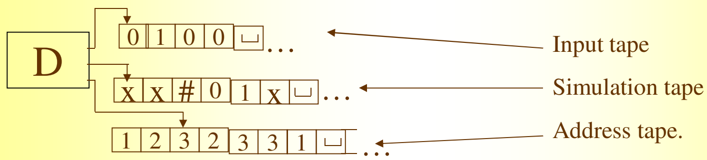
$$\delta : Q \times \Gamma \rightarrow \mathbf{P}(Q \times \Gamma \times \{L, R\})$$

***Theorem 2:*** Let *t(n)* be a function, where $t(n) \geq n$. Then every *t(n)* time non-deterministic TM has an equivalent $2^{O(t(n))}$ time deterministic TM.

- We have seen before that any non-deterministic TM *N* has an equivalent deterministic TM *D*, that simulates it.

# Non-deterministic TMs vs. ordinary TMs

- The simulating deterministic TM $D$ has three tapes.
  - Tape 1 always contains the input string and is never altered.
  - Tape 2 maintains a copy of $N$'s tape on some branch of its non-deterministic computation.
  - Tape 3 keeps track of $D$'s location in $N$'s non-deterministic computation tree.




- Every node in the tree can have at most $b$ children, where $b$ is the size of the largest set of possible choices given by $N$'s transition function.
- Tape 3 contains a string over $\Sigma_b = \{1,2,\ldots,b\}^*$. Each symbol in the string tells us which choice to make next when simulating a step in one branch in $N$'s non-deterministic computation. This gives the address of a node in the tree.
- On an input of length $n$, every branch of $N$'s non-deterministic computation tree has a length of at most $t(n)$. Hence the total number of leaves in the tree is at most $b^{t(n)}$.
- The total number of nodes in the tree is less that twice the maximum number of leaves, i.e. is bounded by $O(t(n))$. Hence the running time of $D$ is $O(t(n)b^{t(n)}) = 2^{O(t(n))}$.
- $D$ has three tapes. Converting it to a single tape TM $S$ at most squares the running time.
- So, the running time of $S$ is $(2^{O(t(n))})^2 = 2^{O(2t(n))} = 2^{O(t(n))}$.