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Computing a median point of a simple rectilinear polygon

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Abstract

Let P be a simple rectilinear polygon with n vertices, endowed with rectilinear metric. Let us assign to points $x_1, \ldots, x_k \in P$ positive weights $w_1, \ldots, w_k$. The median problem consists in the computing the point minimizing the total weighted distances to the given points. We present an $O(n + k \log n)$ algorithm for solving this median problem. If all weighted points are vertices of a polygon P, then the running time becomes $O(n + k)$.

Key words: Computational geometry; Median problem; Rectilinear polygon; Rectilinear distance

1. Introduction

Let P be a simple rectilinear polygon in the plane $\mathbb{R}^2$ (i.e., a simple polygon having all edges axis-parallel) with n edges. A rectilinear path π is a polygonal chain consisting of axis-parallel segments lying inside P. The length of the path π in the L_1-metric is defined as the sum of the length of the segments π consists of. In other words, the length of a rectilinear path in the L_1-metric is equal to its Euclidean length. For any two points u and v in P, the *rectilinear distance* between u and v, denoted by $d(u,v)$, is defined as the length of the minimum length rectilinear path connecting u and v. The *interval* $I(u,v)$ between two points u, v consists of all points z between u and v, that is,

$$I(u,v) = \{z \in P\colon d(u,v) = d(u,z) + d(z,v)\}.$$

Now assume that points $z_1, \ldots, z_k$ of a polygon P have positive weights $w_1, \ldots, w_k$, respectively. The total weighted distance of a point z in P is given by

$$F(z) = \sum_{i=1}^{k} w_i d(z, z_i).$$

A point z of P minimizing this expression is a *median* of P respect to the weight function w, and the set of all medians is the *median set* $Med_w(P)$. By the *median problem* we will mean the problem of finding the median of a polygon P. If all the weighted points are vertices of P then we obtain the *vertex-restricted median problem.*

In this paper we present an $O(n + k)$ time algorithm for solving the vertex-restricted median problem and an $O(n + k \log n)$ time algorithm for the general problem. The median problem in different classes of metric spaces has numerous applications, for example in facility location [13] and as a consensus procedure in group choice and cluster analysis [3,4]. On the other hand, linear time algorithms for finding medians are known only for trees and space $\mathbb{R}^d$ with

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rectilinear distance [9,13]. These algorithms are mainly based on a well-known "majority rule" from the group choice [3,4]. Later similar algorithms have been developed for median graphs and discrete median spaces; see [1,12]. Unfortunately, although these algorithms use the local search and majority rule and avoid the direct computation of distances, their complexity and used storage depends on the time of preprocessing of a median graph. In this case the preprocessing is nothing else than the isometric embedding of a median graph into a hypercube and require more than linear time. We pay attention to these results since our algorithm is also based on a majority rule and on the fact that any simple rectilinear polygon endowed with rectilinear distance is a median space.

2. Median properties of a simple rectilinear polygon

Recall that the metric space (X,d) is a *median space* if every triple of points $u,v,w \in X$ admits a unique "median" point z, such that

$$d(u,v) = d(u,z) + d(z,v),$$
$$d(u,w) = d(u,z) + d(z,w),$$
$$d(v,w) = d(v,z) + d(z,w).$$

A basic example of a median space is the tree equipped with the standard graph-metric. For classical results on median spaces and their particular instances (median semilattices, median algebras, median graphs and median normed spaces) the reader is referred to [2,11,14,15]. The subset M of a metric space (X,d) is *convex* if for any points $u,v \in M$ and $z \in X$ the equality $d(u,z) + d(z,v) = d(u,v)$ implies that $z \in M$. Recall also that the subset M is called *gated* [7], provided every point $x \in X$ admits a *gate* in M, i.e. a point $x_M \in M$ such that $x_M \in I(x,y)$ for all $y \in M$. Any gated subset of a metric space is convex [7]. The converse holds for median spaces:

Lemma 1. *Any convex compact subset of a median space is gated.*

About the proof of this and more general results consult for example [14]. The following result is a well-known property of metric spaces; see [14].

Lemma 2. *If x,y,z,v are points of a metric space (X,d) such that $v \in I(x,y)$ and $z \in I(x,v)$ then $v \in I(z,y)$.*

An axis-parallel segment is called a *cut segment* of a polygon P if it connects two edges of P and lies entirely inside P. Note that any edge or any cut segment of a polygon P is a convex subset of P.

Lemma 3. *A simple rectilinear polygon P equipped with rectilinear distance is a median space.*

Proof. We proceed by induction on the number of vertices of a polygon P. The statement is evident for $n = 4$: for any points u,v,w of a rectangle, the median of these points is a point $p = (x_p, y_p)$, where x_p is the median of x-coordinates and y_p is the median of y-coordinates of u, v and w.

Now assume that $n > 4$ and let c be the cut segment of P with one end-point at the concave vertex of P. Then c cuts P into two rectilinear polygons P' and P'' with at most $n - 1$ vertices each. By induction assumption P' and P'' are median spaces. The segment c is a closed convex subset of each of these subpolygons. By Lemma 1 c is a gated set in P' and P''. Note also that P' and P'' are closed convex subsets of P. Let u, v and w be arbitrary points of P. If all these points belong to the same subpolygon P' or P'' then by induction hypothesis this triple has a unique median. So assume for example that $u \in P'$ and $v,w \in P''$. Denote by u_c the gate of u in c. Choose any point $p \in P''$. Any shortest path from u to p intersects the cut c in some point u'. Since $u_c \in I(u,u')$ and $u' \in I(u,p)$, then we immediately obtain that $u_c \in I(u,p)$, i.e. u_c is a gate for point u in the subpolygon P''. Let z be the median of points u_c, v and w. Since $u_c \in I(u,v) \cap I(u,w)$, z is a median of points u,v,w too. Now assume that z^+ is another median of this triple. Since P'' is convex and $v, w \in P''$ then $z^+ \in P''$. Further, since $u_c \in I(u, z^+)$ then by Lemma 2 we conclude that $z^+ \in I(u_c, v) \cap I(u_c, w)$. Therefore the triple u_c, v, w admits in P'' two median points z and z^+, in contradiction with our induction assumption. □

For any subpolygon P_0 of a polygon P put $w(P_0) = \sum_{x_i \in P_0} w_i$. Then for any cut c and subpolygons P' and P'' defined by this cut we have

$$w(P') + w(P'') = w(P) + w(c).$$

Lemma 4 (Majority rule). *If $w(P') > w(P'')$ then $Med_w(P) \subset P'$, otherwise if $w(P') = w(P'')$ then $Med_w(P) \cap c \neq \emptyset$.*

Proof. First assume that $w(P') > w(P'')$, however the subpolygon P'' contains a median point z. Let z_c be the gate for z in P'. Since for any point $z_i \in P'$ we have $d(z, z_i) = d(z, z_c) + d(z_c, z_i)$, we have

$$\begin{aligned} &F(z) - F(z_c) \\ &= \sum_{i=1}^{k} w_i (d(z, z_i) - d(z_c, z_i)) \\ &= \sum_{z_i \in P'} w_i d(z, z_c) \\ &\quad - \sum_{z_i \in P-P'} w_i (d(z_c, z_i) - d(z, z_i)) \\ &\geqslant (w(P') - w(P'')) d(z_c, z) > 0, \end{aligned}$$

in contradiction with our assumption. Now suppose that $w(P') = w(P'')$ and choose any median point z. Assume for example that $z \in P''$ and let z_c be the gate of z in the subpolygon P'. As in the preceding case we obtain that

$$\begin{aligned} &F(z) - F(z_c) \\ &\geqslant d(z_c, z)(w(P') - w(P'')) = 0, \end{aligned}$$

i.e. $z_c \in c$ is a median point too. □

3. Computing the median of *P*

In this section an algorithm for solving the median problem in a simple rectilinear polygon

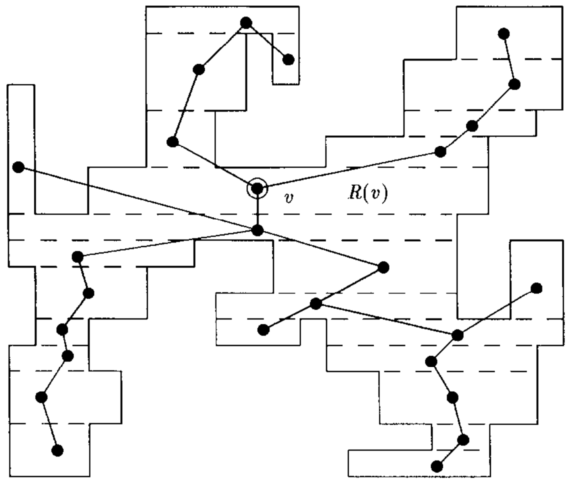

Fig. 1. The tree associated to the partition of P.

P is given. The algorithm is based on a Chazelle algorithm for computing all vertex-edge visible pairs [5] and on a Goldman algorithm for finding the median of a tree [9]. By first algorithm we obtain a decomposition of a polygon P into rectangles, using only horizontal cuts. The dual graph of this decomposition is a tree $T(P)$ [5]: vertices of this tree are the rectangles and two vertices are adjacent in $T(P)$ iff the corresponding rectangles in the decomposition are bounded by a common cut; see Fig. 1. Assign to each vertex of $T(P)$ the weight of their rectangle. In order to compute these weights first we have to compute which rectangles of the decomposition of P contain each of the weighted points. Using one of the optimal point location methods [8,10] this can be done in time $O(k \log n)$ with a structure that uses $O(n)$ storage. Observe that the induced subdivision (decomposition) is monotone, and, hence, the point location structure can be built in linear time. Therefore the weights of vertices of a tree $T(P)$ can be defined in total time $O(k \log n + n)$. For the vertex-restricted median problem this assignment takes $O(n + k)$ time.

Now using the Goldman algorithm [9] compute the median v of a tree $T(P)$. According to majority rule for trees the vertex v has the following property. Let u be some adjacent to v

vertex of $T(P)$. Denote by T_v and T_u the subtrees obtained by deleting the edge (v,u). Then $w(T_v) \geqslant w(T_u)$ [9]. Now we retain to our polygon P. Let $R(v)$ and $R(u)$ are rectangles of the subdivision of P which correspond to vertices v and u. These rectangles have a common part c^* of a some horizontal cut c (it is possible that $c^* = c$). Note that c^* is a cut of polygon P. Let P_v and P_u be the subpolygons defined by c^* and let $R(v) \subset P_v$ and $R(u) \subset P_u$. All rectangles that correspond to vertices from T_v lie in the subpolygon P_v. Hence $w(P_v) \geqslant w(P_u)$. Since such an inequality holds for all rectangles adjacent to $R(v)$ and the rectangle $R(v)$ coincides with the intersection of the subpolygons of the type P_v, then from Lemma 4 we conclude that $Med_w(P) \cap R(v) \neq \emptyset$.

Finally, we concentrate on a finding of the median point from $R(v)$. Assume that $R(v)$ is bounded by the horizontal cuts c' and c'' of our decomposition. Let P' and P'' be the subpolygons of P defined by c' and c'' and disjoint with rectangle $R(v)$. In other words, $P = P' \cup R(v) \cup P''$. Now, for any weighted point z_i we will find its gate g_i in $R(v)$. Note that $g_i \in c'$ if $z_i \in P'$, $g_i \in c''$ if $z_i \in P''$ and $g_i = z_i$ if $z_i \in R(v)$. Further, define the maximal histograms H' and H'' inside P' and P'' with c' and c'' as their bases, respectively; see Fig. 2. (A histogram is a rectilinear polygon that has one distinguished edge, its base, whose length is equal to the sum of the lengths of the other edges that are parallel to it; see for example [6].) The vertical edges of these histograms divide the polygons P' and P'' into subpolygons, called pockets. Consider for example the pockets from P'. Note that all points from the same pocket have one and the same gate. This is a point of a cut c' which has the same x-coordinate with the cut that separates the pocket and histogram H'. Hence it is enough to find the location of weighted points into the pockets. This can be done using the partition of P' and P'' into rectangles by vertical vertex-edge visible pairs. For any point $z_i \in P' \cup P''$ assign the weight w_i to its gate g_i, the weights of points z_i from $R(v)$ remain unchanged. As a result we obtain a median problem in the rectangle $R(v)$. Note that any solution of this problem is a solution of an initial median problem. To see this, observe that for any two points $z', z'' \in R(v)$ we have

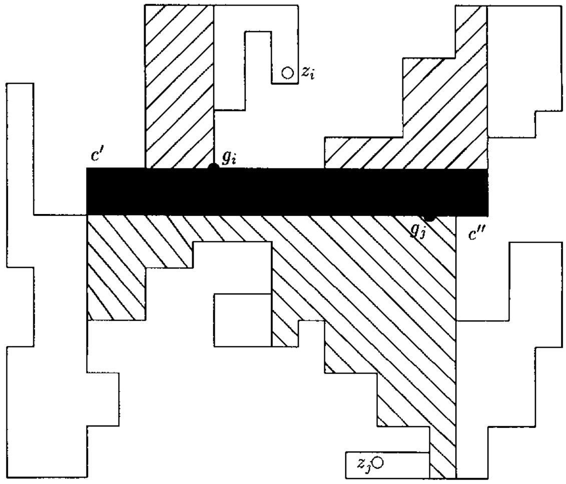

Fig. 2. Histograms H and H'.

$$F(z') - F(z'') = \sum_{i=1}^{k} w_i(d(z',z_i) - d(z'',z_i)) = \sum_{i=1}^{k} w_i(d(z',g_i) - d(z'',g_i)).$$

The new median problem on $R(v)$ may be solved by decomposing into two one-dimensional median problems and applying to them the Goldman algorithm [9].

Summarizing the results of this section, we have the following theorem.

Theorem 5. *The median problem in the simple rectilinear polygon can be solved in time* $O(k \log n + n)$. *The vertex-restricted median problem can be solved in* $O(n + k)$ *time.*

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