NUMERICAL RESULTS FOR SINGULARLY PERTURBED LINEAR AND QUASILINEAR DIFFERENTIAL EQUATIONS USING A COARSE GRID.

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In this paper, we apply classical upwinding schemes |4| and exponential fitting |8|and|10|, denoted by UP and EF respectively, to a number of equations from the literature |1|, |6|, |11|and |12|. The aim is to solve the equations without the use of mesh refinement or other techniques requiring greater computational effort with increasing stiffness.

Another requirement, particularly desirable if the method is to be included in a general package for the solution of differential equations, is that it should also be efficient for non-stiff problems.

We consider problems of the form:

$$\varepsilon u''(x)+a_1(x)u'(x)+a_0(x)u(x)=f(x) \qquad (1)$$

$$u(a)=A,\ u(1)=B \qquad \text{where } a=-1 \text{ or } 0.$$

If $a_0(x)<0$, these satisfy the maximum principle and hence are stable. Both schemes considered reflect this by yielding linear equations of negative definite type, for all h (the mesh size) and $\varepsilon>0$, and hence are uniformly stable. The method for linear equations in |2| is not of negative type for turning point problems and a suitable adaptation is proposed in |1|. Convergence properties for particular cases of (1) are given in |4|, |5|, |8|, |9| and |10|.

As a test for (uniform) convergence we compute

$$M_h^h=\max_i |u_i^h - u_{2i}^{h/2}|,$$

where u_i^h denotes the approximation to $u(x_i)$ obtained from the difference scheme. If $M^h<Ch^p$, and the method is consistent, then by Theorem 2 in |9|, the method is convergent of order p (uniformly in ε if C is independent of ε).

The first class of problems considered is that for which $a_0<0$. All problems of this type from |1| and |11|, except that in Fig. 4, were solved successfully using either UP or EF. However, as is well known, upwinding gives inferior results for $\varepsilon=h$, since the boundary layers are expanded. Examples are given in Figs 1 and 2 and Tables 1 and 2. The results in Table 1 suggest that UP is O(h) convergent uniformly in ε, whereas EF is $O(h^2)$ convergent for $h<<\varepsilon$ and O(h) convergent uniformly in ε.

In general, we cannot expect good results if a_0 is not strictly negative; examples are shown in Fig. 3 and in Table 3. The latter problem is ill-conditioned and varies greatly with small perturbations of the right hand side (cf. |1|). That negativity of $a_0(x)$ does not always guarantee a good rate of convergence is shown in

Boundary and Interior Layers - Computational and Asymptotic Methods.
Proceedings of the BAIL I Conference, (J.J.H. Miller ed.),
Boole Press, Dublin, (1980).

Fig. 4. Finally, in Figs 5 & 6, we show cases where the maximum principle is violated, but EF and UP still work. The results for the example in Fig. 5 are in good agreement with those of Pearson for $\varepsilon=10^{-9}$.

These results for linear problems lead us to expect that the methods will yield good results for quasilinear problems if the linearised equation satisfies the negativity requirement on a_0. In our experiments we used neither a Davidenko process nor a particularly accurate initial guess. Fig. 7 shows one solution to a problem from |6|. This does not have a unique solution, but either solution can be recovered by an appropriate choice of initial guess. Example 8 from |6| was also solved; here, taking $h=1/16$, the errors were less than 10^{-7} for EF and approximately 7×10^{-3} for a centered difference scheme. A similar problem, given in |12|p.354 was solved with error of order 10^{-7} by both methods.

A particularly interesting problem is $\varepsilon u''+uu'-u=0$, with $u(0)=A$ and $u(1)=B$. This is discussed in |3| pp.29-38 and |7|. The solutions exhibit boundary or interior layers depending on the values of A and B, cf. Fig. 8. In the linearised equation, using Newton linearisation, $a_0=w'-1$, where w is the approximation to u computed at the previous iteration. If $w'>1$, a_0 is positive and the Newton process does not converge rapidly or it can even converge to a spurious solution. If the linearisation is changed to one which preserves the negativity of a_0, the correct solution is obtained, cf. Fig. 9.

The methods are also applicable to problems with mixed boundary conditions, an example of which is shown in Fig. 10.

CONCLUSIONS

In general, fitting is more accurate than upwinding, particularly for $h\approx\varepsilon$. It seems, however that upwinding is slightly less sensitive to instabilities in the differential equation. For nonlinear problems, convergence depends primarily on the linearisation chosen. With a correct choice of linearisation, both methods yield good approximations using a coarse grid and a small number of iterations (usually less than 20), in most cases fitting giving more accurate results for the same number of iterations.

In computing the results given below, a DEC-20/60 was used. The programs were written in single precision FORTRAN. The graphs were plotted by joining the values at mesh points with straight lines.

Table 1 (Values of M^h) |11|, p.147

$\varepsilon u''+x^3u'-u=0$, $u(-1)=1$, $u(1)=2$

h	$\varepsilon=10^{-2}$ UP	$\varepsilon=10^{-2}$ EF	$\varepsilon=10^{-6}$ UP&EF
1/32	.0226	.0244	.0262
1/64	.0129	.0105	.0156
1/128	.0071	.0033	.0083
1/256	.0037	.0009	.0043

Table 2 (Values of u_i^h) |1|, p.35 , h=1/20

$10^{-6}u''-xu'-5u=0$, $u(-1)=1$, $u(1)=-\frac{1}{2}$

The solution is $u(x)=e^{-(x+1)/\varepsilon}-\frac{1}{2}e^{(x-1)/\varepsilon}+O(\varepsilon)$

x	Abrahamsson Scheme I	Abrahamsson Scheme II	EF	UP
-0.9	4×10^{-10}	9×10^{-2}	⋮	5×10^{-11}
-0.8	2×10^{-19}	1×10^{-2}	⋮	4×10^{-21}
-0.7	1×10^{-28}	1×10^{-3}	⋮	5×10^{-31}
⋮	$\lvert u_i^h\rvert<10^{-38}$	$\lvert u_i^h\rvert<10^{-5}$	$\lvert u_i^h\rvert<10^{-38}$	$\lvert u_i^h\rvert<10^{-38}$
0.7	-5×10^{-29}	-7×10^{-4}	⋮	-2×10^{-31}
0.8	-9×10^{-20}	-5×10^{-3}	⋮	-2×10^{-21}
0.9	-2×10^{-10}	-5×10^{-2}	⋮	-3×10^{-11}

Table 3 (Values of u_i^h) |1|, p.37

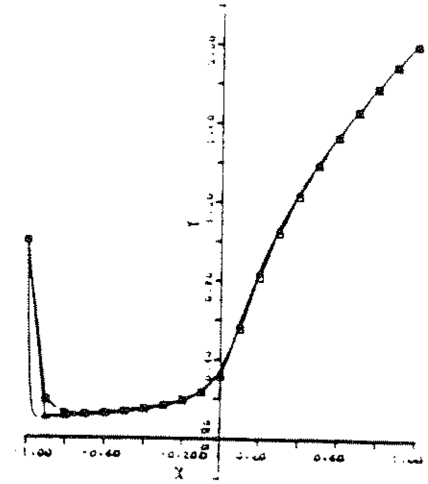
$10^{-6}u''-xu'+5.5u=1/3$, $u(-1)=1$, $u(1)=\frac{1}{2}$

Let A = 1/16.5, then the solution is:

$u(x)=A+(1-A)e^{-(x+1)/\varepsilon}+(\frac{1}{2}-A)e^{(x-1)/\varepsilon}+O(\varepsilon)$

x	ABRAHAMSSON'S SCHEMES I h=.05	I h=.01	II h=.01	UP h=.05	UP h=.01	EF h=.05	EF h=.01
-0.8	.19	1×10^3	.07	.06061	.09	.06061	-.002
-0.6	.09	2×10^2	.0616	"	.066	"	.49
-0.4	.063	2×10^1	.0607	"	.061	"	.06
-0.2	.06065	.52	.06061	"	.06061	"	.0606
0.0	.06061	.06061	.06061	"	.06061	"	.06061
0.2	.06062	.65	.06061	"	.06063	"	.0606
0.4	.061	3×10^1	.06058	"	.063	"	.06
0.6	.069	3×10^2	.0601	"	.09	"	.044
0.8	.10	1×10^3	.057	"	.22	"	-.03

Fig. 1

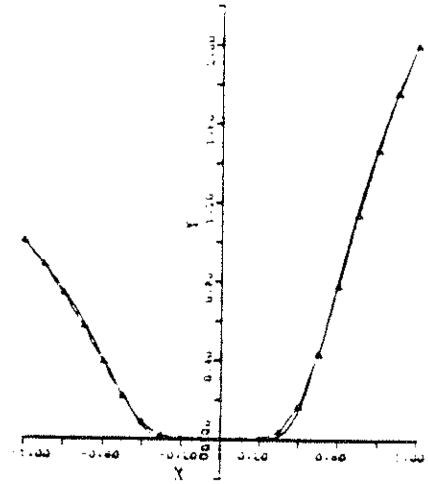


$10^{-2}u''+|x|u'-\frac{1}{2}u=0$, $u(1)=1, u(1)=2$

Δ EF, □ UP, - Exact

$h=1/20$

Fig. 2

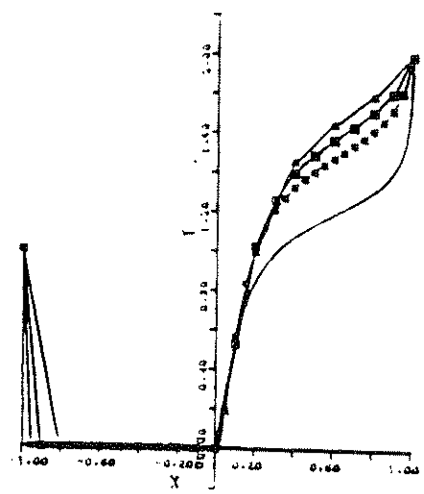


$10^{-6}u''+x^3u'-u=0$, $u(-1)=1, u(1)=2$

Δ EF, - Exact

$h=1/20$

Fig. 3



$10^{-6}u''+\sin^2\pi x u'+(x-1)u=0$

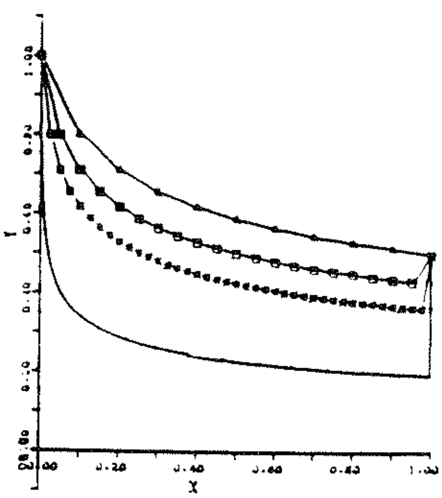
$u(-1)=1$, $u(1)=2$

Δ $h=1/10$, □ $h=1/20$

* $h=1/40$, - $h=1/500$

both EF and UP

Fig. 4



$10^{-6}u''-xu'-\frac{1}{2}u=0$

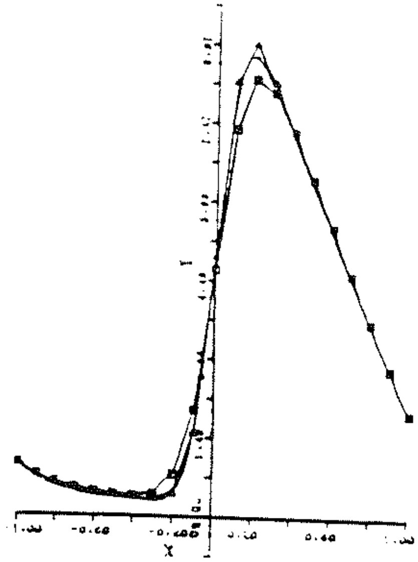
$u(0)=1$, $u(1)=\frac{1}{2}$

Δ $h=1/10$, □ $h=1/20$

* $h=1/40$, - $h=1/500$

both EF and UP

Fig. 5



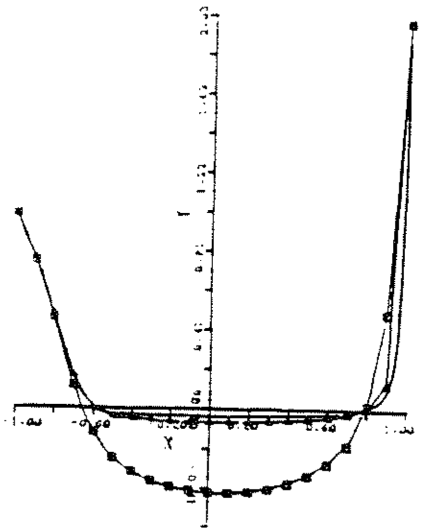
$10^{-2}u''+x\cos(x)u'+(x+x^3)u=0$

$u(-1)=1,\ u(1)=2$

△ EF, □ UP, - Exact

$h=1/20$

Fig. 6



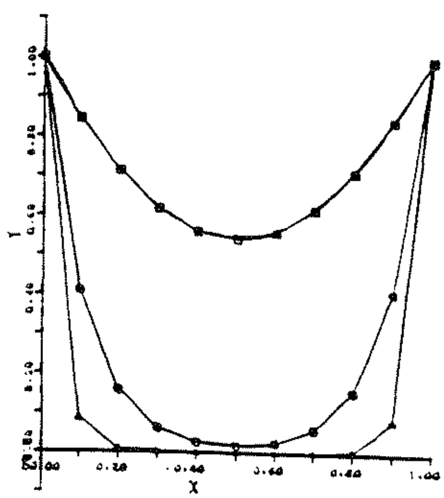
$10^{-2}u''+(\frac{1}{2}-x^2)u'+xu=0$

$u(-1)=1,\ u(1)=2$

△ EF, □ UP, - Exact

$h=1/20$

Fig. 7



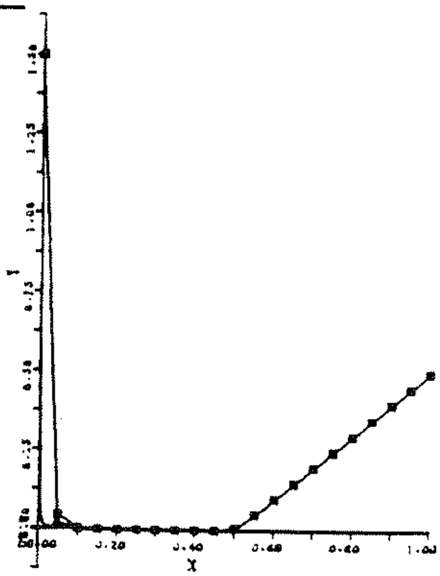
$\varepsilon u''-u+u^3=0,\ u(0)=1, u(1)=1$

□ $\varepsilon=10^{-1}$ EF; - $\varepsilon=10^{-1}$ Exact

o $\varepsilon=10^{-2}$ EF; △ $\varepsilon=10^{-3}$ EF

$h=1/10$

Fig. 8

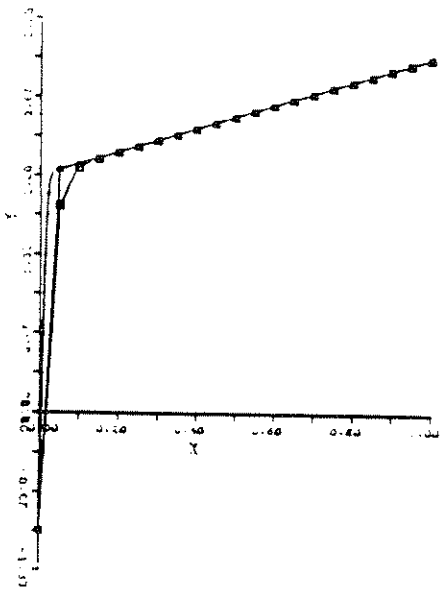


$10^{-4}u''+uu'-u=0,\ u(0)=1.5, u(1)=.5$

△ EF, □ UP, - Exact

$h=1/20$

Fig. 9



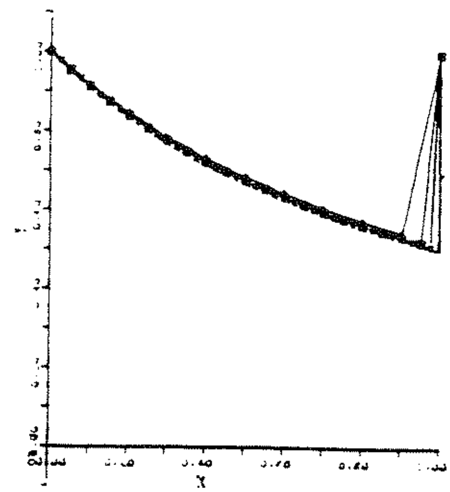
$10^{-2}u''+uu'-u=0$

$u(0)=-1,\ u(1)=3$

△ EF, □ UP, - Exact

$h=1/20$

Fig. 10



$10^{-6}u''-u'-u^2=0$

$u(0)+u'(0)=0,\ u(1)=1$

△ h=1/10, □ h=1/20

○ h=1/40, - h=1/500

both EF and UP

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