SUFFICIENT CONDITIONS FOR THE UNIFORM CONVERGENCE OF DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED TURNING AND NON-TURNING POINT PROBLEMS.

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We consider a singularly perturbed boundary value problem on [d,e] of the form :

$$\epsilon\, u''(x) + a(x)\, u'(x) - b(x)\, u(x) = f(x) \tag{1}$$

$$u(d) = A \quad , \quad u(e) = B. \tag{2}$$

where $a(x)$ is of one sign , or has zeros of finite order. We consider here at most a single zero of $a(x)$ in the interval. However by arguments similar to those of Berger, Han, Kellogg the results may be extended to more than one zero. The problem (1),(2) may have one or more boundary or interior layers depending on values of $a(x)$ and $b(x)$. For sufficiently smooth $a(x)$ we exclude only the case $a(x) \equiv 0$ for which uniformly convergent schemes are proposed in Doolan, Miller, Schilders [4]. In this paper we shall not propose any new schemes for problem (1),(2) but instead derive a set of sufficient conditions for uniform convergence and mention some of the schemes proposed in the literature which satisfy them. Throughout this paper C denotes a generic constant independent of h and ϵ.

Turning Point Case

If $a(x)$ has a single zero this is known as a turning point problem. Without loss of generality we shall take the interval as [-1,1] and the zero of $a(x)$ at zero. In [6] we considered the "constant coefficient" case (i.e. $a(x) = xa$ where a and $b(x)$ are constant) of an attractive stagnant (turning) point

$$a(0)=0, \;\; a'(0)>0, \;\; b(x) \geq 0 \,,\, b(0) > 0 \,, \tag{3}$$
$$\lambda = xb(x)/a(x)\,|_{x=0} > 0 \,, \;\; \lambda \textit{ not an integer.}$$

In this case the solution has an internal layer but no boundary layers. We showed that some schemes including Il'in-Allen-Southwell fitting, Samarskii's scheme, Hemker's schemes and upwinding were uniformly convergent of order min(λ,4/5). We now generalise this result to the following Theorem giving Sufficient Conditions for uniform convergence for the general (non-constant) coefficient case.

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Theorem 1.: *A difference scheme which has the following form*

$$L_\epsilon^h u_i^h \equiv \epsilon\sigma_i D_+ D_- u_i^h + \tau_i\, a(x_i) D_0 u_i^h - \beta_i\, b(x_i) u_i^h = f_i^h \tag{4}$$

for $1 \leq i \leq N$, N even

$$u_0^h = A \qquad , \qquad u_N^h = B \tag{5}$$

is uniformly convergent for the problem (1),(2),(3) if

$$\sigma_i \geq |\tau_i\, a(x_i)|\, \rho/2\ , \quad \tau_i \geq \tau > 0\ , \beta_i \geq 0\ , \beta_0 > 0 \tag{6}$$

$$\text{where } \rho = h/\epsilon$$

$$|\ \sigma_i - 1\ | \leq C\rho(\ |a(x_i)| + h\ ) \tag{7}$$

$$|a(x_i)|\ \ |\ \tau_i - 1\ | \leq Ch \tag{8}$$

$$|\ \beta_i - 1\ | \leq Ch \tag{9}$$

$$|\ f_i^h - f(x_i)\ | \leq Ch \tag{10}$$

and the order of convergence is at least $min(1,\lambda)$

Proof: The proof of this requires a better Classical (Truncation) Error Estimate than in [6]. This is obtained by more careful use of the bound

$$|\ u_\epsilon^{(k)}(x)\ |\ \ \leq\ C(\ |x| + \sqrt{\epsilon}\ )^{[\lambda - k]}$$

for the derivatives. We then prove a Non-Classical error estimate which uses only an $O(\epsilon)$ asymptotic expansion for u(x). These are then combined to give the result.

If the scheme satisfies (6) it is uniformly stable. Conditions (8) - (10) imply that the difference scheme must be a reasonable approximation to the equation (1),(2), whereas (7) is a stronger condition on the fitting factor σ_i. In particular it requires

$$|\ \epsilon\sigma_0 - \epsilon\ | \leq Ch^2.$$

The restriction of the order of convergence to λ for $\lambda < 1$ is intrinsic to the problem and arises from the reduced solution which contains terms of the form $|x|^\lambda$. The restriction to non-integral λ arises in the derivation of the asymptotic expansion, and is due to the linear dependence of the two solutions of the boundary layer equations in this case. The order of convergence shown here is an improvement on the order shown in [6] and is optimum for conditions as weak as (6) - (10). Numerical results show that these rates of convergence are obtained in practice. We remark that , since (4) has three free parameters σ , τ and β any three point difference scheme can be written in this form.

Among additional schemes which satisfy these conditions are Complete Exponential Fitting [3], and Abrahamsson's scheme (2.5) [1]. Schemes which do not satisfy these conditions and are not uniformly convergent include centered differences, the Abrahamsson-Keller-Kreiss box scheme, Abrahamssson's scheme (2.7) [1], and the generalisation of the El-Mistikawy and Werle scheme (II.12.7) [4].

Turning point problems are also considered in Emel'ianov [5], Abrahamsson [1] and

Kellogg [8]. In Berger, Han and Kellogg the comparison function technique is used to give a proof of uniform convergence for a modified El-Mistikawy and Werle scheme. This proof includes the case where λ is an integer and the repulsive stagnant point case $a'(0) < 0$.

Non-Turning Point Case

The sufficient conditions for the non-turning point problem

$$a(x) \geq \underline{a} > 0, \qquad b(x) \geq 0 \tag{11}$$

are more restrictive. We shall take the interval as [0,1]. The solution will then have a boundary lay at $x = 0$. Uniformly convergent schemes for this problem have been considered previously by Carroll and Miller [3], Kellogg and Tsan [8], and Il'in. Miller [11] also deriv sufficient conditions using the double mesh technique. We state sufficient conditions for uniform convergence in this case in the following Theorem.

Theorem 2.: *If a difference scheme of the form*

$$L_\epsilon^h u_i^h \equiv \epsilon\sigma_i D_+ D_- u_i^h + \tau_i a(x_i) D_+ u_i^h - \beta_i b(x_i) u_i^h = f_i^h \tag{12}$$

$$u_0^h = A \quad , \quad u_N^h = B \tag{13}$$

satisfies the conditions

$$\sigma_i \geq 0\,, \quad \tau_i \geq \tau > 0\,, \beta_i \geq 0 \tag{14}$$

$$|\ \tau_i - 1\ | \leq Ch \tag{15}$$

$$|\ \beta_i - 1\ | \leq Ch \tag{16}$$

$$|\ f_i^h - f(x_i)\ | \leq Ch \tag{17}$$

and for any given $\bar{x}$ *s.t.* $0 < \bar{x} \leq 1$

$$|\ \sigma_i - \sigma_B(\rho a_0)\ | \leq C\rho p(\rho) x_i e^{-a(\varsigma)\rho} \quad 0 \leq x_i \leq \bar{x} \tag{18a}$$

where $0 \leq \varsigma \leq x_i + Ch$, $p(\rho)$ *is a polynomial in* r.EQ and $\sigma_B(z) = \dfrac{z}{e^z - 1}$.*EQ (18b) and ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ | sigma sub i ~ -~ 1 ~ | ~ <=~ C rho ~ ~ ~ ~ x bar <=x sub i <=1.*

then (12),(13) is uniformly convergent of order h to the solution of (1),(2),(11) for all $h \leq h_0 < \bar{x}$.

Proof: The proof uses a generalisation of the comparison function techniques of Kellogg and Tsan [8].

The following is a useful Corollary for schemes of a form similar to Complete Fitting.

If (14) - (17) and (18b) hold then (18a) may be replaced by :

$$\left| \frac{\sigma_i}{\tau_i} - \sigma_B(\rho a_o) \right| \leq C\rho p(\rho) x_i\, e^{-a(\varsigma)\rho} \tag{19}$$

Condition (14) is sufficient for uniform stability. Condition (18) implies that the difference scheme must model the boundary layer function accurately in the region where it is varying rapidly, whereas outside this region it is sufficient that the scheme be a good approximation to the differential equation. Among schemes which satisfy these conditions are Il'in-Allen-Southwell fitting, the constant fitting (II.10.2) of Doolan, Miller and Schilders [4], Complete fitting, and the El-Mistikawy and Werle scheme for $b(x) \equiv 0$. One should note that better error estimates may be derived by considering individual schemes as in [9] since schemes which satisfy conditions (14) - (18) need only be classically $O(h)$ as is the case with constant fitting methods.

We remark that all schemes which have been proposed in the literature for this problem attempt to satisfy (18a) in the whole interval.

If $a(x) \leq \underline{a} < 0$, $b(x) \geq 0$ the boundary layer is at $x = 1$. The sufficient conditions are of essentially the same form as (14) - (18).

Multiple Turning Points

If $a(x)$ and $b(x)$ have multiple simultaneous zeros of order p and q respectively at $x = 0$ and no other zeros in the interval, Lundqvist [10] has shown that the general reduced solution is bounded for all sufficiently smooth f(x) only if :

$$p \text{ is odd}, \quad q = 0 \text{ and } \lambda = x^p\, b(x)/a(x)\,|_{x=0} > 0 \tag{20}$$

If, in addition, we require

$$b(x) \geq 0 \quad \text{and} \quad b(0) > 0 \ , \tag{21}$$

in order that the maximum principle holds, and that $a(x)$, $b(x)$ be sufficiently smooth it can be shown that the problem is a regular perturbation problem and thus the derivatives of the solution u_ϵ are bounded on the whole interval. Thus if a scheme is consistent and uniformly stable it is uniformly convergent. We recall that for a scheme of the form (4),(5) condition (6) is sufficient for uniform stability.

If $q = 0$ and (21) holds then the solution will have one or more boundary layers. If p is odd and $\lambda < 0$ then it will have boundary layers at each end of the interval. If p is even then it will have a layer at the left (right) end of the interval if λ is positive (negative) and the solution of the reduced equation will satisfy the other boundary condition.

Proofs of the above results will appear in [7].

Conclusions:

Finally, as a consequence of the above results, it is sufficient for uniform convergence that a scheme be uniformly stable, be fitted in the neighbourhood of each boundary and "upwinded" elsewhere. Thus, for example, a scheme which is Complete Exponential Fitting in the neighbourhood of each boundary and upwinding in the interior will be O(h) convergent. An advantage of such a scheme would be the elimination of evaluations of the fitting factor in the interior region.

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