CRITERIA FOR UNIFORM CONVERGENCE

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Abstract : We present results which characterise the behaviour of a linear non-selfadjoint singular perturbation problem. We give criteria for uniform convergence for a non-turning, simple turning point and one multiple turning point case and indicate the uniform methods for higher order cases. The consequences for quasi-linear problems are discussed.
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We consider the following singularly perturbed boundary value problem:

$$Lu(x) \equiv \epsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x) \quad l < x < r \tag{1a}$$

$$u(l) = A \ , \ u(r) = B \tag{1b}$$

where we assume

$$b(x) \geq 0 \tag{2}$$

in order that (1) satisfy a maximum principle and be uniformly stable. We consider a class of difference schemes characterized as follows:

$$\epsilon_i^{\pm} D_+ D_- u_i^h + a_i^h D_{\pm} u_i^h - b_i^h u_i^h = f_i^h \tag{3a}$$

$$u_{-N}^h = A \ , \ u_N^h = B \ . \tag{3b}$$

where

$$D_{\pm} u_i^h = \begin{cases} D_+ u_i^h & \text{if } a_i^h > 0 \\ D_- u_i^h & \text{if } a_i^h < 0 \ . \end{cases}$$

1. Non-Turning Point and Simple Turning Point Problems

In Farrell [4] [5] sufficient conditions were derived for a scheme of this form to be uniformly convergent for a non-turning point problem specified by

$$a(x) \geq \underline{a} > 0 \tag{4}$$

In this case, the problem exhibits an exponential boundary layer at $x = l$. The necessary conditions and sufficient conditions each specify that the scheme must model this well in the region where it dominates the behavior of the solution. (cf Farrell [4] [5]). This is what has been termed exponential fitting in the literature.

Theorem 1.1: *If a difference scheme has the form (3) where ϵ_i^+, a_i^h, b_i^h, f_i^h are bounded and satisfy the conditions*

$$\epsilon_i^+ \geq 0 \ , \ a_i^h \geq a^h > 0 \ , \ b_i^h \geq 0 \tag{I}$$

$$| \ a_i^h - a(x_i) \ | \leq Ch \tag{II}$$

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$$| \; b_i^h - b(x_i) \; | \leq Ch \qquad \text{(III)}$$

$$| \; f_i^h - f(x_i) \; | \leq Ch \qquad \text{(IV)}$$

and for any given $\bar{x}$ *such that* $0 < \bar{x} \leq 1$

$$| \sigma_i^+ - \sigma(a(0)\rho) | \leq C\rho p(\rho) x_i e^{-a(\varsigma)\rho} + Ch\,\sigma(a(0)\rho) \qquad 0 \leq x_i \leq \bar{x} \qquad \text{(Va)}$$

where $0 \leq \varsigma \leq x_i + Ch$, $p(\rho)$ *is a polynomial in* ρ

and $\sigma(z) = \dfrac{z}{e^z - 1}$

and $| \; \epsilon_i^+ - \epsilon \; | \leq Ch \qquad \bar{x} \leq x_i \leq 1 \, .$ (Vb)

then (S) is uniformly convergent of order h *to the solution of (1), for all* $h \leq h_0 < \bar{x}$.

These conditions are satisfied by many schemes including constant Il'in fitting [2], Il'in fitting [7] and complete exponential fitting.

For a simple turning point problem, with turning point at $x = t$,

$$a(t) = 0, \;\; a'(t) \neq 0 \qquad (5)$$

we can have either a repulsive turning point $a'(t) < 0$ or an attractive turning point $a'(t) > 0$. In either case, we require the additional condition $b(t) > 0$. In the former case, Berger, Kellogg and Han [1] have shown that a similar form of fitting is required to that in the non-turning point case. For an attractive turning point, the behavior is characterized by the parameter

$$\lambda = xa(x)/b(x)\,|_{x=t} \, . \qquad (6)$$

In this case, the problem exhibits an internal layer of corner or cusp type. This does not require exponential fitting but only a "stronger" version of upwinding, which propagates the correct boundary conditions. Thus we may replace (Va) and (Vb) by

$$| \epsilon_i^{\pm} - \epsilon | \leq Ch(\,| \, a(x_i) \, | + h\,) \qquad \text{(V)}$$

We remark that the restriction of order of convergence to λ for $\lambda < 1$ is intrinsic to the problem and arises from the solution of the reduced equation which contains a term of the form $| \, x - t \, |^{\lambda}$. This rate is attained in practice by schemes such as upwinding, Il'in's scheme, Abrahamsson's scheme and Samarski's scheme.

2. Repeated Simple Turning Points

If we consider a problem, in which $a(x)$ has more than one simple turning point, the exponential fitting is required in the vicinity of an exponential layer if it exists, but "strong upwinding" is sufficient elsewhere. To be precise if $a(l) < 0$ then fitting is required at $x = l$ and if $a(r) < 0$ then fitting is required at $x = r$. Also if $a(x)$ has an attractive turning point then the uniform rate of convergence is limited to min $(\lambda, 1)$ where λ is defined by (6).

3. Multiple Turning Points

We now consider the case when $a(x)$ has multiple (simultaneous) zeros. We say $a(x)$ has a zero of order p at t if

$$a^{(i)}(t) = 0, \;\; 0 \leq i \leq p-1 \;\text{ and }\; a^{p}(t) \neq 0 \, . \qquad (7)$$

To analyze these more general problems, we must first determine the conditions under which the solution of these problems is uniformly stable and also where and of what type are the boundary or interior layers. We say a turning point is a boundary turning point if $t = l$ or r. Uniform stability results are given in the following lemmas, for cases with even and odd order turning points respectively.

Lemma 3.1: *If u is the solution of (1), $a(x)$, $b(x)$ are continuous, $a(x)$ has one turning point at t, either at the boundary or of even order and*

$$b(t) > 0 \tag{8}$$

then u satisfies a uniform stability result of the form:

$$|u(x)|_\infty \leq \max(|A|, |B|) + C|f(x)|_\infty . \tag{9}$$

Theorem 3.2: *If u is a solution of (1), (2) and*

$$(x - t)\,a(x) + b(x) \geq \mu > 0 \qquad l \leq x \leq x_r \tag{10}$$

then u satisfies a uniform stability result of the form (9).

This is a quite general result. To rewrite it in a form more familiar from turning point problems we adopt a format similar to Lemma 3.1.

Corollary 3.3: *If u is a solution of (1), a and b are continuous, $a(x)$ has only a single turning point, t, of odd order and*

$$(i) \quad a(x)/(x-t)^p \geq \underline{a} > 0 \tag{11}$$

$$(ii) \quad b(t) > 0$$

then u satisfies a uniform stability result of the form (9).

We now examine the nature of the solution of (1) in greater detail. In particular we determine the existence and location of boundary layers. We first define a norm on $C^k\,[l, r]$ by

$$|g(x)|_k = \sum_{i=0}^{k} |\, g^{(i)}(x)\,|_\infty \tag{12}$$

By a generalisation of a lemma in [8] one can show that, if $a(l) < 0$ then there is no boundary layer at l and similarly if $a(r) > 0$ there is no boundary layer at r.

To characterise the behaviour of the solution in the neighbourhood of a turning point, we proceed as follows. At each turning point we must, by Lemma 3.1 and Corollary 3.3 have $b(t) > 0$ in order that the problem be stable. The nature of the reduced solution near a turning point of order p depends on the parameter:

$$\lambda = \lambda(t) = \frac{b(t)}{a^{(p)}(t)} \tag{13}$$

Note that, by the stability criteria, $b(t) > 0$ and hence $|\lambda| > 0$. We require that the turning points are discrete in the sense that there exists a neighborhood of each turning point not containing any other turning point.

We remark that for a problem with a number of turning points, t_i, $i = 1, k$, the result is true for all of them if we replace λ and $b(t)$ by $\max\lambda(t_i)$ and $\min b(t_i)$. Each of these exists and is finite.

In particular for a general problem we require $b(x) \geq \underline{b} > 0$ for stability and thus $b(t) \geq \underline{b} > 0$.

We are now in a position to state the major result on the behavior of the solution near a turning point. For convenience we assume the turning point is at $x = 0$.

Theorem 3.4: *Let* $a \in C^{p+1}[-1, 1] \cap C^m[-1, 1]$, b *and* $f \in C^m[-1, 1]$, $m \geq 1$, $0 \leq \epsilon \leq 1$

$$|a^{(p)}(x)| \geq |a^{(p)}(0) / 2| \quad -1 \leq x \leq 1 \tag{14}$$

If

(i) $b(0) > 0$, $a(x)$ *has a zero of order* $p > 1$ *at* $x = 0$, *and in addition, if* p *is odd,* $\lambda > 0$.

or

(ii) $b(x) \geq \underline{b} > 0$ *on* $[-1, 1]$ *and* $a(x)$ *has a zero of order* p *at* $x = 0$ *and* $\lambda < 0$

then for $|x| \leq 1/2$

$$|u^{(i)}(x)| \leq C \qquad 1 \leq i \leq m \tag{15}$$

where C *depends only on the set* T *defined by:*

$$\{ |a|_m, |a|_{p+1}, |b|_m, |f|_m, \lambda, b(t), C_\epsilon, |A|, |B|, m \}$$

and C_ϵ *is the stability constant.*

We have examined the behavior of problems having one or more turning points in the interior. Depending on the sign of $a(x)$ at the boundary, the solution will have a boundary layer or will differ little from the solution of the reduced equation at the boundary. With the exception of the simple turning point case, they will be smooth in the interior. Stability criteria for the difference scheme (3) can be derived in a similar manner to those above and are essentially analogous in form.

4. Turning point of odd order $p > 1$ at $x = 0$ with $\lambda > 0$ on $[-1, 1]$.

Since $\lambda > 0$ and $b(x) \geq 0$, we have by (13)

$$b(0) > 0 \text{ and } a^{(p)}(0) > 0 \tag{16}$$

It follows from (14) and (15), that $a(-1) < 0$ and $a(1) > 0$ and hence, by [8], that in a neighborhood of the endpoints

$$|u^{(i)}(x)| \leq C, \ 1 \leq i \leq m, \ 1 - \delta \leq |x| \leq 1 \tag{17}$$

Also, in any internal interval not including zero, one can show that

$$|u^{(i)}(x)| \leq C \qquad 1 \leq i \leq m \tag{18}$$

Finally by (14) and (16) and $\lambda > 0$, we can apply Theorem 3.4 (i) to give

$$|u^{(i)}(x)| \leq C \qquad 1 \leq i \leq m, \qquad |x| \leq 1/2 \tag{19}$$

Combining (17), (18), (19) we have the following theorem.

Theorem 4.1: *If (2), (8), (13) and (14) hold,* p *odd,* $p > 1$ *and* $\lambda > 0$ *then the solution of (1)* *satisfies*

$$| u^{(i)}(x) | \leq C \quad 1 \leq i \leq m, \ -1 \leq x \leq 1$$

Thus the solution in this case has no rapidly varying behavior anywhere in the interval. It is thus a regular perturbation problem rather than a singularly perturbed problem. In fact we can apply an argument similar to that in [1] on [-1, 0) and it's analogue for $a(1) > 0$ on (0, 1] to give the following result.

Lemma 4.2: *If (2), (8), (13) and (14) hold, p odd, $p > 1$ and $\lambda > 0$ then the solution of (1) satisfies*

$$| u(x) - u_0(x) | \leq C\epsilon \qquad 0 < | x | \leq 1$$

where $u_0(x)$ is the solution to the reduced equation satisfying $u_0(-1) = A$ for $x < 0$ and $u_0(1) = B$ for $x > 0$.

In fact we could extend this result to $x = 0$ also provided we define $u_0(0)$ as the limit $\lim_{\delta \to 0} u_0(\delta)$. Lundquist [9] showed that if the coefficients $a(x)$ and $b(x)$ were analytic then the reduced solution was bounded on the whole interval. This can also be used to give an explicit regular expansion for the solution. As a consequence of Theorem 4.1 any scheme which is uniformly stable and consistent is also uniformly convergent. Thus any of the schemes satisfying the sufficient conditions for either the non-turning or simple turning point problem are also uniformly convergent for this problem.

5. Turning point of odd order p with $\lambda < 0$.

Since $\lambda < 0$ and $b(x) \geq 0$, we have by (13)

$$b(0) > 0 \quad \text{and} \quad a^{(p)}(0) < 0 \tag{20}$$

It follows from (14) and (20), that $a(-1) > 0$ and $a(1) < 0$. In this case we require the additional condition

$$b(x) \geq b > 0, \ -1 < x < 1 \tag{21}$$

for the stability of the solution. By similar arguments to those used earlier we can show that the solution exhibits boundary layers of exponential type at both boundaries.

Theorem 5.1: *If (2), (8), (13), (14) and (21) hold, p odd and $\lambda > 0$ then the solution of (1) satisfies*

$$| u^{(i)}(x) | \leq C + C\epsilon^{-1}e^{-2\alpha(x+1)/\epsilon} + C\epsilon^{-1}e^{-2\alpha(1-x)/\epsilon},$$

for all $-1 \leq x \leq 1$.

6. Turning point of even order with $\lambda < 0$ or $\lambda > 0$

Since $\lambda < 0$ and $b(x) \geq 0$ we have

$$b(0) > 0 \tag{22}$$

In addition, since the turning point is of even order,

$$a(x) \geq 0 \qquad -1 \leq x \leq 1$$

and thus $a(-1) > 0,\ a(1) > 0$. Thus the problem has a boundary layer only at $x = -1$.

Theorem 6.1: *If (2), (8), (13) and (14) hold, p even, $\lambda < 0$ then the solution of (1) satisfies, for $1 \le i \le m$,*

$$| u^{(i)}(x) | \le C + C\epsilon^{-1}e^{-2\alpha(x+1)/\epsilon} \qquad -1 \le x \le 1 .$$

We thus see the behavior is similar to the non-turning point problem with $a(x) \ge \underline{a} > 0$. The case with $\lambda > 0$ is entirely analogous except the boundary layer is at the right-hand end of the interval.

We hypothesize that a natural generalization of the schemes which are uniformly convergent for the non-turning point problem are also uniformly convergent for these problems. Numerical results confirm this hypothesis. For quasi-linear problems a similar situation exists. If the problem exhibits boundary layer or internal layer behavior of the type which exists in linear problems, then the same methods yield good results. If the behavior is exhibited only by non-linear problems then the results are less satisfactory. In [5] we illustrated this for the problem,

$$\epsilon u'' + uu' - u = 0$$
$$u(0) = A \quad , \quad u(1) = B ,$$

which is considered in Howes [6]. This problem exhibits different behavior depending on the boundary values, A and B. Among these are hyperbolic tangent and cotangent (exponential) layers, at $x = 0$, shock layers, corner layers and transition layers. We linearize the problem first and then discretize this linearized problem. It is obvious that using Newton's method for linearisation is often inappropriate. This is because the linearised equation does not exhibit the same type of behavior as the non-linear problem. In practice, only the hyperbolic cotangent and some of the transition layers are modelled with an accuracy approaching uniform convergence.

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