A Uniform Convergence Result for a Turning Point Problem

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Abstract: We survey a number of methods for obtaining accurate solutions of a model turning-point problem. These include fitted schemes and schemes with *a-priori* or *a-posteriori* refined meshes. We then describe a new, uniformly $O(h)$, fitted scheme. This scheme is constructed using a variation of the approach of El-Mistikawy and Werle by replacing certain coefficient functions in the original differential equation by piecewise constants and then solving this approximate problem exactly. The resulting three-point difference scheme is truly uniform $O(h)$ and does not suffer from the degradation of all other known fitted schemes on uniform meshes where the uniform convergence rate can be (depending on the problem) $O(h^\lambda)$ with $0 < \lambda < 1$. The scheme is easy to describe and analyze (using known stability results); however it is somewhat complicated to compute, since the difference coefficients involve parabolic cylinder functions. A complete analysis and numerical results are presented.

1. Introduction. We consider a model turning-point problem of the form

$$\begin{aligned} -\epsilon u'' - xa(x)u' + b(x)u &= f(x), \quad -1 < x < 1 \\ u(-1) = A, &\qquad u(1) = B. \end{aligned} \tag{1}$$

Here ϵ is a ("small") positive parameter, and the functions a, b, and f are "sufficiently smooth." In addition, we will impose the following conditions on the coefficient functions, which will be assumed throughout the rest of the paper:

1. $a(x) > 0$, $a(0) = 1$, and
2. $b(x) \geq 0$, $b(0) > 0$.

Under these conditions, the problem (1) is well posed for $\epsilon > 0$ (and in fact possesses a maximum principle) and has a single, isolated turning point of *cusp type* at $x = 0$. Setting $a(0) = 1$ is a convenient normalization. As $\epsilon \to 0$, the unique solution of (1) converges to the solution of the *reduced* ($\epsilon = 0$) equation, which is characterized by the parameter

$$\lambda := b(0)$$

and admits the representation (see [1])

$$u(x) = v(x) + w(x)|x|^\lambda \begin{cases} c_1, & -1 \leq x \leq 0 \\ c_2, & 0 \leq x \leq 1. \end{cases}$$

Here c_1 and c_2 are constants, and v and w are smooth, with $w(x) > 0$. We are especially interested in the case $0 < \lambda < 1$.

Such problems have been rather carefully studied. Stability results, and in particular their dependence on λ, have been examined in [1]. *A-priori* estimates on the solution and its derivatives have been obtained in [4]. And asymptotic analyses can be found in [6], [7], [9], [10], and [18]. We will make use of many of these known results is what follows.

Numerical methods for problems like (1), and more general related problems that exhibit similar solution behavior but which may be nonlinear or systems of equations, have been considered by several authors. For fitted schemes, Emel'ianov [6] has shown the Allen-Southwell-Il'in scheme to be

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uniformly $O(h^{\min\{\lambda,2/3\}})$; Niijima [17] has analyzed an exponentially fitted version of the Engquist-Osher scheme and proven uniform $O(h)$ convergence in an L^1 norm; Berger, Han, and Kellogg [4] have shown the El-Mistikawy-Werle scheme to be uniformly $O(h^{\min\{\lambda,1\}})$; and Farrell [8], [9], and [10] has given sufficient conditions for various other schemes (including standard upwinding) to have this same uniform rate of convergence.

Using a Petrov-Galerkin framework, O'Riordan and Stynes [20] have constructed a scheme on a uniform mesh, and Stynes [19] an adaptive scheme on a quasi-uniform mesh, both of which are proven to be uniformly $O(h)$ with respect to an L^1 norm. Lorenz has analyzed schemes for such problems and in [16] gives a higher (classical) order modification of the Engquist-Osher scheme. Graded-mesh schemes have been developed by Liseikin [15] and Vulanović [21]; while first order vector systems have been studied by Ascher [3], Kreiss, Nichols, and Brown [14], and Brown and Lorenz [5].

2. A new uniformly $O(h)$ fitted scheme. In [4], the following scheme is analyzed. Write the problem (1) in the form

$$Lu := -\epsilon u'' - p(x)u' + q(x)u = f(x),$$

and assume that p has a single simple zero at $x = 0$, $p'(0) = 1$, and $q(x) > 0$. The parameter λ (denoted β in [4]) is then given by $\lambda = q(0)$. Construct a uniform grid of mesh-width h, and replace p, q, and f by functions $\bar{p}$, $\bar{q}$, and $\bar{f}$ that are piecewise constant on this grid so that

$$\|p - \bar{p}\|_\infty + \|q - \bar{q}\|_\infty + \|f - \bar{f}\|_\infty \le Ch.$$

Let $e := u - \bar{u}$ denote the difference between the true solutions of the problems $Lu = f$ and $\bar{L}\bar{u} = \bar{f}$ (with identical Dirichlet boundary conditions). Then the nodal values of $\bar{u}$ can be computed by solving a three-point difference equation (the coefficients of which involve exponential functions), and it is proved in [4] that

$$\|e\|_{h,\infty} \le Ch^{\min\{\lambda,1\}}.$$

This is somewhat disturbing, since in the non-turning-point case, such a discretization is uniformly $O(h^2)$. It is not difficult to identify where problems arise. First recall that the behavior of the solution u of (1) in the case of $\epsilon << 1$ and $0 < \lambda < 1$ is approximately modeled by the "cusp-like" function $(x^2 + \epsilon)^{\lambda/2}$. In particular, we have the following bounds, which are weaker consequences of estimates established in [4]:

$$|u^{(k)}(x)| \le C(x^2 + \epsilon)^{\frac{\lambda-k}{2}}, \quad k = 0, 1, \dots . \tag{2}$$

Next, note that both L and $\bar{L}$ possess maximum principles. One consequence of this property of $\bar{L}$ is the validity (uniformly in h and ϵ) of the stability inequality

$$\|v\|_\infty \le C\{\|\bar{L}v\|_\infty + |v(-1)| + |v(1)|\} \tag{3}$$

for all sufficiently smooth v. We note that stronger stability estimates are possible (see, for example, [1]), but we will not require them here.

Now the error, e, satisfies

$$\bar{L}e = -(\bar{p} - p)u' + (\bar{q} - q)u + (f - \bar{f}),$$

from which now follows

$$\|e\|_\infty \le C\{\|p - \bar{p}\|_\infty \|u'\|_\infty + \|q - \bar{q}\|_\infty \|u\|_\infty + \|f - \bar{f}\|_\infty\}.$$

The last two terms above present no problem. Using the *a-priori* bounds (2), one can establish

$$\|u'\|_\infty = O(\epsilon^{\frac{\lambda-1}{2}}).$$

So when $\epsilon = h^2$,

$$\|p - \bar{p}\|_\infty \|u'\|_\infty = O(h^\lambda).$$

In numerical experiments, this is exactly the rate of convergence that is observed. We note that a modification of this scheme done in [4] effectively handles the cases $\epsilon = h^p$ with $p > 2$, which for the unmodified scheme above can get even worse.

This deterioration of the uniform convergence rate can be circumvented in the following way. Suppose that one has an approximation $\bar{p}$ to p of the quality

$$|(p - \bar{p})(x)| \leq Ch|x|.$$

Then it would follow that

$$\|(p - \bar{p})u'\|_\infty \leq Ch \max_{-1 \leq x \leq 1} \{|x|(x^2 + \epsilon)^{\frac{\lambda-1}{2}}\} \leq C'h.$$

This degree of approximation can be achieved by writing our problem in the form (1) and approximating a, b, and f (instead of p, q, and f) by piecewise constants. We have proven the following.

Theorem: *Let the approximate problem $\bar{L}\bar{u} = \bar{f}$ be constructed by replacing the functions a, b, and f in (1) by functions $\bar{a}$, $\bar{b}$, and $\bar{f}$ that satisfy*

$$\|a - \bar{a}\|_\infty + \|b - \bar{b}\|_\infty + \|f - \bar{f}\|_\infty \leq Ch.$$

Then the error, $e := u - \bar{u}$, between the solution of the original problem (1) and that of the approximate problem (with the same boundary conditions) satisfies

$$\|e\|_\infty \leq Ch, \qquad 0 < \epsilon < \infty, \quad 0 < h \leq 1.$$

We note that the only ingredients required to make this proof work are the uniform stability (3) of the approximate operators, the uniform boundedness of the true solution, $\|u\|_\infty$, and the *a-priori* estimate $|u'(x)| \leq C(x^2 + \epsilon)^{(\lambda-1)/2}$. We now consider the characterization and construction of the exact discretization for this approximate problem.

3. Exact discretization of the approximate problem. We suppose that we have a given uniform mesh $x_i = ih$, $i = -n, \ldots, n$, $h = 1/n$, and piecewise-constant approximations to a (and b and f) given by

$$\bar{a}(x) = \bar{a}_i, \quad x_i < x < x_{i+1}$$

and satisfying $\|a - \bar{a}\|_\infty \leq Ch$ (and similarly for b and f). Typically, one would take $\bar{a}_i = (a(x_i) + a(x_{i+1}))/2$ and so on, but numerical evidence indicated a better choice for the two points $i = -1$ and $i = 0$ to be given by $\bar{a}_{-1} = \bar{a}_0 = a(x_0)$, which makes the coefficient function $\bar{a}$ continuous at the origin.

We now wish to solve (for nodal values) the problem

$$\begin{gathered} \bar{L}\bar{u} := -\epsilon\bar{u}'' - x\bar{a}(x)\bar{u}' + \bar{b}(x)\bar{u} = \bar{f}(x), \quad -1 < x < 1 \\ \bar{u}(-1) = A, \qquad \bar{u}(1) = B. \end{gathered} \tag{4}$$

Constructing an exact discretization for such a problem can be looked at from various points of view: utilizing L-splines [12], using a Petrov-Galerkin framework (as in [20]), using *patch functions*, using Marchuk type integral relations, etc. They all lead to the same thing, and that is a three-point formula that can be described in terms of the patch functions $\{\psi_i\}$. These are associated with the local formal adjoint operator $\bar{L}^\dagger$ and satisfy

$$\bar{L}^\dagger\psi_i := -\epsilon\psi_i'' + x\bar{a}(x)\psi_i' + (\bar{a} + \bar{b})(x)\psi_i = 0, \qquad x_{i-1} < x < x_i, \quad x_i < x < x_{i+1}$$

$$\psi_i(x_{i-1}) = \psi_i(x_{i+1}) = 0, \qquad \psi_i(x_i) = 1.$$

In terms of these functions, we have the following theorem, the proof of which follows by integration by parts of

$$\int_{x_{i-1}}^{x_{i+1}} \bar{f}\,\psi_i = \int_{x_{i-1}}^{x_{i+1}} \bar{L}\bar{u}\,\psi_i = \cdots .$$

Theorem: *The solution of the approximate problem (4) satisfies the difference scheme*

$$\alpha_{i,-1}\bar{u}(x_{i-1}) + \alpha_{i,0}\bar{u}(x_i) + \alpha_{i,1}\bar{u}(x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} \bar{f}\psi_i, \qquad i = -n+1, \ldots, n-1,$$

where ψ_i is the local patch function defined above and the difference coefficients are given by

$$\begin{aligned}
\alpha_{i,-1} &= -\epsilon\psi_i'(x_{i-1}^+), \\
\alpha_{i,0} &= -\epsilon\{\psi_i'(x_i^+) - \psi_i'(x_i^-)\} + x_i\{\bar{a}(x_i^+) - \bar{a}(x_i^-)\}, \text{ and} \\
\alpha_{i,1} &= \epsilon\psi_i'(x_{i+1}^-).
\end{aligned}$$

We make some observations. First, the nonsingularity of the associated tri-diagonal matrix follows from the well-posedness of the $\bar{L}$ problem; in fact the discrete problem inherits analogous stability properties (see, for example, [12]). Second, the computation of the right-hand sides for the discrete equations can be simplified in the following way. Since $\bar{f}$ is piecewise constant, we have

$$\int_{x_{i-1}}^{x_{i+1}} \bar{f}\psi_i = \bar{f}(x_i^-)\int_{x_{i-1}}^{x_i} \psi_i + \bar{f}(x_i^+)\int_{x_i}^{x_{i+1}} \psi_i.$$

Also, integrating both sides of $\bar{L}\psi_i = 0$ from x_{i-1} to x_i and from x_i to x_{i+1} yields the relations

$$-\epsilon\{\psi_i'(x_i^-) - \psi_i'(x_{i-1}^+)\} + \bar{a}(x_i^-)x_i + \bar{b}(x_i^-)\int_{x_{i-1}}^{x_i} \psi_i = 0$$

and

$$-\epsilon\{\psi_i'(x_{i+1}^-) - \psi_i'(x_i^+)\} + \bar{a}(x_i^+)x_i + \bar{b}(x_i^+)\int_{x_{i-1}}^{x_i} \psi_i = 0.$$

So the needed integrals of the patch functions can be obtained from the already needed left and right derivatives of the ψ_i's at the mesh points.

Solutions of the homogeneous ψ-equations can be obtained in terms of parabolic cylinder functions (as in [4] and [18]). On a given subinterval, write

$$-\epsilon v'' + x\bar{a}v' + (\bar{a} + \bar{b})v = 0 \quad \text{as} \quad -\hat{\epsilon}v'' + xv' + \beta v = 0,$$

with $\hat{\epsilon} := \epsilon/\bar{a}$ and $\beta := (\bar{a} + \bar{b})/\bar{a}$.

Seek solutions in the form

$$v(x) = \exp\left(\frac{\tilde{x}^2}{4}\right)\tilde{v}(\tilde{x}), \quad \tilde{x} = \frac{x}{\sqrt{\hat{\epsilon}}} .$$

This will then be a solution if and only if $\tilde{v}$ satisfies

$$-\tilde{v}'' + \left(\frac{\tilde{x}^2}{4} + \beta - \frac{1}{2}\right)\tilde{v} = 0,$$

which is the defining equation of the parabolic cylinder functions and has a basis of solutions given by $U(a, \tilde{x})$ and $V(a, \tilde{x})$ with $a = \beta - 1/2$ (see [2]).

The patch functions, ψ_i, can then be expressed as linear combinations of the functions

$$v_1(x) = \exp\left(\frac{\tilde{x}^2}{4}\right)U(\beta - 1/2, \tilde{x}) \quad \text{and} \quad v_2(x) = \exp\left(\frac{\tilde{x}^2}{4}\right)V(\beta - 1/2, \tilde{x}).$$

Using the recurrence relations

$$U'(a, x) + \frac{x}{2}U(a, x) = -\left(a + \frac{1}{2}\right)U(a+1, x)$$

and

$$V'(a, x) + \frac{x}{2}V(a, x) = V(a+1, x),$$

one can express the derivatives of v_1 and v_2 as

$$v_1'(x) = -\frac{\beta}{\sqrt{\epsilon}} \exp\left(\frac{\tilde{x}^2}{4}\right) U(\beta + 1/2, \tilde{x}) \quad \text{and} \quad v_2'(x) = \frac{1}{\sqrt{\epsilon}} \exp\left(\frac{\tilde{x}^2}{4}\right) V(\beta + 1/2, \tilde{x}),$$

so that formulas for all of the difference coefficients and right-hand-side weights can be given in terms of the values $U(\beta - 1/2, \tilde{x}_i)$, $U(\beta + 1/2, \tilde{x}_i)$, $V(\beta - 1/2, \tilde{x}_i)$, and $V(\beta + 1/2, \tilde{x}_i)$, $i = -n, \ldots, n$.

4. Numerical results and concluding remarks. Some numerical experiments were conducted to appraise the performance of this scheme on some model problems. We include a table of results for the problem

$$-\epsilon u'' - xe^x u' + .5e^{-x} u = (x+3)^{-1}, \qquad -1 < x < 1$$
$$u(-1) = 1, \qquad u(1) = 2\ .$$

An exact formula for the true solution of this problem is not known, and so the approximate errors in the numerical tests were estimated by comparison against a more accurate approximation, computed using the same discretization on sufficiently finer grids—this was determined by experimentation and typically involved four or eight times as many mesh points.

The above problem seems fairly typical; it has no special symmetries or whatever, and it does not impose any "premeditated" structure on the solution. The limiting ($\epsilon \to 0$) solution has an $|x|^{.5}$-type cusp at $x = 0$ (here $\lambda = 1/2$), and it satisfies

$$u(-1) = 1\ , \quad u(0) = 2/3\ , \quad \text{and} \quad u(1) = 2\ .$$

The numerical tests reported below are for experiments where both ϵ and h tend to zero simultaneously (but at differing rates). Tabulated are the errors, $\|e\|_{h,\infty}$, in the mesh sup-norm along with the approximate convergence rate $p = \log_2(\|e(h)\|_{h,\infty}/\|e(h/2)\|_{h,\infty})$. In theory this rate should always be at least one for any of these approaches to zero, and this indeed is observed in the table. In contrast to the non-turning-point problems, where the maximum errors (as a function of ϵ) occur around $\epsilon = h$, here the rate seems to decrease until $\epsilon = h^3$ or so.

Numerical results and approximate convergence rates

	$\epsilon = 1$		$\epsilon = h$		$\epsilon = h^2$		$\epsilon = h^3$		$\epsilon = h^4$	
n	$\|e\|_{h,\infty}$	p	$\|e\|_{h,\infty}$	p	$\|e\|_{h,\infty}$	p	$\|e\|_{h,\infty}$	p	$\|e\|_{h,\infty}$	p
4	.34(-2)	2.3	.94(-2)	1.6	.19(-1)	1.0	.31(-1)	1.1	.32(-1)	1.4
8	.70(-3)	2.3	.31(-2)	1.8	.95(-2)	1.2	.14(-1)	1.4	.12(-1)	1.9
16	.15(-3)	2.2	.91(-3)	1.9	.41(-2)	1.3	.52(-2)	1.6	.32(-2)	1.2
32	.32(-4)	2.1	.24(-3)	2.0	.17(-2)	1.3	.17(-2)	1.5	.14(-2)	1.2
64	.73(-5)	2.0	.62(-4)	2.0	.69(-3)	1.3	.61(-3)	1.4	.61(-3)	1.2
128	.18(-5)	2.0	.15(-4)	2.1	.28(-3)	1.3	.23(-3)	1.1	.26(-3)	1.2
256	.46(-6)	2.0	.37(-5)	2.1	.12(-3)	1.2	.11(-3)	1.1	.11(-3)	1.2
512	.12(-6)	2.0	.84(-6)	2.0	.50(-4)	1.2	.49(-4)	1.1	.50(-4)	1.2
1024	.31(-7)		.21(-6)		.21(-4)		.22(-4)		.23(-4)	

At this point, this result seems to be primarily of theoretical interest. However criticisms of schemes involving special functions, like the El-Mistikawy-Werle formula, apply here even more so, since parabolic cylinder functions are not as widely available on computing systems and do not have as many nice properties as exponential functions to simplify their handling. Furthermore, this type of approach does not readily generalize to higher dimensions. We hope to report in the future on higher-order uniform schemes for model problems like those considered here using the more general framework of [11] and [13].

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