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SUFFICIENT CONDITIONS FOR THE UNIFORM CONVERGENCE OF A DIFFERENCE SCHEME FOR A SINGULARLY PERTURBED TURNING POINT PROBLEM*

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Abstract. A number of results exist in the literature for singularly perturbed differential equations without turning points. In particular a number of difference schemes have been proposed that satisfy a stronger than normal convergence criteria known as uniform convergence. This guarantees that the schemes model the boundary layers well. We wish to examine whether these schemes will also be uniformly convergent, if the equation has turning points. To this end we derive sufficient conditions for uniform convergence which are satisfied not only by these schemes but by a more general class of schemes. We show that the rate of convergence is determined by a characteristic parameter of the problem which may be less than one. We confirm these theoretical results by numerical calculations.

Key words. uniformly convergent difference scheme, singularly perturbed differential equation, turning point

AMS(MOS) subject classifications. primary 65L10; secondary 34E05, 34E20

1. Introduction. We shall consider the following singularly perturbed two-point boundary value problem having an isolated turning-point at $x=0$:

$$L_\varepsilon u_\varepsilon \equiv \varepsilon u_\varepsilon''(x)+a(x)u_\varepsilon'(x)-b(x)u_\varepsilon(x)=f(x), \qquad -1\leqq x\leqq 1, \tag{1.1a}$$

$$u_\varepsilon(-1)=A, \qquad u_\varepsilon(1)=B, \tag{1.1b}$$

where a, b, f are in $C^k[-1,1]$, $0<\varepsilon\leqq\varepsilon_0$, and

$$a(0)=0, \qquad a'(0)>0. \tag{1.2a}$$

In order that the solution of (1.1) satisfy a maximum principle, we require that

$$b(x)\geqq 0, \qquad b(0)>0. \tag{1.2b}$$

We also impose the following restriction which ensures that there are no other turning-points in the interval $[-1, 1]$

$$|a'(x)|\geqq|a'(0)/2|, \qquad -1\leqq x\leqq 1. \tag{1.2c}$$

Under conditions (1.2a)-(1.2c) the solution of (1.1) has an internal layer at $x=0$. The smoothness of the solution is determined by the characteristic parameter λ

$$\lambda = xb(x)/a(x)|_{x=0}. \tag{1.3}$$

We shall restrict our attention to cases where λ is not an integer. We illustrate the solution, for a problem with $a(x)=x$, $b(x)=\lambda$, and $f(x)=0$ in Fig. 1 and Fig. 2. Without loss of generality, we shall assume in the remainder of this paper that $A=B=0$.

Singular perturbation problems have been widely used in the literature as model equations for convection-diffusion equations

$$\Delta u(x,y)+R\frac{\partial}{\partial x}u(x,y)=f(x,y),$$

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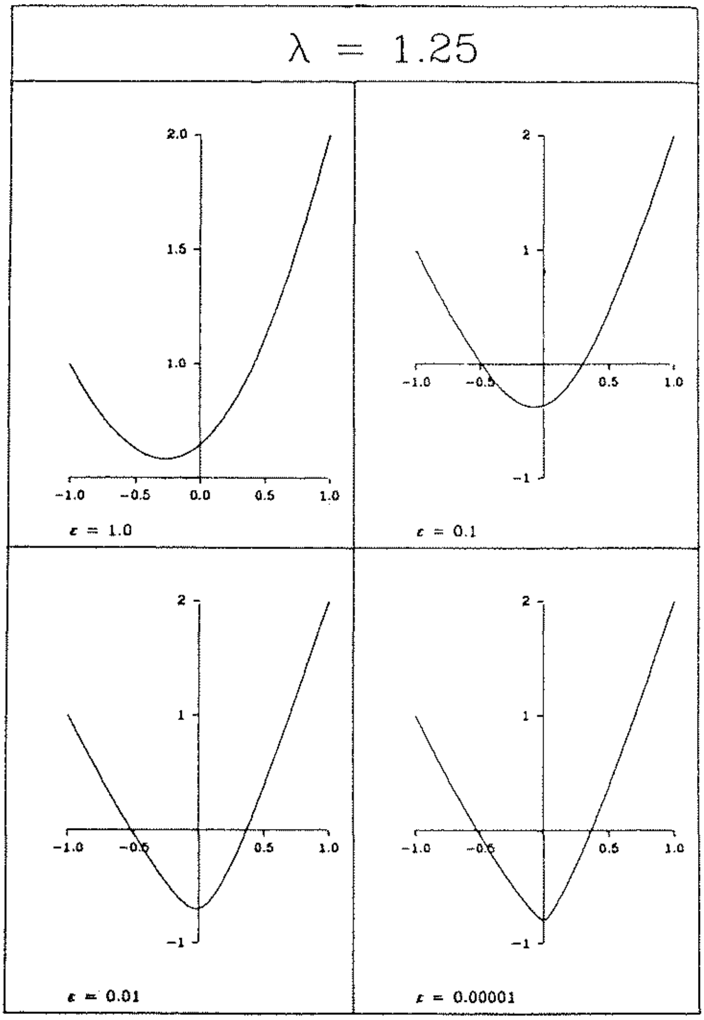

FIG. 1

where the Reynolds number R may be large. Many methods have been proposed for their solution. We are concerned here with difference schemes that satisfy the unusually strong convergence criteria known as *uniform convergence*, that is,

$$|u_i^h - u(x_i)| \leqq Ch^p,$$

where u_i^h is the solution of the difference scheme, $p > 0$ and C is independent of both h and ε. Previous works have concentrated on the nonturning point case, where

$$a(x) \geqq \underline{a} > 0.$$

This exhibits a boundary layer at $x = -1$. Uniform convergence is sufficient to guarantee that the problem can be solved accurately on a coarse mesh and that the boundary layer will be resolved accurately. It is well known that classical methods do not satisfy this criteria. For example, the centered difference method is unstable, unless $a(x_i)h/2\varepsilon < 1$, and thus a fine mesh spacing proportional to ε is required (Fig. 3).[1]

[1] Figures 3, 4, and 5 are for a problem on [0, 1] with boundary layer at $x = 0$ (from [20, pp. 220–221]).

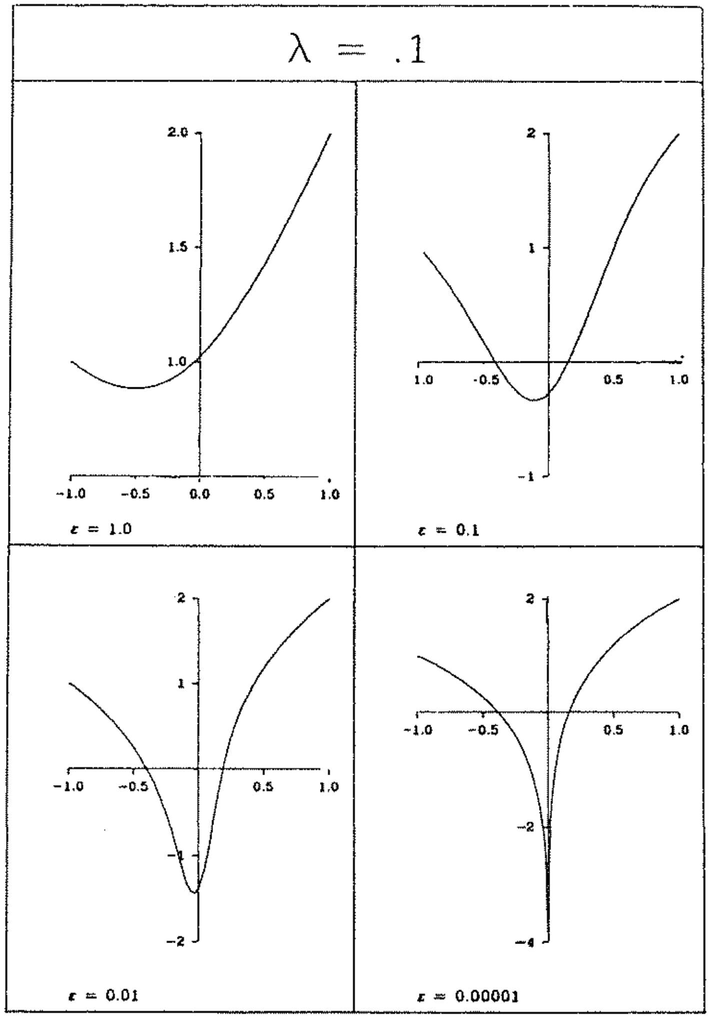

FIG. 2

On the other hand, employing an upwinded scheme, which uses a directed difference for approximating the first derivative, is stable but models the boundary layer badly (Fig. 4). In fact, such schemes are good for approximating the behaviour of the solution outside the boundary layer but, as the uniform mesh spacing is decreased and thus points fall in the boundary layer region for the first time, the error, measured by the discrete l_∞ norm on the mesh points, initially increases. When the mesh spacing is decreased sufficiently the error eventually begins to decline again. To solve these problems a class of schemes known as exponentially fitted schemes was proposed. These satisfy the above criteria of *uniform convergence*. The results for one of these schemes, known as Il'in–Allen–Southwell fitting, are shown in Fig. 5. This scheme is given by

$$\varepsilon\tau(a(x_i)h/2\varepsilon)D_+D_-u_i^h + a(x_i)D_0u_i^h - b(x_i)u_i^h = f(x_i) \quad \text{where } \tau(z) = z \coth(z).$$

A set of sufficient conditions for uniform convergence, which classify the manner in which schemes must be fitted, is given in Farrell [10], [11].

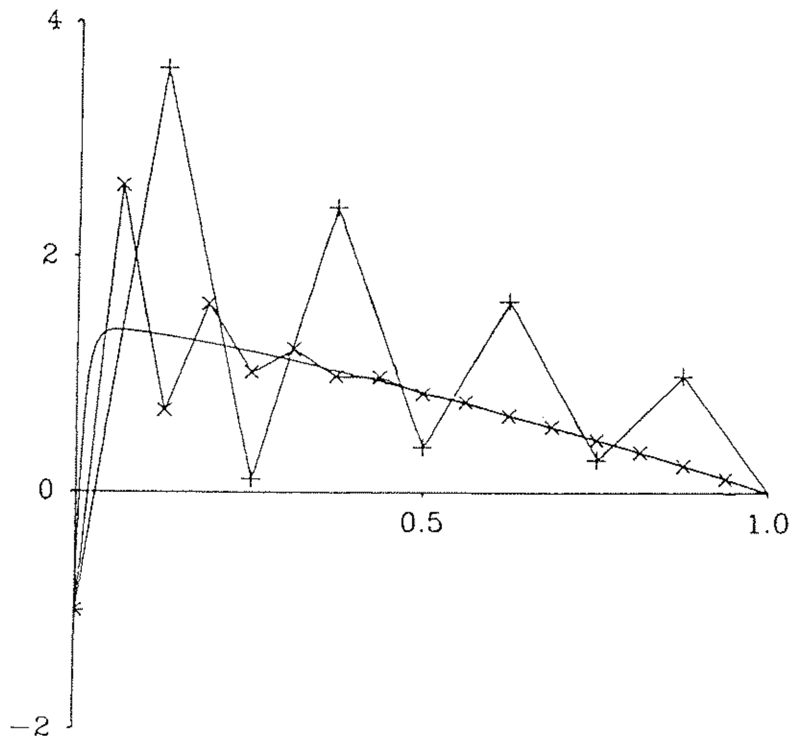

FIG. 3. *Centered differences.*

It is desirable to know if the same fitted methods are also accurate for more general problems. Miller [16] investigated the self-adjoint problem, obtained by setting $a(x)=0$ in (1.1). He showed that, in this case, a different type of fitting is required. Berger, Han, and Kellogg [4], [5] considered the turning-point case with

$$a(0)=0, \qquad a'(0)<0.$$

This exhibits boundary layers at both ends and the same fitted methods, interpreted correctly, can be shown to be uniformly convergent for this case. The turning-point problem dealt with in this paper is essentially different, in that it does not have a boundary layer of exponential type but rather an internal layer of cusp type, the boundary layer function of which is the Weber parabolic cylinder function. The analysis is therefore considerably more complex.

Four results exist, in the literature, for specific fitted schemes. This problem has been considered previously by Emel'ianov [8] who showed that the Il'in–Allen–Southwell scheme was uniformly convergent of order $h^{\min(\lambda,2/3)}$ for $0<\lambda<1$. We employ a method of proof, which involves asymptotic expansions, similar to the approach used there. Farrell [9] showed that a number of schemes are uniformly convergent for the problem (1.1) with $a(x)=ax$, where a is a constant, and $b(x)=\lambda$. Niijima [17] proved that the complete exponential fitting scheme was uniformly convergent, for problem (1.1), using an argument involving discrete Green's functions. Berger, Han, and Kellogg [4], [5], have proved analytic results for the bounds on the derivatives of this problem. These are valid for the case where λ is an integer in addition to the nonintegral case considered here. Using these bounds they show that a modified version of El-Mistakawy and Werle's scheme is uniformly convergent of order $h^{\min(\lambda,1)}$. This

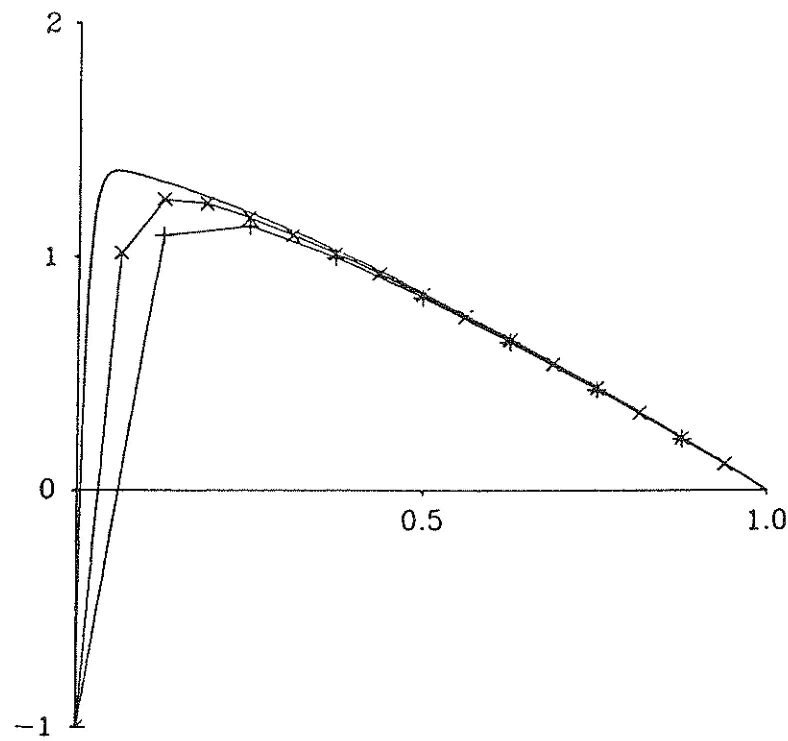

FIG. 4. *Upwinded scheme.*

proof involved the use of a comparison equation with piecewise constant coefficients. Abrahamsson [1], [2] produced extensive analytic results on the nature of the solution of this and other turning point problems and in addition proved results concerning the *nonuniform* covergence of difference schemes for these problems.

In this paper, we generalise the results of [9]. The sufficient conditions for uniform convergence derived are satisfied, not only by fitted schemes, but also by a large class of schemes of upwinded type. To be precise, we shall consider a class of difference schemes of the form

$$L_\varepsilon^h u_i^h \equiv \varepsilon_i^\pm D_+ D_- u_i^h + a_i^h D_\pm u_i^h - b_i^h u_i^h = f_i^h, \tag{1.4a}$$

$$u_{-N}^h = 0, \qquad u_N^h = 0, \tag{1.4b}$$

where $\pm$ in (1.4a) indicates taking ε_i^+ and D_+ if $a_i^h \geqq 0$, and ε_i^- and D_- if $a_i^h \leqq 0$. We shall assume additionally that

$$\varepsilon_i^\pm = \varepsilon\sigma_i^\pm, \quad a_i^h = \alpha_i a(x_i), \quad b_i^h = \beta_i b(x_i). \tag{1.5}$$

We shall also find it convenient in the convergence proofs to rewrite (1.4) in central difference form

$$L_\varepsilon^h u_i^h \equiv \varepsilon_i^0 D_+ D_- u_i^h + a_i^h - b_i^h u_i^h = f_i^h, \tag{1.6}$$

where $\varepsilon_i^0 = \varepsilon\sigma_i^0$ and $\varepsilon_i^\pm$ are related by

$$\varepsilon_i^0 = \varepsilon_i^\pm \pm h a_i^h/2. \tag{1.7}$$

We shall impose conditions on a_i^h, b_i^h, f_i^h, and $\varepsilon_i^\pm$ and show that these are sufficient for uniform convergence of the solution of (1.4), (1.5) to the solution of (1.1), (1.2).

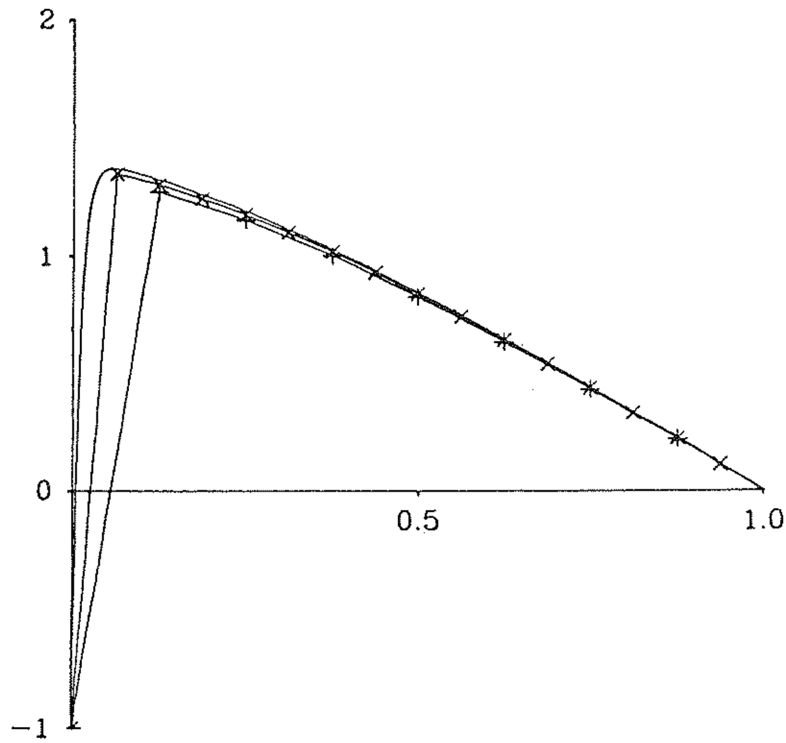

FIG. 5. *Il'in–Allen–Southwell scheme.*

In § 2 we state the sufficient conditions and discuss their significance. Section 3 contains results concerning the exact solution of (1.1), (1.2) including an asymptotic expansion and bounds on the solution and its derivatives. The proof of the sufficient conditions is given in § 4. Numerical computations are presented in § 5 which confirm our results. The Appendix contains three technical lemmas, in which we prove bounds required in § 4.

2. Sufficient conditions for uniform convergence. We shall first state the main theorem of the paper which gives the sufficient conditions for uniform convergence of the scheme (1.4). We will also discuss the significance of these conditions and list some of the difference schemes which satisfy them. A proof of this theorem is deferred until § 4.

THEOREM 2.1 (Uniform convergence). *Let u_ε be the solution of the isolated simple turning point problem*

$$L_\varepsilon u_\varepsilon \equiv \varepsilon u_\varepsilon''(x)+a(x)u_\varepsilon'(x)-b(x)u_\varepsilon=f(x), \qquad -1<x<1,$$

$$u_\varepsilon(-1)=0, \qquad u_\varepsilon(1)=0,$$

where the coefficients satisfy (1.2) *and* (1.3), *and let u_i^h be the solution of*

$$L_\varepsilon^h u_i^h \equiv \varepsilon_i^\pm D_+D_-u_i^h+a_i^h D_\pm u_i^h-b_i^h u_i^h=f_i^h, \qquad -N<i<N,$$

$$u_{-N}^h=0, \qquad u_N^h=0,$$

where the coefficients $\varepsilon_i^\pm$, $a_i^h=\alpha_i a(x_i)$, $b_i^h=\beta_i b(x_i)$ and f_i^h are bounded and are such that *the scheme is uniformly stable, e.g.,*

(I) $$\varepsilon_i^{\pm} > 0, \quad \alpha_i \geqq \underline{\alpha} > 0, \quad \beta_i \geqq \underline{\beta} > 0,$$

(II) $$|a_i^h - a(x_i)| \leqq Ch,$$

(III) $$|b_i^h - b(x_i)| \leqq Ch,$$

(IV) $$|f_i^h - f(x_i)| \leqq Ch,$$

(V) $$|\varepsilon_i^{\pm} - \varepsilon| \leqq Ch(|a(x_i)| + h);$$

then for $h \leqq h_0$, $\varepsilon \leqq \varepsilon_0$,

$$|u_\varepsilon(x_i) - u_i^h| \leqq Ch^{\min(\lambda, 1)}, \qquad -1 \leqq x_i \leqq 1$$

where C is independent of h and ε.

Remarks on the conditions (I)–(V). Condition (I) is a condition for the matrix of the scheme to be of positive type and hence for the scheme to be uniformly stable. Note that the conditions are analogous to those for the stability of the differential equation.

Conditions (II)–(IV) are "consistency" type conditions which state that the scheme should not vary much from the form of the differential equation, i.e., the coefficients should only be $O(h)$ perturbations of the coefficients of the differential equation.

Condition (V) effectively states that $\varepsilon_i^{\pm}$ must be an order h^2 approximation to ε near the turning point but need only be order h away from it. Alternatively we may say that a better order of approximation is required in the region where the internal layer is situated.

The restriction of the order of convergence to $\min(\lambda, 1)$ reflects the fact that the reduced difference scheme only approximates the reduced differential equation with order λ for $\lambda < 1$. This is apparent if we consider an equation with

$$a(x) = x, \quad b(x) = \lambda, \quad f(x) = 0$$
$$u(-1) = u(1) = 1$$

which has a reduced solution of $|x|^\lambda$.

We note that unlike the nonturning point case discussed in Farrell [10], [11], no exponential fitting is required in this case for uniform convergence. This reflects the fact that the solution of the turning-point problem does not exhibit boundary layers and is thus smoother than the solution of the nonturning point problem.

It is easily seen that a wide range of schemes satisfy these conditions. In particular, we can show that Il'in–Allen–Southwell fitting

$$\varepsilon\tau(a(x_i)h/2\varepsilon)D_+D_-u_i^h + a(x_i)D_0u_i^h - b(x_i)u_i^h = f(x_i),$$

where $\tau(z) = z \coth(z)$, satisfies the conditions. Using $\tau(z) = \sigma(2z) + z$ and $\sigma(z) - 1 \leqq z$ we can write the scheme in the form (1.4) where

$$\sigma_i^{\pm} = \sigma(|a(x_i)|h/2\varepsilon), \quad a_i^h = a(x_i), \quad b_i^h = b(x_i), \quad f_i^h = f(x_i)$$

and hence

$$|\varepsilon_i^{\pm} - 1| \leqq \varepsilon|\sigma_i^{\pm} - 1| \leqq \varepsilon|a(x_i)|h/\varepsilon = h|a(x_i)|.$$

Similarly the generalizations in Farrell [11], which use $\sigma(a(\nu_i)h/2\varepsilon)$, $0 \leqq x_i - Ch \leqq \nu_i \leqq x_i + Ch$, also satisfy these conditions. Similar but somewhat longer arguments show that complete exponential fitting (Shishkin and Titov [18], Carroll [6], Carroll and Miller [7]) satisfies them. A more interesting result is that the simple upwinded scheme

$$\varepsilon_i^{\pm} = \varepsilon, \quad a_i^h = a(x_i), \quad b_i^h = b(x_i), \quad f_i^h = f(x_i)$$

also satisfies them trivially. Similarly two other special schemes, from the literature, for convection-diffusion problems also satisfy them. That of Sarmarskii (Gushkin and Shchennikov [12], Kellogg and Tsan [15]), is given by (1.4) with

$$\varepsilon_i^{\pm}=\frac{\varepsilon}{(1+|a(x_i)|h/2\varepsilon)},\quad a_i^h=a(x_i),\quad b_i^h=b(x_i),\quad f_i^h=f(x_i).$$

We can see that it satisfies the sufficient conditions using

$$\frac{1}{(1+z)}-1=\frac{z}{(1+z)}\leqq z.$$

The schemes, proposed by Hemker [13], [14], are given by (1.4) with

$$\varepsilon_i^{\pm}=\varepsilon+\tfrac{1}{2}(\nu_i\pm 1)a(x_i)h,\quad a_i^h=a(x_i),\quad b_i^h=b(x_i),\quad f_i^h=f(x_1)$$

where $|\nu_i|\leqq 1$ and

$$\frac{-1}{1+\nu_i}<\frac{a(x_i)h}{2}<\frac{1}{1-\nu_i}.$$

The result folllows by observing that

$$(\nu_i\pm 1)\leqq 2,\qquad \varepsilon_i^{\pm}>\varepsilon+\left(\frac{-2\varepsilon}{a(x_i)h}\right)\frac{a(x_i)h}{2}\geqq 0.$$

The modification of the Abrahamsson-Keller-Kreiss box scheme proposed by Abrahamsson [2], which we shall refer to as Abrahamsson's scheme 1, is defined as follows:

if $a_i>0$,

$$\varepsilon D_+D_-u_i^h+a_{i+1/2}D_+u_i^h-\tfrac{1}{2}b_{i+1/2}(u_i^h+u_{i+1}^h)=f_{i+1/2},\qquad \gamma_i>0,$$

$$\varepsilon D_+D_-u_i^h+a_iD_+u_i^h-b_iu_i^h=f_i,\qquad \gamma_i\leqq 0,$$

if $a_i=0$,

$$\varepsilon D_+D_-u_i^h-b_iu_i^h=f_i,$$

if $a_i<0$,

$$\varepsilon D_+D_-u_i^h+a_{i-1/2}D_-u_i^h-\tfrac{1}{2}b_{i-1/2}(u_i^h+u_{i-1}^h)=f_{i-1/2},\qquad \tilde{\gamma}_i>0,$$

$$\varepsilon D_+D_-u_i^h+a_iD_-u_i^h-b_iu_i^h=f_i,\qquad \tilde{\gamma}_i\leqq 0;$$

where, $a_{i\pm 1/2}\equiv a(x_{i\pm 1/2})$ etc.,

$$\gamma_i=\frac{\varepsilon}{h^2}+\frac{a_{i+1/2}}{h}-\frac{1}{2}b_{i+1/2}\quad\text{and}\quad \tilde{\gamma}_i=\frac{\varepsilon}{h^2}-\frac{a_{i-1/2}}{h}-\frac{1}{2}b_{i-1/2}.$$

This was shown in [2] to be $O(h^2+\varepsilon h)$ outside the boundary layer. It can easily be shown to satisfy the sufficient conditions, since

$$\varepsilon_i^{\pm}=\varepsilon,\quad a_i^h=a_{i\pm 1/2}\mp hb_{i\pm 1/2}/2 \text{ or } a_i,\quad b_i^h=b_{i\pm 1/2}\text{ or } b_i.$$

All these latter schemes are not uniformly convergent for the nonturning point problem.

3. Analytic results. In this section we will present some analytic results including an order ε asymptotic expansion and bounds on the solution of the equation, together with its derivatives. We shall first introduce the following notation. Expand $a(x_i)$, $b(x_i)$ in Taylor expansion about zero; then

$$a(x_i)=x_i(a_0+a_1x_i+\tilde{a}_2x_i^2), \qquad b(x_i)=b_0+b_1x_i+\tilde{b}_2x_i^2,$$

where

$$a_0=\frac{a(x)}{x}\bigg|_{x=0}, \quad a_1=\frac{d}{dx}\left[\frac{a(x)}{x}\right]\bigg|_{x=0}, \quad \tilde{a}_2=\frac{1}{2}\frac{d^2}{dx^2}\left[\frac{a(x)}{x}\right]\bigg|_{x=\eta},$$

$$b_0=b(0), \quad b_1=b'(0), \quad \tilde{b}_2=b''(\zeta)/2, \quad 0\leqq \zeta,\ \eta\leqq x_i.$$

Let $u_0(x)$ be the solution of the reduced equation

$$L_0u_0(x)\equiv a(x)u_0'(x)-b(x)u_0(x)=f(x), \qquad 0<|x|<1,$$

$$u_0(-1)=0, \qquad u_0(1)=0.$$

In addition, we define a polynomial approximation, to $u_0(x)$, in the turning point region by

$$y_0(x)=d_0+d_1x+d_2x^2$$

where

$$-b_0d_0=f_0,$$

$$-b_1d_0+(a_0-b_0)d_1=f_1,$$

$$-b_2d_0+(a_1-b_1)d_1+(2a_0-b_0)d_2=f_2.$$

Then, letting $w(x)$ be the solution of the homogeneous reduced equation, for $x>0$ and

$$g(x)=f(x)-L_0y_0(x),$$

we define

$$d=-\left[d_0+d_1+d_2-\int_0^1\frac{g(s)\,ds}{w(s)a(s)}\right].$$

We can define $\tilde{d}$ similarly, using a solution to the homogeneous reduced equation fo $x<0$. In addition, it is clear that

$$w(x)=\exp\left[-\int_x^1\frac{T(s)}{s}\,ds\right]$$

where $T(s)=sb()s/a(s)$. Thus

$$w(x)=\exp\left[-\int_x^1\frac{T(s)-T(0)}{s}\,ds\right]\exp\left[\int_x^1\frac{T(0)}{s}\,ds\right]$$

$$=x^{T(0)}m(x)=x^{\lambda}m(x),$$

where

$$m(x)=\exp\left[-\int_x^1\frac{T(s)-T(0)}{s}\,ds\right].$$

Using this notation the following asymptotic expansion for the solution of (1.1), (1.2) was derived in Emel'ianov [8] and appeared also in Farrell [11].

THEOREM 3.1. *If $u_\varepsilon(x)$ is the solution of* (1.1) *and $\tilde{u}_\varepsilon(x)$ is given by*

(3.1)
$$\tilde{u}_\varepsilon(x) = u_0(x) + \varepsilon^{\lambda/2} v_0(\zeta) + \varepsilon^{(\lambda+1/2)} v_1(\zeta) - \begin{cases} dm(0)(x^\lambda + T'(0)x^{\lambda+1}), & x \geqq 0, \\ \tilde{d}m(0)((-x)^\lambda + T'(0)(-x)^{\lambda+1}), & x \leqq 0, \end{cases}$$

where d, $\tilde{d}$, and $m(0)$ are constants independent of ε, $\zeta = x/\varepsilon^{1/2}$, $T(s) = sb(s)/a(s)$, and v_0, v_1 satisfy

$$v_0''(\zeta) + \zeta a_0 v_0'(\zeta) - b_0 v_0(\zeta) = f_0, \tag{3.2}$$

$$v_1''(\zeta) + \zeta a_0 v_1'(\zeta) - b_0 v_1(\zeta) = \zeta f_1 + \zeta(b_1 v_0 - a_1 v_0' \zeta), \tag{3.3}$$

then

$$|u_\varepsilon(x) - \tilde{u}_\varepsilon(x)| \leqq C\varepsilon, \qquad -1 \leqq x \leqq 1.$$

In addition to this asymptotic expansion we shall also require bounds on the solution and its derivatives.

Emel'ianov [8] has shown that the following bounds hold for the solution of the reduced equation $u_0(x)$,

$$|u_0^{(i)}(x)| \leqq C(1 + |x|^{\lambda - i}). \tag{3.4}$$

In addition we have the following bounds for the derivatives of the solution $u_\varepsilon(x)$ of (1.1) and for the derivatives of the first and second internal layer functions $v_0(x)$ and $v_1(x)$. These bounds first appeared in Berger et al. [4], [5] and Emel'ianov [8], respectively.

The following theorems which give bounds on the solution $u_\varepsilon(x)$ of (1.1) and its derivatives are proved in Berger et al. [4], [5]. The proof involves examining the solution near the turning point $x = 0$, transforming the equation to that for the parabolic cylinder functions, and using the properties of these functions given in Abramowitz and Stegun [3].

We shall require the following conditions on the coefficients of the equation (1.1)

$$\begin{gathered} a(x) \in C^2[-1, 1], \qquad b(x), f(x) \in C^1[-1, 1], \\ 0 < \varepsilon \leqq 1, \\ b(x) \geqq \underline{b} > 0, \qquad -1 \leqq x \leqq 1. \end{gathered} \tag{3.5}$$

Further let β_l, β_s be fixed positive constants

$$\beta_l < 1 < \beta_s. \tag{3.6}$$

We are now in a position to state the first theorem.

THEOREM 3.2. *Assume* (1.2), (1.3), (3.5), *and* (3.6) *and, in addition, that $\beta_l < \lambda < \beta_s$, $a(x), b(x), f(x) \in C^k[-1, 1]$ where $k \geqq 2$. Then there is a constant C, depending only on*

$$S(k) = \{\|a\|_2, \|b\|_1, \|f\|_1, \underline{b}, \beta_l, \beta_s, |A|, |B|, \|a\|_k, \|b\|_k, \|f\|_k, k\},$$

where $\|\cdot\|_m$ is the norm in $C^m[-1, 1]$, such that $u_\varepsilon(x)$, the solution of (1.1) *satisfies*

$$|u_\varepsilon^{(i)}(x)| \leqq C(|x| + \varepsilon^{1/2})^{\lambda - i} I(x, \varepsilon, \lambda), \qquad i = 1, \cdots, k+1, \tag{3.7}$$

for $-1 \leqq x \leqq 1$, where

$$I(x, \varepsilon, \beta) = \int_{x^2+\varepsilon}^{6} s^{-(\beta-1)/2}\, ds.$$

Proof. The proof appears in Berger et al. [5, Thm. 2.7, p. 469].

The estimate (3.7) does not give a tight bound for the higher derivatives, when $\lambda > 1$, since $I(x, \varepsilon, \lambda)$ increases with λ. The following theorem gives an improved estimate for this case.

THEOREM 3.3. *Let $\lambda = m+\beta$, where m is a positive integer and $\beta_l < \lambda < \beta_s$. In addition let* (1.2), (3.5), *and* (3.6) *hold and $a(x)$, $b(x)$, $f(x) \in C^{m+k}[-1, 1]$ where $k \geqq 2$. Then there exists a constant C, depending only on $S(m+k)$, such that $u_\varepsilon(x)$, the solution of* (1.1) *satisfies*

$$(3.8)\qquad \begin{aligned} &|u_\varepsilon^{(i)}(x)| \leqq C, \qquad i=1,\cdots,m,\\ &|u_\varepsilon^{(i)}(x)| \leqq C(|x|+\varepsilon^{1/2})^{\lambda-i} I(x,\varepsilon,\beta), \qquad i=m+1,\cdots,m+k+1, \end{aligned}$$

for $-1 \leqq x \leqq 1$.

Proof. The proof appears in Berger et al. [5, Thm. 2.8, p. 469].

These theorems are valid whether λ is an integer or not. For λ not an integer, we may simplify the results by noting that for $0<\beta<1$

$$I(x,\varepsilon,\beta) = \frac{2}{1-\beta}\{6^{(1-\beta)/2} - (x^2+\varepsilon)^{(1-\beta)/2}\} \leqq \frac{2}{1-\beta}\{6^{(1-\beta)/2} + 2^{(1-\beta)/2}\} \leqq C(\beta).$$

Hence we have the following, by combining Theorems 3.2 and 3.3.

COROLLARY 3.4. *Assume* (1.2), (3.5), (3.6) *and in addition $\lambda > 0$ not an integer, $a(x)$, $b(x)$, $f(x) \in C^k[-1,1]$ where $k \geqq 2$. Then there is a constant C, depending only on $S(k)$ and λ, such that $u_\varepsilon(x)$, the solution of* (1.1), (6.2), *satisfies*

$$(3.9)\qquad |u_\varepsilon^{(i)}(x)| \leqq C[1+(|x|+\varepsilon^{1/2})^{\lambda-i}], \qquad i=1,\cdots,k+1,$$

for $-1 \leqq x \leqq 1$.

We remark that this result could be proved directly. The analysis in this case would be similar but less complicated.

Finally we have the following estimates for the first and second boundary layer functions, $v_0(x)$ and $v_1(x)$, and their derivatives. These appeared in Emel'ianov [8]. The former estimate also appears as an intermediate result in the proof of Theorem 3.2.

THEOREM 3.5. *Let $v_0(x)$ and $v_1(x)$ be the first and second boundary layer functions of the equations* (1.1), (6.2); *then under the conditions of Corollary* 3.4 *we have*

$$(3.10)\qquad |v_0^{(i)}(x)| \leqq C[1+(|x|+\varepsilon^{1/2})^{\lambda-i}], \qquad i=0,1,2,$$

$$(3.11)\qquad |v_1^{(i)}(x)| \leqq C[1+(|x|+\varepsilon^{1/2})^{\lambda+1-i}], \qquad i=0,1,2.$$

In the next section we shall make extensive use of these estimates, together with the asymptotic expansion, in order to obtain the error estimates for difference schemes.

4. Proof of the sufficient conditions. The proof consists of obtaining two separate estimates for the truncation error, the first using the traditional approach but retaining powers of ε explicitly, the second using the approximation $\tilde{u}_\varepsilon(x)$ given in § 3. Using uniform stability we thus get two bounds for the error, which we then combine to give the uniform error estimate.

We shall assume throughout that $\varepsilon < \varepsilon_0$ (given) and that the scheme is uniformly stable. A sufficient condition for this written in terms of α_i and β_i is that

$$(4.1)\qquad \varepsilon_i^{\pm} \geqq 0, \quad \beta_i \geqq \underline{\beta} > 0, \quad \alpha_i \geqq \underline{\alpha} > 0.$$

It follows from this that $a_i^h \neq 0$ for $i \neq 0$ and $b_0^h > 0$. Let $M \equiv (m_{ij})_{-N \leqq i,j \leqq N}$ be the matrix of the scheme. If $i>0$, then $a_i^h > 0$, the off-diagonal entries of M are given by $\varepsilon_i^+/h^2 \geqq 0$, $\varepsilon_i^+/h^2 + a_i^h/h \geqq 0$ and the row sum of row i is $-b_i^h \leqq 0$ and a similar result holds for $i<0$. For $i=0$, the off-diagonal entries are $\varepsilon_i^+/h^2 \geqq 0$ and the row sum is $-b_0^h < 0$. Further, it is clear that the graph of the matrix is connected, since for $i<0$ all nonzero elements can be connected to $m_{-N,-N}$ and for $i>0$ to $m_{N,N}$. The matrix is thus irreducibly diagonally dominant, hence the negative of an M-matrix and consequently inverse negative (Varga [19]).

In order to establish stability, we define the comparison function $\phi_i = (x_i^2 - 4)/2$. Now

$$\begin{aligned} L^h\phi_i &= \varepsilon_i^h + a_i^h x_i - b_i^h[x_i^2-4]/2 \\ &= \varepsilon_i^h + x_i a_i^h + b_i^h - b_i^h[x_i^2-2]/2 \\ &\geqq x_i a_i^h + b_i^h = x_i\alpha_i a(x_i) + \beta_i b(x_i) \\ &\geqq x_i \underline{\alpha} a(x_i) + \underline{\beta} b(x_i) \geqq \min(\underline{\alpha}, \underline{\beta})[x_i a(x_i) + b(x_i)] \geqq \min(\underline{\alpha}, \underline{\beta})\mu > 0, \end{aligned}$$

where μ depends only on $a(x)$ and $b(x)$, since $b(0) > 0$ and $xa(x) > 0$ for $x \neq 0$ and $a(x)$ and $b(x)$ continuous. Uniform stability follows by applying the maximum principle to

$$w_i = \pm u_i^h + C_1|f_i^h|_\infty \phi_i,$$

where C_1 is chosen sufficiently large.

We shall also require the following estimates, which are a direct consequence of condition (V):

$$\begin{aligned} |\sigma_i^\pm - 1| &\leqq C\rho(\varepsilon^{-1/2}|a(x_i)| + \rho) \leqq C\rho(|x_i|/\varepsilon^{1/2} + \rho), \\ |\sigma_i^0 - 1| &\leqq C\rho(\varepsilon^{-1/2}|a(x_i)| + \rho) \leqq C\rho(|x_i|/\varepsilon^{1/2} + \rho), \end{aligned} \tag{4.2}$$

$$|a(x_i)(\alpha_i - 1)| \leqq Ch, \tag{4.3}$$

where $\rho = h/\varepsilon^{1/2}$.

LEMMA 4.1 (Classical consistency). *Let conditions* (I)–(V) *of Theorem* 2.1 *hold. Then the truncation error of the scheme* (1.5), (1.6) *applied to* (1.1), (1.2) *is*

$$\tau_i^h = |L_\varepsilon^h(u_\varepsilon(x_i) - u_i^h)| \leqq C\varepsilon^{\lambda/2}\left(\frac{h^3}{\varepsilon^2} + \frac{h^2}{\varepsilon^{3/2}} + \frac{h}{\varepsilon^{1/2}}\right) + Ch.$$

Proof.

$$\begin{aligned} |\tau_i^h| &= |L_\varepsilon^h u_\varepsilon(x_i) - L_\varepsilon u_\varepsilon(x_i)| + |f_i^h - f(x_i)| \\ &= |\varepsilon_i^0(D_+D_-u_\varepsilon - u_\varepsilon''(x_i))| + |(\varepsilon_i^0 - \varepsilon)||u_\varepsilon''(x_i)| + |a_i^h(D_0 u_\varepsilon - u_\varepsilon'(x_i))| \\ &\quad + |(a_i^h - a(x_i))u_\varepsilon'(x_i)| + |(b_i^h - b(x_i))u_\varepsilon(x_i)| + |f_i^h - f(x_i)| \\ &\leqq |\varepsilon_i^0 h^2 u^{(4)}(\zeta_i^1)| + |a_i^h h^2 u^{(3)}(\zeta_i^2)| + |h(|a(x_i)| + h)u_\varepsilon''(x_i)| + Ch|u_\varepsilon'(x_i)| + Ch. \end{aligned}$$

We require a better estimate for $a(x_i)u_\varepsilon''(x_i)$ than that given by (3.9). By differentiating (1.1) we may show that

$$|a(x_i)u_\varepsilon''(x_i)| \leqq C(\varepsilon u_\varepsilon^{(3)}(x_i) + u_\varepsilon'(x_i) + u_\varepsilon(x_i) + C).$$

Thus we have, using (3.9),

$$\begin{aligned} |\tau_i^h| &\leqq C(\varepsilon + h)h^2|u^{(4)}(\zeta_i^1)| + C(\varepsilon h + h^2)|u^{(3)}(\zeta_i^2)| + Ch^2|u_\varepsilon''(x_i)| + Ch|u_\varepsilon'(x_i)| + Ch \\ &\leqq C\varepsilon^{\lambda/2}\left(\frac{h^3}{\varepsilon^2} + \frac{h^2}{\varepsilon^{3/2}} + \frac{h}{\varepsilon^{1/2}}\right) + Ch. \qquad \square \end{aligned}$$

Since we have assumed the scheme is uniformly stable, classical convergence is an immediate consequence, as is uniform convergence for the special case $\lambda > 4$.

THEOREM 4.2 (Classical convergence theorem). *If the scheme* (1.5), (1.6) *is uniformly stable and in addition satisfies* (I)–(V) *then*

$$|u_i^h - u_\varepsilon(x_i)| \leqq C\varepsilon^{\lambda/2}\left(\frac{h^3}{\varepsilon^2}+\frac{h^2}{\varepsilon^{3/2}}+\frac{h}{\varepsilon^{1/2}}\right)+Ch.$$

We shall now use the asymptotic expansion of § 3 to derive a further estimate of the error. We shall consider only the case $x \geqq 0$, since $x<0$ is analogous.

THEOREM 4.3 (Nonclassical convergence). *If u_i^h is the solution of a difference scheme* (1.4) *satisfying the conditions* (I)–(V), *$u_\varepsilon(x)$ is the solution of* (1.1), (1.2) *and $h \leqq h_0$, $\varepsilon \leqq \varepsilon_0$ then*

$$|u_\varepsilon(x_i) - u_i^h| \leqq C(h^{\min(\lambda,1)}+\varepsilon), \qquad -1 \leqq x_i \leqq 1.$$

Proof. We restrict ourselves first to the region $x \geqq h$, the case $x=0$ will be dealt with later. Applying L_ε^h to $\tilde{u}_\varepsilon(x_i)-u_i^h$, where $\tilde{u}_\varepsilon$ is the asymptotic expansion of § 3 and u_i^h is the solution of (1.5), (1.6), we obtain

$$L_\varepsilon^h(\tilde{u}_\varepsilon(x_i)-u_i^h) = L_\varepsilon^h\tilde{u}_\varepsilon(x_i) - f_i^h = L_\varepsilon^h\tilde{u}_\varepsilon(x_i) - L_0u_0(x_i) + f(x_i) - f_i^h$$

and thus

$$|L_\varepsilon^h(\tilde{u}_\varepsilon(x_i)-u_i^h)| \leqq |L_\varepsilon^h\tilde{u}_\varepsilon(x_i) - L_0u_0(x_i)| + |f_i^h - f(x_i)|.$$

Since the latter term is bound by Ch, it therefore remains to bound

$$T_i^h = L_\varepsilon^h\tilde{u}_\varepsilon(x_i) - L_0u_0(x_i).$$

Writing $\tilde{u}_i$ for $\tilde{u}_\varepsilon(x_i)$

$$T_i^h = \varepsilon_i^0 D_+D_-\tilde{u}_i + a_i^h D_0\tilde{u}_i - (b_i^h - b(x_i))\tilde{u}_i - a(x_i)u_0'(x_i) - b(x_i)(\tilde{u}_i - u_0(x_i)).$$

Now substituting explicitly for $\tilde{u}_i$, using (3.1), expanding $a(x_i)$ and $b(x_i)$ in Taylor expansions around 0, using the notation of § 3, and regrouping

$$\begin{aligned}
T_i^h = {}& \varepsilon_i^0 D_+D_-[u_0(x_i) - dm(0)(x_i^\lambda + T'(0)x_i^{\lambda+1})] + \varepsilon_i^0\varepsilon^{\lambda/2}D_+D_-[v_0(x_i)+\varepsilon^{1/2}v_1(x_i)]\\
&+ x_i(a_0 + x_i\tilde{a}_1)\alpha_i D_0[u_0(x_i) - dm(0)(x_i^\lambda + T'(0)x_i^{\lambda+1})]\\
&- x_i(a_0 + x_i\tilde{a}_1)u_0'(x_i) - (b_i^h - b(x_i))\tilde{u}_i\\
&+ x_i(a_0 + x_ia_1 + x_i^2\tilde{a}_2)\alpha_i\varepsilon^{\lambda/2}D_0v_0(x_i) + x_i(a_0 + \tilde{a}_1x_i)\alpha_i\varepsilon^{(\lambda+1)/2}D_0v_1(x_i)\\
&- (b_0 + b_1x_i + \tilde{b}_2x_i^2)[\varepsilon^{\lambda/2}v_0(x_i) - dm(0)x_i^\lambda]\\
&- (b_0 + \tilde{b}_1x_i)[\varepsilon^{(\lambda+1)/2}v_1(x_i) - dm(0)T'(0)x_i^{\lambda+1}].
\end{aligned}$$

Recall that, if $\zeta = x/\varepsilon^{1/2}$ and $\rho = h/\varepsilon$, then

$$\frac{d}{d\zeta}v_0(\zeta) = \varepsilon^{1/2}\frac{d}{dx}v_0(x/\varepsilon^{1/2}), \qquad D_0^\zeta v_0(\zeta) = \varepsilon^{1/2}D_0^x v_0(x/\varepsilon^{1/2}),$$

$$\frac{d^2}{d\zeta^2}v_0(\zeta) = \varepsilon\frac{d^2}{dx^2}v_0(x/\varepsilon^{1/2}), \qquad D_+^\zeta D_-^\zeta v_0(\zeta) = \varepsilon D_+^x D_-^x v_0(x/\varepsilon^{1/2}).$$

We shall henceforth write $v_0'(\zeta)$ for $(d/d\zeta)v_0(\zeta)$ and similarly for D_0^ζ, etc.

Now writing the expansions in terms of ζ, collecting terms of equal order in ε and using the relations

$$b_0x^\lambda = a_0x\frac{dx^\lambda}{dx}, \qquad b_0x^{\lambda+1} = a_0x\frac{dx^{\lambda+1}}{dx} - a_0x^{\lambda+1},$$

$$b_0v_0(\zeta) = v_0''(\zeta) + \zeta a_0v_0'(\zeta), \qquad b_0v_1(\zeta) = v_1''(\zeta) + \zeta a_0v_1'(\zeta) + \zeta^2a_1v_0'(\zeta) - b_1\zeta v_0(\zeta),$$

of which the latter two follow from the defining relations for v_0 and v_1, we may rewrite this as

$$T_i^h = \sum_{i=1}^{11} S_i$$

where

$$S_1 = (b(x_i) - b_i^h)\tilde{u}_i + x_i a_0 \alpha_i \left[D_0 Y(x) - \frac{d}{dx} Y(x) \right]$$
$$+ x_i^2 \tilde{a}_1(\alpha_i - 1) u_0'(x_i) + \varepsilon^{(\lambda+1)/2} a_0 \zeta_i (\alpha_i - 1) v_1'(\zeta),$$
$$S_2 = \varepsilon_i^0 D_+ D_- Y(x),$$
$$S_3 = \varepsilon^{\lambda/2}(\alpha_i - 1)[a_0 v_0'(\zeta_i) + \varepsilon^{1/2} a_1 \zeta_i v_0'(\zeta_i)],$$
$$S_4 = x_i^2 \tilde{a}_1 \alpha_i (D_0 u_0(x_i) - u_0'(x_i)),$$
$$S_5 = \varepsilon^{\lambda/2} a_0 \zeta_1 \alpha_i (D_0 v_0(\zeta_i) - v_0'(\zeta_i)),$$
$$S_6 = \varepsilon^{(\lambda+1)/2} a_0 \zeta_i \alpha_i (D_0 v_1(\zeta_i) - v_1'(\zeta_i)),$$
$$S_7 = \varepsilon^{\lambda/2}(\sigma_i^0 - 1) v_0''(\zeta_i),$$
$$S_8 = \varepsilon^{\lambda/2}[\sigma_i^0(D_+ D_- v_0(\zeta_i) - v_0''(\zeta_i))],$$
$$S_9 = \varepsilon^{(\lambda+1)/2}[(\sigma_i^0 - 1) v_1''(\zeta_i) + \sigma_i^0(D_+ D_- v_1(\zeta_i) - v_1''(\zeta_i))],$$
$$S_{10} = \varepsilon^{(\lambda+1)/2}[a_1 \zeta_i^2 \alpha_i (D_0 v_0(\zeta_i) - v_0'(\zeta_i)) - dm(0)(a_1 \zeta_i^2 \alpha_i D_0 \zeta_i^2 - b_1 \zeta_i^{\lambda+1}$$
$$+ a_0 T'(0) \zeta_i^{\lambda+1})],$$
$$S_{11} = \varepsilon^{(\lambda+2)/2}[\zeta_i^3 \alpha_i \tilde{a}_2 D_0(v_0(\zeta_i) - dm(0)\zeta_i^\lambda) - \zeta_i^2 \tilde{b}_2(v_0(\zeta_i) - dm(0)\zeta_i^\lambda)$$
$$+ \zeta_i^2 \alpha_i \tilde{a}_1 D_0(v_1(\zeta_i) - dm(0) T'(0) \zeta_i^{\lambda+1}) - \zeta_i \tilde{b}_1(v_1(\zeta_i) - dm(0) T'(0) \zeta_i^{\lambda+1})],$$

and

$$Y(x) = u_0(x) - dm(0) x^\lambda - dm(0) T'(0) x^{\lambda+1}.$$

It can be shown that the S_i satisfy the following bounds:

$$S_i \leqq Ch, \qquad i = 1, 4, 6, 9,$$
$$S_i \leqq C(h + \varepsilon), \qquad i = 2, 11,$$
$$S_i \leqq Ch^{\min(\lambda, 1)}, \qquad i = 3, 5, 7, 8, 10.$$

We shall defer the proof of these bounds to Lemma A.1 in the Appendix. Now combining the bounds on S_1 to S_{11}, and (IV), for $x_i \geqq h$, we have

$$|L_\varepsilon^h(\tilde{u}_\varepsilon(x_i) - u_i^h)| \leqq T_i^h + |f_i^h - f(x_i)| \leqq \sum_{i=1}^{11} S_i + Ch \leqq C(h^{\min(\lambda,1)} + \varepsilon). \tag{4.4}$$

For $x_i \leqq -h$ the same bound is also valid. Thus there remains only the case $x_i = 0$. Using (3.4) we can see that $a(0) u_0'(0) = 0$ and hence

$$L_0 u_0(0) = -b_0 u_0(0).$$

Thus, using $f(0) = L_0 u_0(0) = -b_0 u_0(0)$ and (IV),

$$\begin{aligned} |L_\varepsilon^h(\tilde{u}_\varepsilon(0) - u_0^h)| &\leqq |L_\varepsilon^h \tilde{u}_\varepsilon(0) - f(0)| + |f_0^h - f(0)| \\ &\leqq |L_\varepsilon^h \tilde{u}_\varepsilon(0) + b_0 u_0(0)| + Ch. \end{aligned} \tag{4.5}$$

Now, at $\zeta_i = 0$, the defining equations for v_0 and v_1 become

$$v_i'' - b_0 v_i(0) = 0, \qquad i = 0, 1.$$

Hence, using (3.1) and rearranging the terms,

$$|L_\varepsilon^h \tilde{u}_\varepsilon(0) + b_0 u_0(0)| \leqq \sum_{i=1}^{6} |T_i|. \tag{4.6}$$

We can show that the following bounds hold for these terms

$$T_1 = \varepsilon_0^0 D_+ D_- Y(x_0) \leqq C(\varepsilon + h^2),$$
$$T_2 = a_0^h D_0 \tilde{u}_\varepsilon(x_0) \leqq C h^{\min(\lambda,1)},$$
$$T_3 = (b_0^h - b_0)\tilde{u}_\varepsilon(x_0) \leqq Ch,$$
$$T_4 = \varepsilon^{\lambda/2}(\sigma_0^0 - 1) D_+ D_- v_0(\zeta_0) \leqq C h^{\min(\lambda,2)},$$
$$T_5 = \varepsilon^{\lambda/2}(D_+ D_- v_0(\zeta_0) - v_0''(\zeta_0)) \leqq C h^\lambda,$$
$$T_6 = \varepsilon^{(\lambda+1)/2}(D_+ D_- v_1(\zeta_0) - v_1''(\zeta_0)) + \varepsilon^{(\lambda+1)/2}(\sigma_0^0 - 1) D_+ D_- v_1''(\zeta_0) \leqq C h^{\min(\lambda,1)}$$

and $Y(x)$ is a smooth function as before. We will defer the proof of these bounds to Lemma A.3. By (4.5), (4.6), and the bounds on T_i, $i = 1, 6$,

$$|L_\varepsilon^h(\tilde{u}_\varepsilon(0) - u_0^k)| \leqq C(h^{\min(\lambda,1)} + \varepsilon). \tag{4.7}$$

We may now use these results to produce a new error estimate.

Since $u_{-N}^h = u_N^h = 0$ and since from a consideration of the asymptotic expansion similar to Theorem 3.1

$$|\tilde{u}_\varepsilon(\pm 1)| \leqq C\varepsilon,$$

we have

$$|\tilde{u}_\varepsilon(x_i) - u_i^h| \leqq C\varepsilon, \qquad i = -N, N.$$

By (4.1), a scheme of the form (1.4) is stable. Thus, using (4.4) and (4.7),

$$|\tilde{u}_\varepsilon(x_i) - u_i^h| \leqq C \max_{-N<i<N} |L_\varepsilon^h(\tilde{u}_\varepsilon(x_i) - u_i^h)| + C\varepsilon \leqq C(h^{\min(\lambda,1)} + \varepsilon).$$

Also, by Theorem 3.1,

$$|\tilde{u}_\varepsilon(x_i) - u(x_i)| \leqq C\varepsilon$$

and hence

$$|u(x_i) - u_i^h| \leqq C(h^{\min(\lambda,1)} + \varepsilon).$$

This concludes the proof of Theorem 4.3. □

Finally by combining Theorem 4.3 with Therorem 4.2 we can show the following uniform convergence result.

Proof of Theorem 2.1. Consider the case $\lambda > 1$. Then applying Theorem 4.2 for $h \leqq \varepsilon$, we obtain

$$|u_\varepsilon(x_i) - u_i^h| \leqq C\left\{\frac{h^3}{\varepsilon^{3/2}} + \frac{h^2}{\varepsilon} + h\right\} C(h^{3/2} + h) \leqq Ch \tag{4.8}$$

and, by Theorem 4.3 for $h \geqq \varepsilon$,

$$|u_\varepsilon(x_i) - u_i^h| \leqq C(h + \varepsilon) \leqq Ch. \tag{4.9}$$

Similarly for $0<\lambda<1$, applying Theorem 4.2 for $h^\lambda \leqq \varepsilon$,

$$\begin{aligned}|u_\varepsilon(x_i)-u_i^h| &\leqq C(h^3\varepsilon^{(\lambda-4)/2}+h^2\varepsilon^{(\lambda-3)/2}+h\varepsilon^{(\lambda-1)/2}+h)\\ &\leqq C(h^{(6+\lambda(\lambda-4))/2}+h^{(4+\lambda(\lambda-3))/2}+h^{(2+\lambda(\lambda-1))/2}+h)\end{aligned}$$

and $\lambda^2-\lambda+2>\lambda^2+1>2\lambda$ and thus

$$|u_\varepsilon(x_i)-u_i^h|\leqq Ch^\lambda. \tag{4.10}$$

Finally, if $0<\lambda<1$ and $h^\lambda \geqq \varepsilon$ we have by Theorem 4.3,

$$|u_\varepsilon(x_i)-u_i^h|\leqq C(h^\lambda+\varepsilon)\leqq Ch^\lambda. \tag{4.11}$$

Theorem 2.1 now follows by combining (4.8), (4.9), (4.10), and (4.11). □

5. Numerical results. In this section we present numerical results for a number of schemes which satisfy the sufficient conditions of Theorem 2.1 and also for some schemes which do not. The graphs each show the results on meshes of width $h=\frac{1}{8}$ and $\frac{1}{16}$ and also an accurate approximation to the exact solution obtained on a fine mesh ($h=1/1024$). Except for Fig. 8, which is for $\varepsilon=0.01$ and $\lambda=1.25$, they are all for the following problem (from [5, p. 487]), which has a turning point at $x=\frac{1}{2}$,

$$\varepsilon u''(x)+((x-0.5)/\lambda+0.3121(x-0.5)^2/\lambda)u'(x)-(1+0.2764(x-0.5))u(x)=f(x),$$

$$u(0)=1.2062, \qquad u(1)=2.2003,$$

with $\varepsilon=0.00001$ and $\lambda=0.25$. The right-hand side $f(x)$ is chosen, so that the solution and its derivatives, exhibit the behaviour given in Corollary 3.4. Figures 6, 7, and 8 show the results of the Il'in–Allen–Southwell scheme, Abrahamsson's scheme 1 [2],

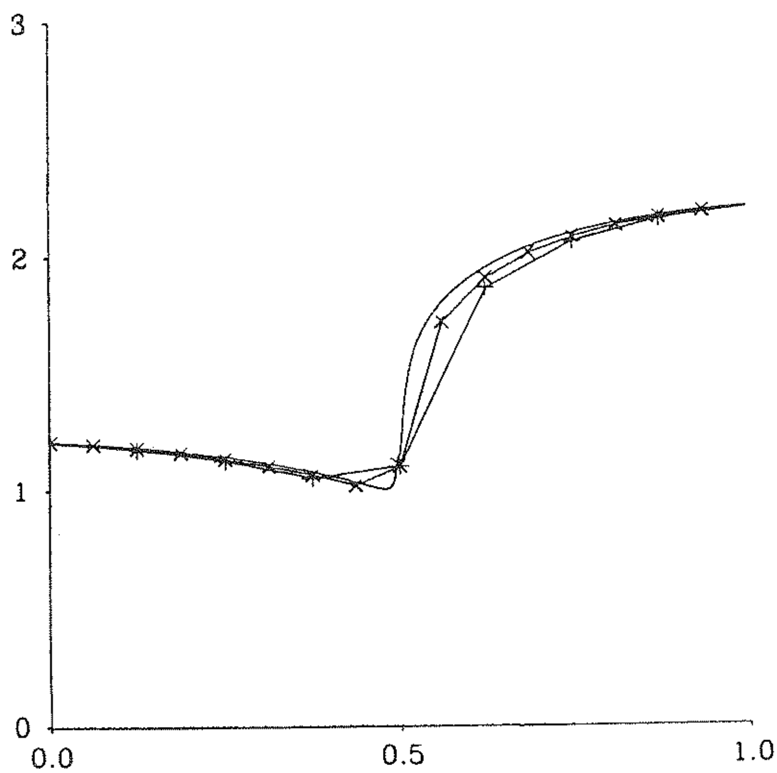

FIG. 6. *Il'in-Allen-Southwell scheme.*

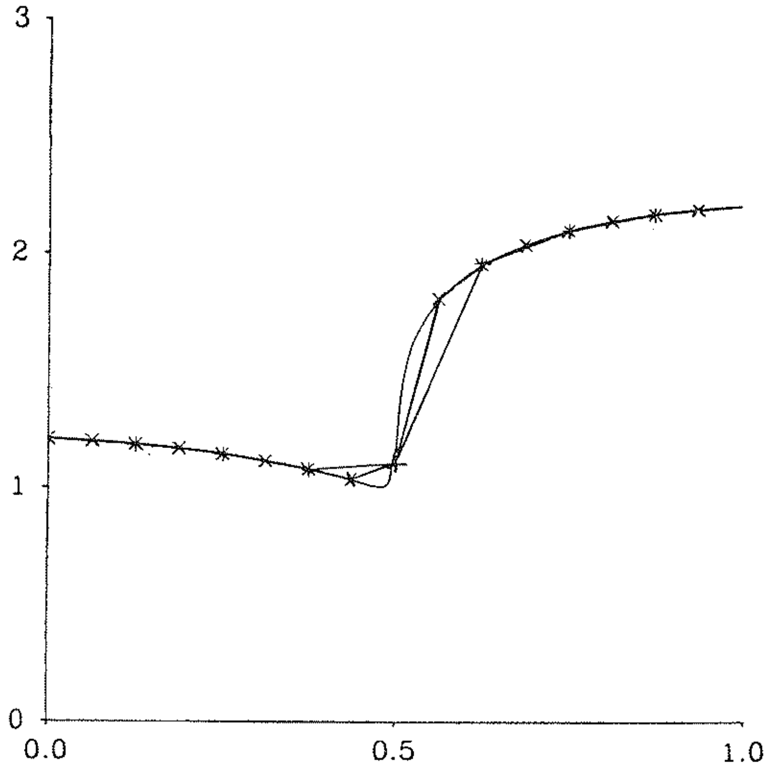

FIG. 7. *Abrahamsson scheme.*

described in § 2, and Complete Exponential Fitting, respectively. All of these satisfy the sufficient conditions. We do not show graphical results for any of the other schemes satisfying the sufficient conditions, since they are virtually indistinguishable from these. Figures 9 and 10 illustrate the solution for centered differences and for the following scheme proposed by Abrahamsson [2] for nonlinear problems, which we shall refer to as Abrahamsson's scheme 2:

$$\varepsilon\left(1+\frac{\mu h}{\varepsilon}\right)D_+D_-u_i^h+a(x_i)D_0u_i^h-b(x_i)u_i^h=f(x_i),$$

where

$$\mu \geqq \tfrac{1}{2}\max_{-1\leqq x\leqq 1}|a(x_i)|.$$

Neither of these schemes satisfies the conditions of Theorem 2.1. It is easily seen that these do not solve the problem accurately. Numerical and graphical results for this test problem, together with many more general problems may be found in Farrell [11].

To determine more accurately whether a scheme is in practice uniformly convergent, we calculate an experimental rate of uniform convergence p as follows:

$$p=\operatorname*{mean}_{h\in H}[\ln(e^{2h})-\ln(e^h)]/\ln(2),$$

where

$$\varepsilon^h=\max_{\varepsilon\in E}(\max_{0\leqq i\leqq N}|u_i^{2h}-u_{2i}^h|),$$

$$H=\{1/2^j\,|\,j=3,\cdots,9\},\qquad E=\{1/2^j\,|\,j=0,\cdots,jred\}$$

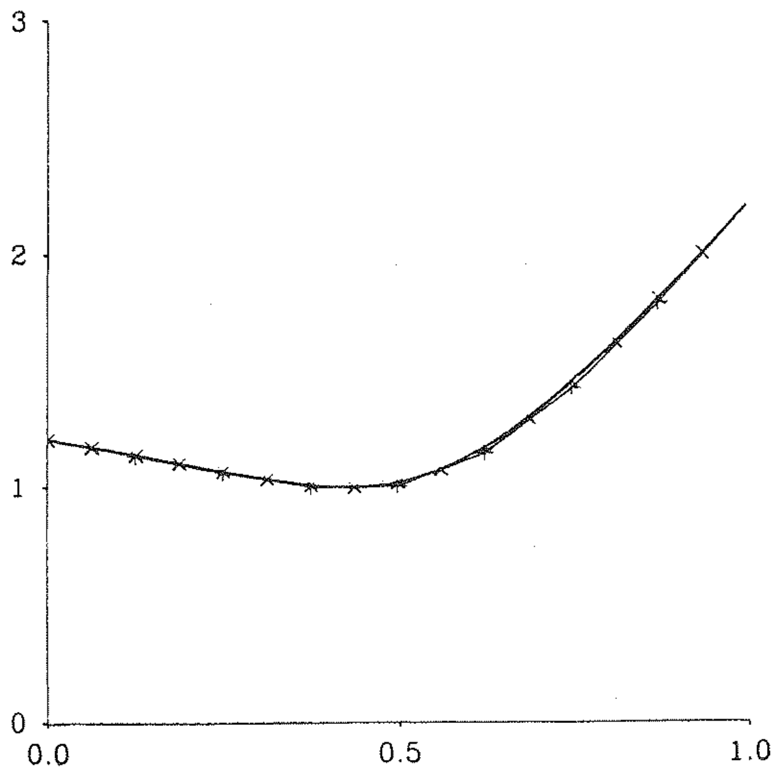

FIG. 8. *Complete exponential fitting.*

and *jred* is chosen so that $\varepsilon = 1/2^{jred}$ is a value at which the rate of convergence stabilizes, which normally occurs when, to machine accuracy, we are solving the reduced problem. A further discussion on the effectiveness of this as a measure of uniform convergence may be found in Farrell [11]. Tables 1–3 give the experimental and theoretical rates of convergence for a number of schemes for various values of λ.

The experimental rate of classical convergence given here is the average rate for $\varepsilon = 1/2$. This is the reason that the upwinded schemes, including that of Abrahamsson, have a classical rate of 1.00. The Generalized Constant Il'in scheme is a new scheme, which uses $\sigma_i^{\pm} = \sigma(\rho|a(0)|)$ near $x = 0$, $\sigma_i^{\pm} = \sigma(\rho|a(1)|)$ near $x = 1$, and the appropriate directed difference approximation on each side of $x = \frac{1}{2}$. The next, Il'in averaged $a(x)$ uses $\sigma_i^{\pm} = \sigma(\rho(|a(x_{i+1}) + a(x_i)|)/2)$. These tables illustrate that the rate of convergence predicted for the schemes, which satisfy the sufficient conditions, is achieved in practice. For a general scheme of this type the predicted rate min $(\lambda, 1)$ is the best attainable rate, that is, there exist schemes that only achieve the predicted rate. We also note, that as stated earlier, one can only attain uniform convergence of order h^{λ} for $\lambda < 1$. In fact, if we try to evolve schemes that attain a higher rate of uniform convergence for a particular turning point problem, their performance for other more general problems will deteriorate. Naturally, if we consider only classical (nonuniform) convergence outside the internal layers, these schemes will attain higher orders, usually h or in certain cases h^2, as illustrated by the graphical results.

Appendix. Bounds on the terms S_i and T_i. In this section we give detailed proofs of the bounds on S_i and T_i appearing in § 4. Before we proceed to bound the terms

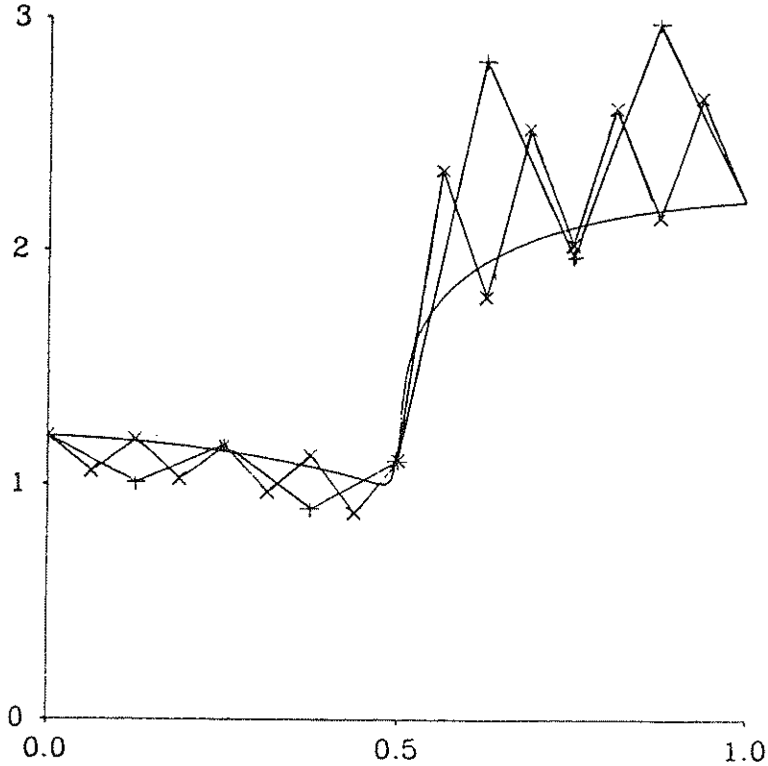

FIG. 9. *Centered differences.*

S_i we prove the following technical lemma which is a generalization of that in Emel'ianov [8].

LEMMA A.1. *Let* $A \geqq 1$, $B \geqq 0$, $l \leqq 1$, *and* $\lambda \neq k-1$, $k-2$, *then*

$$\left| \int_{\zeta - l\rho}^{\zeta + l\rho} \int_{\zeta}^{z} \frac{dy}{(Ay+B)^{k-\lambda}} \, dz \right| \leqq M \frac{\rho^2}{(A\zeta + B)^{k-\lambda}} \quad \textit{for all } \zeta \geqq 2\rho.$$

Proof. By integrating twice explicitly and Taylor expanding about ζ we can show

$$\left| \int_{\zeta - l\rho}^{\zeta + l\rho} \int_{\zeta}^{z} \frac{dy}{(Ay+B)^{k-\lambda}} \, dz \right| = \left| \frac{Al^3 \rho^3 (\lambda - k)}{6} [(A\zeta + B + \theta_1 l\rho)^{\lambda - k - 1} + (A\zeta + B - \theta_2 l\rho)^{\lambda - k - 1}] \right|$$

where $0 \leqq \theta_1$, $\theta_2 \leqq 1$. Now, since $\zeta \geqq 2\rho$,

$$A\zeta + B \pm \theta l\rho \geqq \frac{2A - \theta l}{2} \zeta \geqq \frac{A\zeta}{2}$$

and

$$\frac{\rho}{A\zeta + B \pm \theta l\rho} \leqq \frac{2\rho}{A\zeta} \leqq \frac{1}{A}.$$

Also, if $\lambda < k$,

$$(A\zeta + B \pm \theta l\rho)^{\lambda - k} \leqq \left(\frac{A\zeta}{2} + B \right)^{\lambda - k} \leqq C_1 (A\zeta + B)^{\lambda - k}$$

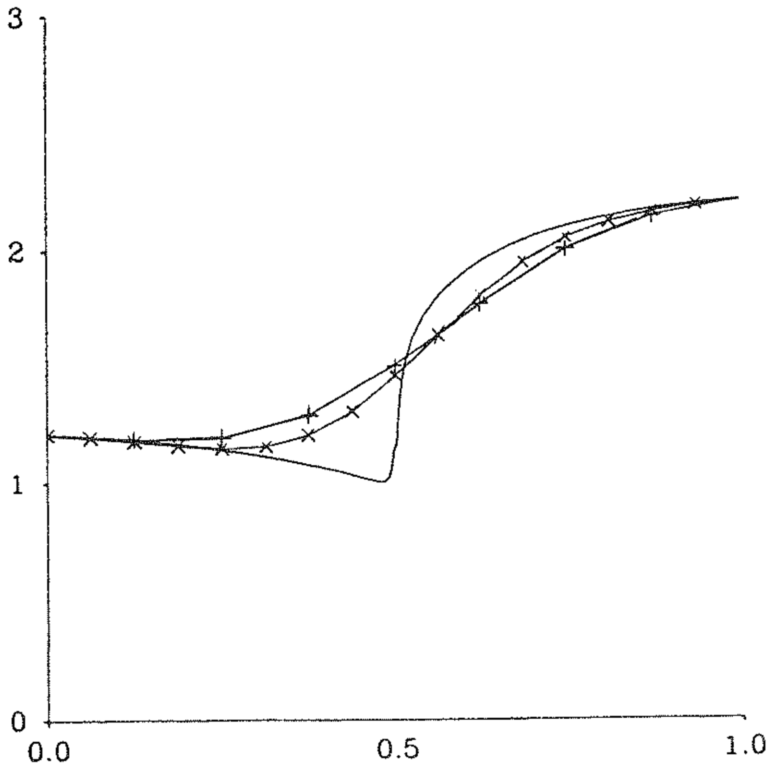

FIG. 10. *Abrahamsson scheme 2.*

TABLE 1
$\lambda = 0.1$.

Scheme	Uniform Exper.	Uniform Theory	Classical Exper.	Classical Theory
Il'in fitting	0.098	0.10	1.96	2.00
Il'in averaged $a(x)$	0.090	0.10	2.18	2.00
Complete fitting	0.092	0.10	1.97	2.00
Gen. const. Il'in	0.095	0.10	1.05	1.00
Upwinding	0.098	0.10	0.91	1.00
Samarkskii	0.098	0.10	1.85	2.00
Abrahamsson scheme 1	0.090	0.10	0.94	1.00
El-Mistakawy-Werle (Gen)	0.200	0.10	2.00	2.00

and, if $\lambda > k$,

$$(A\zeta + B \pm \theta l\rho)^{\lambda-k} \leqq ((A+\tfrac{1}{2})\zeta + B)^{\lambda-k} \leqq C_2(A\zeta+B)^{\lambda-k}.$$

The result follows with $M = l^3|\lambda - k| \max(C_1, C_2)/3$.

LEMMA A.2. *Let S_1 to S_{11} be defined as in § 4, then*

$$S_i \leqq Ch, \qquad i = 1, 4, 6, 9,$$

$$S_i \leqq C(h+\varepsilon), \qquad i = 2, 11,$$

$$S_i \leqq Ch^{\min(\lambda,1)}, \qquad i = 3, 5, 7, 8, 10.$$

TABLE 2
$\lambda = 0.25$.

Scheme	Uniform Exper.	Uniform Theory	Classical Exper.	Classical Thery
Il'in fitting	0.25	0.25	2.00	2.00
Il'in averaged $a(x)$	0.30	0.25	2.10	2.00
Complete fitting	0.25	0.25	2.00	2.00
Gen. const. Il'in	0.25	0.25	1.27	1.00
Upwinding	0.25	0.25	0.85	1.00
Samarskii	0.25	0.25	2.01	2.00
Abrahamsson scheme 1	0.25	0.25	0.93	1.00
El-Mistakawy-Werle (Gen)	0.24	0.25	2.00	2.00

TABLE 3
$\lambda = 1.25$.

Scheme	Uniform Exper.	Uniform Theory	Classical Exper.	Classical Theory
Il'in fitting	0.91	1.00	2.00	2.00
Il'in averaged $a(x)$	1.16	1.00	2.00	2.00
Complete fitting	0.96	1.00	2.00	2.00
Gen. const. Il'in	0.91	1.00	1.08	1.00
Upwinding	0.87	1.00	0.99	1.00
Samarskii	0.91	1.00	2.00	2.00
Abrahamssson scheme 1	0.94	1.00	0.96	1.00
El-Mistakawy-Werle (Gen)	1.33	1.00	0.83	2.00

Proof.

Bound on S_1. By (III) and the boundedness of $\tilde{u}_\varepsilon$

$$|(b(x_i)-b_i^h)\tilde{u}_i| \leqq Ch. \tag{A.1}$$

Also $Y(x)$ is a smooth function and hence

$$\left| x_i a_0 \alpha_i \left[D_0 Y(x) - \frac{d}{dx} Y(x) \right] \right| \leqq Ch. \tag{A.2}$$

Finally, using (3.4) and (4.3), we obtain

$$|x_i(\alpha_i - 1)\tilde{a}_1 x_i u_0'(x_i)| \leqq Ch, \tag{A.3}$$

and by (3.11) and (4.3)

$$|\varepsilon^{(\lambda+1)/2} a_0 \zeta_i (\alpha_i - 1) v_1'(\zeta)| \leqq C\varepsilon^{\lambda/2} h[1 + (|\zeta_i| + 1)^\lambda] \leqq Ch. \tag{A.4}$$

Combining (A.1)-(A.4) we have

$$|S_1| \leqq Ch.$$

Bound on S_2. Since $Y(x)$ is smooth we have, using (V),

$$|S_2| \leqq C|\varepsilon_i^0| \leqq C(h+\varepsilon).$$

Bound on S_3. By (3.10), (4.3), we have using $|x_i| \leqq 1$ and $\varepsilon \leqq \varepsilon_0$

$$|S_3| \leqq C\varepsilon^{\lambda/2} h \varepsilon^{-1/2} |v_0'(\zeta_i)| \leqq Ch\varepsilon^{(\lambda-1)/2}(|\zeta_i|+1)^{\lambda-1} \leqq Ch(|x_i| + \varepsilon^{1/2})^{\lambda-1}.$$

Now if $\lambda<1$, $(|x_i|+\varepsilon^{1/2})^{\lambda-1}\leqq Ch^{\lambda-1}$ and if $\lambda>1$, $(|x_i|+\varepsilon^{1/2})^{\lambda-1}\leqq C$. Hence

$$|S_3|\leqq Ch^{\min(\lambda,1)}.$$

Bound on S_4. Consider first $x_i=h$. Then, using (3.4),

$$\text{(A.5)}\quad |S_4|\leqq Ch^2\frac{1}{2h}\left|\int_0^{2h}\int_h^y \frac{d^2}{dz^2}u_0(z)\,dz\,dy\right|\leqq Ch\left|\int_0^{2h}\int_h^y \zeta^{\lambda-2}\,dz\,dy\right|\leqq Ch^{\lambda+1}.$$

Now, if $x_i\geqq h$, we have using Lemma A.1 with $\zeta=x_i$, $r=h$, $l=1$, $A=1$, $B=0$,

$$\begin{aligned}|S_4|&\leqq C\left|x_i(|x_i|+h)h^{-1}\int_{x_i-h}^{x_i+h}\int_{x_i}^{z}u_0''(y)\,dy\,dz\right|\\ &\leqq C\left|x_i^2h^{-1}\int_{x_i-h}^{x_i+h}\int_{x_i}^{z}\frac{dy}{y^{2-\lambda}}\,dz\right| &\text{(A.6)}\\ &\leqq C|x_i^2h^{-1}h^2x_i^{\lambda-2}|\leqq Ch.\end{aligned}$$

Thus, combining (A.5) and (A.6), we get

$$|S_4|\leqq Ch.$$

Bound on S_5. Consider first $x_i=h$ $(\zeta_i=\rho)$. Then, using (4.3) and (3.10), we obtain

$$\begin{aligned}|S_5|&\leqq C\varepsilon^{\lambda/2}\rho(|D_0v_0(\zeta_i)|+|v_0'(\zeta_i)|)\\ &\leqq C\varepsilon^{\lambda/2}\left(\int_0^{2\rho}|v_0'(z)|\,dz+|v_0'(\rho)|\right)\\ &\leqq C\varepsilon^{\lambda/2}\left[\int_0^{2\rho}(z+1)^{\lambda-1}\,dz+\rho(\rho+1)^{\lambda-1}\right].\end{aligned}$$

Now, if $\lambda<1$, then $(z+1)^{\lambda-1}\leqq z^{\lambda-1}$, $0\leqq z<2\rho$ and hence

$$|S_5|\leqq C\varepsilon^{\lambda/2}\left[\int_0^{2\rho}z^{\lambda-1}\,dz+\rho\rho^{\lambda-1}\right]\leqq C\varepsilon^{\lambda/2}\rho^{\lambda}=Ch^{\lambda}.$$

If $\lambda>1$,

$$|S_5|\leqq C\varepsilon^{\lambda/2}[\rho(\rho+1)^{\lambda-1}]\leqq Ch.$$

Hence

$$|S_5|\leqq Ch^{\min(\lambda,1)}. \tag{A.7}$$

Suppose now that $x_i\geqq 2h$ $(\zeta_i\geqq 2\rho)$. Then, using (4.3), (3.10), Lemma A.1 with $A=B=1$, $k=2$, and $\zeta_i/(\zeta_i+1)\leqq 1$, we have

$$\begin{aligned}|S_5|&\leqq C\varepsilon^{\lambda/2}(\zeta_i+\rho)\rho^{-1}\left|\int_{\zeta_i-\rho}^{\zeta_i+\rho}\int_{\zeta_i}^{y}v_0''(z)\,dz\,dy\right|\\ &\leqq C\varepsilon^{\lambda/2}\zeta_i\rho^{-1}\int_{\zeta_i-\rho}^{\zeta_i+\rho}\int_{\zeta_i}^{y}(|z|+1)^{\lambda-2}\,dz\,dy\\ &\leqq C\varepsilon^{\lambda/2}\zeta_i\rho^{-1}\rho^2(\zeta_i+1)^{\lambda-2}\leqq Ch(x_i+\varepsilon^{1/2})^{\lambda-1}.\end{aligned}$$

If, $\lambda<1$ then,

$$|S_5|\leqq Chx_i^{\lambda-1}\leqq Ch^{\lambda}, \tag{A.8}$$

whereas if $\lambda>1$

$$|S_5|\leqq Ch. \tag{A.9}$$

Thus, combining (A.7)-(A.9),

$$|S_5| \leqq Ch^{\min(\lambda,1)}.$$

Bound on S_6. By analogous argument to that for S_5, using (3.11) instead of (3.10), we can show

$$|S_6| \leqq Ch.$$

Bound on S_7. By (4.2) and (3.10)

$$\begin{aligned}|S_7| &\leqq C\varepsilon^{\lambda/2}\rho(\zeta_i+\rho)(\zeta_i+1)^{\lambda-2} \leqq C^{(\lambda-1/2)}h(\zeta_i+1)^{\lambda-1}\\ &\leqq Ch(x_i+\varepsilon^{1/2})^{\lambda-1} \leqq Ch^{\min(\lambda,1)}.\end{aligned}$$

Bound on S_8. This is a more difficult result to prove, particularly for the case $x_i = h$. We shall show it first for $x_i \geqq 2h$ $(\zeta_i \geqq 2\rho)$.

By (4.2), (3.10), and Lemma 4.1 with $A = B = 1$, $k = 3$ we have

$$\begin{aligned}|S_8| &\leqq C\varepsilon^{\lambda/2}(1+\rho(\zeta_i+\rho))\left[\frac{1}{\rho}\int_0^1 \int_{\zeta_i+l\rho}^{\zeta_i+l\rho} \int_{\zeta_i}^{y} |v_0^{(3)}(z)|\,dz\,dy\,dl\right]\\ &\leqq C\varepsilon^{\lambda/2}(1+\rho\zeta_i)\rho(\zeta_i+1)^{\lambda-3} \leqq C\varepsilon^{\lambda/2}\rho(\zeta_i+1)^{\lambda-1} \leqq Ch^{\min(\lambda,1)}.\end{aligned} \tag{A.10}$$

To obtain a bound for $x_i = h$ $(\zeta_i = \tau)$ we must consider the cases $\rho \leqq 1$ and $\rho > 1$ separately. For $\rho \leqq 1$ we have using (4.2) and (3.10)

$$\begin{aligned}|S_8| &\leqq C\varepsilon^{\lambda/2}\sigma_i\rho|v_0^{(3)}(\theta\rho)| \quad \text{some } \theta, 0<\theta<1\\ &\leqq C\varepsilon^{\lambda/2}(1+\rho^2)\rho(\theta\rho+1)^{\lambda-3} \leqq C\varepsilon^{(\lambda-1)/2}h(\theta\rho+1)^{\lambda-1} \leqq Ch^{\min(\lambda,1)}.\end{aligned} \tag{A.11}$$

If $\rho > 1$ then again using (4.2) and (3.10)

$$\begin{aligned}|S_8| &\leqq C\varepsilon^{\lambda/2}(1+\rho^2)(|D_0(\zeta_i)| + |v_0''(\zeta_i)|)\\ &\leqq C\varepsilon^{\lambda/2}\rho^2\left[\frac{1}{\rho}\int_0^1 \int_{(1-y)\rho}^{(1+y)\rho} |v_0''(z)|\,dz\,dy + |v_0''(\rho)|\right]\\ &\leqq C\varepsilon^{\lambda/2}\rho\left[\int_0^1 \int_{(1-y)\rho}^{(1+y)\rho} (1+z)^{\lambda-2}\,dz\,dy + (1+\rho)^{\lambda-1}\right].\end{aligned}$$

Considering the cases $\lambda < 2$ and $\lambda > 2$ separately we can show that

$$|S_8| \leqq Ch^{\min(\lambda,2)}. \tag{A.12}$$

Thus, by (A.10)-(A.12),

$$|S_8| \leqq Ch^{\min(\lambda,1)}.$$

Bound on S_9. By analogous arguments to those for S_7 and S_8, using (3.11) instead of (3.10), we can show that

$$|S_9| \leqq Ch.$$

Bound on S_{10}. Using the definition of $T(s)$, in Theorem 3.1,

$$b_1 - T'(0)a_0 = b_1 - \frac{b_1 - \lambda a_1}{a_0}\,a_0 = \lambda a_1$$

and hence

$$\zeta_i^{\lambda+1}(b_1 - T'(0)a_0) = a_1\zeta_i^2\frac{d\zeta_i^\lambda}{d\zeta}.$$

Thus

$$\begin{aligned}&|\varepsilon^{(\lambda+1)/2}[a_1\zeta_i^2\alpha_i D_0\zeta_i^\lambda-\zeta_i^{\lambda+1}(b_1-T'(0)a_0)]|\\ &\leqq\left|\varepsilon^{(\lambda+1)/2}a_1\zeta_i^2\left[\alpha_i\left(D_0\zeta_i^\lambda-\frac{d\zeta_i^\lambda}{d\zeta}\right)+(\alpha_i-1)\frac{d\zeta_i^\lambda}{d\zeta}\right]\right|\\ &\leqq C\varepsilon^{(\lambda+1)/2}\rho\zeta_i^\lambda=Ch|x_i|^\lambda\leqq Ch\end{aligned}$$

using (4.3), Lemma 4.1 for $x_i\geqq 2h$, and explicit calculation for $x_i=h$. There remains

$$|\varepsilon^{(\lambda+1)/2}\zeta_i^2\alpha_i a_1 D_0(v_0(\zeta_i)-v_0'(\zeta_i))|=\left|\varepsilon^{1/2}\frac{a_0}{a_1}\zeta_i S_5\right|\leqq Ch^{\min(\lambda,1)}.$$

Hence

$$|S_{10}|\leqq Ch^{\min(\lambda,1)}.$$

Bound on S_{11}. We consider first $x_i=h$ $(\zeta_i=\rho)$. By explicit calculation of the differences, using (3.10) and (3.11) we can show that

$$|S_{11}|\leqq Ch. \tag{A.13}$$

To prove the result for $x_i\geqq 2h$ $(\zeta_i\geqq 2\rho)$ we proceed in a manner similar to that in § 3. Consider the cases $x_i\geqq\varepsilon^s$ and $x_i\leqq\varepsilon^s$ separately, where we shall choose

$$0<s<\frac{1}{\lambda+2}<\frac{1}{2}. \tag{A.14}$$

If $x_i\geqq\varepsilon^s$ we have $\zeta_i\geqq\varepsilon^{s-1/2}$ and hence $\zeta\to\infty$ as $\varepsilon\to 0$. Using the asymptotic properties of v_0 and v_1 and integrating explicitly,

$$\begin{aligned}|S_{11}|\leqq C\varepsilon^{\lambda/2+1}&\left[\zeta_i^3\frac{1}{\rho}\int_{\zeta_i-\rho}^{\zeta_i+\rho}\left|\frac{d}{dz}(v_0(z)-dm(0)z^\lambda)\right|dz\right.\\ &\left.+\frac{\zeta_i^2}{\rho}\int_{\zeta_i-\rho}^{\zeta_i+\rho}\left|\frac{d}{dz}(v_1(z)-dm(0)T'(0)z^\lambda)\right|dz+\zeta_i^2\zeta_i^{\lambda-2}+\zeta_i\zeta_i^{\lambda-1}\right]\\ \leqq Cx_i^\lambda\varepsilon\frac{\zeta_i}{\rho}&[(1+\omega)^{\lambda-2}-(1-\omega)^{\lambda-2}+(1+\omega)^{\lambda-1}-(1-\omega)^{\lambda-1}]\\ &+C\varepsilon x_i^\lambda\quad\text{where }\omega=\rho/\zeta_i.\end{aligned}$$

Thus, Taylor expanding and using $\omega\leqq\frac{1}{2}$, we have

$$(1+\omega)^{\lambda-2}-(1-\omega)^{\lambda-2}\leqq C\omega,$$

and hence

$$|S_{11}|\leqq C\varepsilon. \tag{A.15}$$

There remains the case $x_i\leqq\varepsilon^s$. Using (3.10) and (3.11) we have

$$\begin{aligned}|S_{11}|\leqq C\varepsilon^{\lambda/2+1}&\left[\frac{\zeta_i^3}{2\rho}\int_{\zeta_i-\rho}^{\zeta_i+\rho}(z+1)\lambda^{\lambda-1}\,dz\right.\\ &\left.+\frac{\zeta_i^2}{2\rho}\int_{\zeta_i-\rho}^{\zeta_i+\rho}(z+1)^\lambda\,dz+\zeta_i^2(\zeta_i+1)^\lambda+\zeta_i(\zeta_i+1)^{\lambda+1}\right].\end{aligned}$$

Integrating explicitly, rewriting in terms of $1+\omega$, where $\omega=\rho/(\zeta_i+1)$ and expanding as before we can show

(A.16) $|S_{11}| \leqq C\varepsilon^{\lambda/2+1}[\zeta_i^2(\zeta_i+1)^\lambda+\zeta_i(\zeta_i+1)^{\lambda+1}] \leqq C(x_i+\varepsilon^{1/2})^{\lambda+2} \leqq C\varepsilon^{s(\lambda+1)} \leqq C\varepsilon$

the latter by (A.14).

Combining (A.13), (A.15), and (A.16) we have

$$|S_{11}| \leqq C(h+\varepsilon).$$

LEMMA A.3. *Let T_1 to T_6 be defined as in § 4, then*

$$T_1 \leqq C(\varepsilon+h^2), \qquad T_2 \leqq Ch^{\min(\lambda,1)},$$
$$T_3 \leqq Ch, \qquad T_4 \leqq Ch^{\min(\lambda,2)},$$
$$T_5 \leqq Ch^\lambda, \qquad T_6 \leqq Ch^{\min(\lambda,1)}.$$

Proof. Since $Y(x_i)$ is smooth

$$|T_1| \leqq C\varepsilon_0^0 \leqq C[\varepsilon+h(a(0)+h)] \leqq C(\varepsilon+h^2).$$

Now, using (3.9) and $a(0)=0$,

$$|T_2| = \left|(a(0)+Ch)\frac{1}{2h}\int_{-h}^{h}(|z|+\varepsilon^{1/2})^{\lambda-1}\,dz\right| \leqq C\left|\int_0^h (z+\varepsilon^{1/2})^{\lambda-1}\,dz\right|.$$

If $\lambda>1$, we have

$$|T_2| \leqq Ch \max_{0\leqq z\leqq h}(z+\varepsilon^{1/2})^{\lambda-1} \leqq Ch,$$

and if $\lambda<1$

$$|T_2| \leqq C\left|\int_0^h z^{\lambda-1}\,dz\right| \leqq Ch.$$

Thus

$$|T_2| \leqq Ch^{\min(\lambda,1)}.$$

Since $\tilde{u}_\varepsilon(x_0)$ is bounded, we have, by (III),

$$|T_3| \leqq Ch.$$

Using (4.2), $a(x_0)=0$, and (3.10), we get

$$|T_4| \leqq C\varepsilon^{\lambda/2}\rho^2\rho^{-1}\left|\int_0^1\int_0^{y\rho}(|z|+1)^{\lambda-2}\,dz\,dy\right|.$$

Now in a similar manner to T_2 we have

$$|T_4| \leqq Ch^\lambda \quad \text{for } \lambda<2.$$

If $\lambda>2$

$$|T_4| \leqq C\varepsilon^{\lambda/2}\rho^2(\rho+1)^{\lambda-2}.$$

Considering the case $\rho\leqq 1$ and $\rho\geqq 1$ separately we can show

$$|T_4| \leqq Ch^{\min(\lambda,2)}.$$

To bound T_5 we again write it in integral form, use (3.10) and $(|\omega|+1)^{-1}\leqq 1$, and integrate explicitly to give

$$|T_5| \leqq C\varepsilon^{\lambda/2}\rho^{-1}\left|\int_0^1\int_{-y\rho}^{+y\rho}\int_0^z|\omega|^{\lambda-1}\,dz\,dy\right| \leqq C\varepsilon^{\lambda/2}\rho^{-1}\rho^{\lambda+1}=Ch^\lambda.$$

Using similar arguments to those in T_4 and T_5 we may show

$$|T_6| \leqq Ch^{\min(\lambda,1)}.$$

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