NUMERICAL AND APPLIED MATHEMATICS
C. Brezinski (editor)
J.C. Baltzer AG, Scientific Publishing Co. © IMACS, 1989
pp. 681-686

FLOW CONFORMING ITERATIVE METHODS FOR CONVECTION DOMINATED FLOWS

Paul A. FARRELL
Department of Mathematical Sciences, Kent State University, Kent, OH 44242, U.S.A.

Abstract: We examine a number of variants of the Gauss-Seidel method for the solution of linear systems arising from covection dominated flow problems. We show that, because of the dominant role played by the characteristics in such problems, the rate of convergence depends critically on following the flow direction and thus the spectral radius is an inappropriate measure in this case. We introduce the Symmetric Gauss-Seidel method and demonstrate that it performs well for one and two dimensional examples. We also discuss a number of other flow conforming methods.

1. Introduction

Convection diffusion problems are a class of problems, whose numerical solution exhibits significant difficulties, particularly when the diffusion coefficient is small. This case is called *convection dominated flow.* The difficulties in obtaining the solution are due to the prescence of sharp boundary and/or interior layers, which degrade the accuracy of standard difference schemes. One approach to solving these problems is to employ a uniform mesh and either an upwind scheme or an *(exponentially) fitted* scheme which attempts to model the boundary layer or interior layer accurately [1][2][3][4]. All these methods respect the natural direction of flow in the problem, as given by the characteristics of the differential equation. If the difference scheme does not conform to this inherent direction in the problem, a satisfactory solution will not be obtained without considerable computational effort. An obvious question which arises is whether this directionality in the solution need also be conformed to by an iterative method used to solve the linear systems, arising from the difference scheme.

In this work, we consider the linear systems which arise as a result of the finite difference discretization, on a uniform mesh, of linear singularly perturbed differential equations in one and two dimensions. Such problems are theoretical models for the physical phenomena of convection dominated flow. We shall show that considerable benefits are derived by using iterative schemes which conform to the natural direction of the problem. In fact, when the diffusion coefficient is sufficiently small, a correctly conforming method will converge after only a few iterations. For this reason, the spectral radius, which measures the asymptotic rate of convergence, is not the most suitable measure of the appropiateness of a scheme, since it is essentially direction independent.

2. One Dimensional Problems

We first analyse a sample one dimensional problem :

$$-\epsilon u''(x) + p(x)u'(x) + r(x)u(x) = f(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = 0.$$

We assume that $r(x) \geq 0$ and that $r(x)$ and $p(x)$ have no simultaneous zeros. In this case, ϵ plays the role of the diffusion coefficient and we shall assume that the flow is convection dominated, that is $\epsilon << 1$, whereas $p(x)$ is of order 1 over most of the region under consideration. The behaviour of the problem will depend on the zeros, if any, of $p(x)$ in the interval $[0,1]$. If $p(x) > 0 (< 0)$ the flow is to the right (left) and there is an exponential boundary layer at the left (right). We reflect this fact by discretizing using an upwinded scheme in the forward (backward) direction or an exponentially fitted scheme. On a uniform grid of size $h = 1/(n+1)$ the upwinded difference scheme is :

$$-b_i u_{i-1} + a_i u_i - c_i u_{i+1} = f_i, \quad 1 \leq i \leq n, \tag{1}$$

$$u_0 = u_n = 0,$$

where the coefficients a_i, b_i, c_i are given by

$$\begin{aligned} b_i &= \frac{\epsilon}{h^2} + \frac{1}{2h}(|p_i| + p_i) \geq 0, \\ a_i &= \frac{2\epsilon}{h^2} + \frac{1}{h}|p_i| + r_i \geq 0, \\ c_i &= \frac{\epsilon}{h^2} + \frac{1}{2h}(|p_i| - p_i) \geq 0, \end{aligned}$$

and $p_i = p(x_i), r_i = r(x_i)$ and $f_i = f(x_i)$. An exponentially fitted scheme would also be of the form (1), with the coefficients a_i, b_i, c_i given by

$$\begin{aligned} b_i &= \frac{\epsilon_i}{h^2} + \frac{1}{2h}(|p_i| + p_i) \geq 0, \\ a_i &= \frac{2\epsilon_i}{h^2} + \frac{1}{h}|p_i| + r_i \geq 0, \\ c_i &= \frac{\epsilon_i}{h^2} + \frac{1}{2h}(|p_i| - p_i) \geq 0, \end{aligned}$$

and

$$\epsilon_i = \epsilon\sigma(\frac{|p_i|h}{\epsilon}), \quad \text{where} \quad \sigma(z) = \frac{z}{e^z - 1}.$$

The analysis and results are similar in both cases. However, it is clear that $\epsilon_i < \epsilon$. Thus the exponentially fitted method behaves like an upwinded scheme with smaller ϵ. This makes the difference scheme more accurate than the upwinded scheme. In physical terms, we can view the use of an upwinded scheme, rather than the (unstable) centered difference scheme, as introducing *artificial viscosity* into the approximation. This normally leads to diffusion of the boundary layers. Exponentially fitted schemes effectively seek to use precisely the correct amount of artificial viscosity. For simplicity, we shall do our analysis in terms of the upwinded scheme. However it is clear that analogous results exist for exponentially fitted schemes. In fact, the error reduction involved will be even faster, since, in general, this is proportional to ϵ for the upwinded scheme and ϵ_i for the exponentially fitted scheme.

In either case the matrix $A \in R^{n\times n}$, of the difference scheme is a tridiagonal matrix. It is easily shown that this matrix is an *irreducibly diagonally dominant M-matrix* [5, p23] under the above assumptions, and thus that $A^{-1} \geq 0$, where the inequality is meant component-wise.

Now consider the matrix splitting of A, into a diagonal D, a lower triangular L, and an upper triangular U matrix, $A = D - L - U$. The forward (backward) point Gauss-Seidel methods, which we shall call FGS and BGS respectively, may now be written as

$$M_f u^{k+1} = N_f u^k + f, \text{ where } M_f = D - L,\ N_f = U,$$

$$M_b u^{k+1} = N_b u^k + f, \text{ where } M_f = D - U,\ N_f = L.$$

In either case, the splitting $A = M - N$ is a *regular splitting* of the matrix A and $\rho(M^{-1}N) < 1$ [5, Theorem 3.14]. Thus both iterative methods are convergent with the same *asymptotic convergence rate*, since $\rho(M_f^{-1}N_f) = \rho(M_b^{-1}N_b) = \rho(D^{-1}(L+U))^{1/2}$. However although the asymptotic convergence rate is a good measure of the ultimate rate of convergence of these methods, Han et al. [6] [7] have shown that the initial rate of convergence for the two methods differ considerably. It is well known that, in general, the error, as measured in the L_∞ norm, may increase initially. For this reason, in [6] [7], the initial error reduction was calculated by considering the matrix norm of the iteration matrix, $||M^{-1}N||_\infty$.

2.1 Non-turning point problems

If $p(x)$ is of one sign, that is $\hat{p} \geq p(x) \geq \check{p} > 0$ or $\hat{p} \leq p(x) \leq \check{p} > 0$, the problem is said to be a non-turning point problem. In this case, it is shown in [6] [7] that for mesh spacings h which satisfy $\epsilon \leq .8\hat{p}h$ the error reduction for the *forward* Gauss-Seidel (FGS) is given by

$$||M^{-1}N||_\infty \leq \frac{2\epsilon}{\check{p}h^2}, \tag{2}$$

and thus if $\epsilon < \check{p}h^2/2$ the error reduction is monotonic. The analysis is performed by writing

$$M^{-1}N = (I - D^{-1}L)^{-1}D^{-1}U$$

and then estimating the norms of $D^{-1}L$ and $D^{-1}U$. These are respectively strictly lower and strictly upper triangular matrices, with a non-zero sub(super)-diagonal whose entries are given by

$$\alpha_i = \frac{\delta + (|p_i| + p_i)/2}{2\delta + |p_i| + r_i h},\quad \beta_i = \frac{\delta + (|p_i| - p_i)/2}{2\delta + |p_i| + r_i h}, \tag{3}$$

where $\delta = \epsilon/h$. Note that (2) implies very rapid convergence if $\epsilon \ll h$. The *backward* Gauss-Seidel (BGS), on the other hand does not exhibit this property. Conversely, if $p(x) \leq 0$, BGS exhibits rapid convergence under appropriate conditions and FGS does not. Thus, if one solves the linear system in the natural direction given by the characteristics, one achieves rapid convergence. We shall see later that similar problems arise if we assume $p(x)$ has one or more zeros.

We shall now consider another iterative method which we shall call Symmetric Gauss-Seidel (SGS). This consists of an application of FGS, followed by an application of BGS given by:

$$M_f u^{k+1/2} = N_f u^k + f, \quad \text{where } M_f = D - L, N_f = U,$$
$$M_b u^{k+1} = N_b u^{k+1/2} + f, \quad \text{where } M_f = D - U, N_f = L.$$

Thus the iteration matrix is $M_b^{-1}N_bM_f^{-1}N_f$. This may be considered as a special case of the Symmetric Successive Over Relaxation Method (SSOR) with relaxation parameter $\omega = 1$. Thus, as remarked in [8], provided we store intermediate results, it requires no more calculations per iteration than the FGS. Note that the FGS sweep is guaranteed to reduce the error, but the BGS might increase it again. We shall show that this is not the case, although the error reduction is only as good as that for FGS.

Theorem 1: If the mesh spacing h satisfies $\epsilon < .8\hat{p}h$ the error reduction for the Symmetric Gauss-Seidel (SGS) is given by

$$||M^{-1}N||_\infty \leq \frac{2\epsilon}{\check{p}h^2}.$$

Proof: This is similar to that of [6] [7]. To simplify notation we write $||.||$ for $||.||_\infty$.

$$\begin{aligned} ||M^{-1}N||_\infty &= ||M_b^{-1}N_bM_f^{-1}N_f|| \\ &= ||I - D^{-1}U||||D^{-1}L||||M_f^{-1}N_f||. \end{aligned} \tag{4}$$

Using the fact that $D^{-1}U \leq D_0^{-1}U_0$ and $D^{-1}L \leq D_0^{-1}L_0$, where $A_0 = D_0 - L_0 - U_0$ is the matrix corresponding to A with $r_i = 0, \forall i$. Using similar arguments to [6] [7], we have

$$||M_f^{-1}N_f|| \leq \frac{2n\delta}{\hat{p}}\frac{2\delta+\hat{p}}{\delta}\frac{\delta}{2\delta+\check{p}}, \tag{5}$$

$$||D^{-1}L|| \leq ||D_0^{-1}L_0|| = \frac{\delta+\hat{p}}{2\delta+\hat{p}}, \tag{6}$$

$$\begin{aligned} ||I - D^{-1}U|| &\geq 1 - ||D^{-1}U|| \geq 1 - ||D_0^{-1}U_0|| \\ &= 1 - \frac{\delta}{2\delta+\check{p}} = \frac{\delta+\check{p}}{2\delta+\check{p}}. \end{aligned} \tag{7}$$

Using (7), it follows that,

$$||(I - D^{-1}U)^{-1}|| = 1/||I - D^{-1}U|| \leq \frac{2\delta+\check{p}}{\delta+\check{p}}. \tag{8}$$

Combining (4), (8), (6) and (5) we get

$$||M^{-1}N||_\infty \leq \frac{2\delta+\check{p}}{\delta+\check{p}}\frac{\delta+\hat{p}}{2\delta+\hat{p}}\frac{2n\delta}{\hat{p}}\frac{2\delta+\hat{p}}{\delta}\frac{\delta}{2\delta+\check{p}} \leq \frac{2n\delta}{\check{p}}. \quad \square$$

Thus SGS achieves the same rate of convergence as the FGS, which solves in the characteristic direction. It is easily shown that SGS also achieves the same rate of convergence as BGS for the problem with $p(x) \le \tilde{p} < 0$. This is illustrated in Figure 1 and Table 1, which are for the problem

$$-\epsilon u''(x) - u'(x) = -2, \quad 0 < x < 1, \qquad (9)$$

$$u(0) = -1, \quad u(1) = 1,$$

with $\epsilon = .000001$. It should be noted that this problem

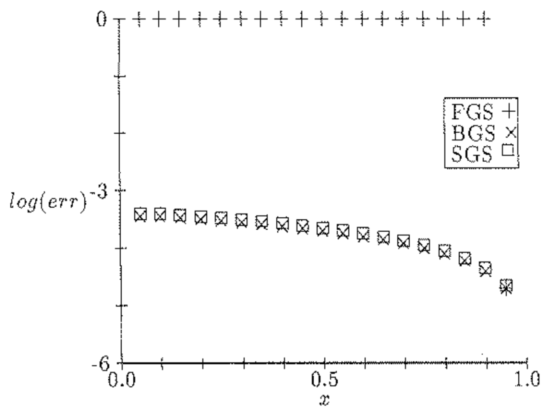

Figure 1: Log of error after first iteration for problem (9)

	Method		
1/h	FGS	BGS	SGS
20	20	2	2
100	101	3	3

Table 1: Problem (9), Iterations to achieve error 10^{-5}

has an exact solution $u(x) = 2x - 1$ and thus exhibits no boundary layers. For all the results exhibited, the initial guess is chosen so that the initial error at each point is 1. Figure 1 gives the log of the error after one iteration at each point of a grid with $h = 1/20$. For this problem, we can choose $b = .5 \times 10^{-4}$ and expect an error of approximately 10^{-4} after the first iteration. The FGS method shows almost no error reduction except at the right-most point where it is 10^{-4}. A similar reduction propagates across the grid at subsequent iterations, requiring a number of iterations of the order of the number of grid-points to reach the left-hand boundary and thus reduce the L_∞ norm significantly. This is illustrated in Table 1.

An alternative method, to exponential fitting, for modeling the solution accurately in the boundary layer, is to refine the mesh there. The rapid convergence results above do not hold in this case, since they depend crucially on the ratio of ϵ to h. However, even in this case, Symmetric Gauss Seidel converges faster that a Gauss Seidel scheme performed in the direction opposite to the flow.

2.2 Turning-point problems

If $p(x)$ has one or more zeros, the problem is said to be a turning point problem. We shall, for convenience, assume that the zero x^* of $p(x)$ lies in (x_k, x_{k+1}). There are two interesting cases here. The first is when $p(0) > 0, p(1) < 0$ and $p'(x) < 0$ in $[0,1]$. In this case the zero x^* corresponds to a *sink* with the characteristic flow towards x^*. For this case, Han et al. [6] [7], used a method which swept to the right while $p_i > 0$ and then sweeps to the left while $p_i < 0$.

The second interesting case is when $p(0) < 0, p(1) > 0$ and $p'(x) > 0$ in $[0,1]$. In this case the zero x^* of $p(x)$ corresponds to a *source* and the characteristic flow is directed outward from x^*. In [6] [7], a special block Gauss-Seidel method is proposed, in which the block of D around the turning-point is a 2×2 block, involving the k-th and $(k+1)$-th rows and columns and the others are 1×1. In both the sink and source cases, it is shown in [6] [7] that the special scheme proposed satisfies a result similar to that for the case where $p(x)$ is of one sign. All of these special schemes correspond to a reordering of the mesh-points and it is in this context that it is generalized in [6] to the two-dimensional case. An alternative method for the *source* problem, called the Flow Directed Point Iteration (FDPI) method, is proposed in [7]. This does not involve the 2×2 block. In it the mesh is swept from left to right, going from $k+1$ to n, and then swept from right to left, going from k to 1. The error analysis for this method is not as favourable. However, it is easier to generalize to two dimensions.

It should be noted that neither FGS nor BGS satisfies a monotonicity or rapid convergence result. The SGS, on the other hand, performs well in both cases. This is illustrated in Figure 2 and Table 2 for the sink problem:

$$-\epsilon u''(x) - (x - 1/2)u'(x) + 3/2u(x) = 1, \quad 0 < x < 1, \quad (10)$$

$$u(0) = -1, \quad u(1) = 1.$$

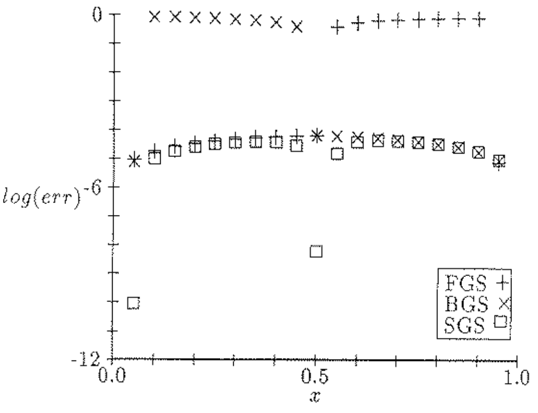

Figure 2: Log of error after first iteration for problem (10)

	Method		
1/h	FGS	BGS	SGS
20	10	10	2
100	50	50	3

Table 2: Problem (10), Iterations to achieve error 10^{-5}

We can prove a theorem similar to Theorem 1, for this case also.

Theorem 2: Let (1) be a sink problem, with the zero of $p(x)$,x^*, in the interval (x_l, x_{l+1}) and hence, $\hat{p} \geq p(x) \geq \mu h$. Then, if ϵ satisfies $\epsilon \leq \min(.8\hat{p}h, \mu h^3/2)$, the L_∞ norm of the error e of the SGS method satisfies:

$$||e^{m+1}|| \leq \frac{4\epsilon}{\mu h^3}||e^m||\,.$$

Proof: Consider first the forward sweep. The new error $e^{m+1/2}$ is given by:

$$(I - D^{-1}L)e^{m+1/2} = D^{-1}Ue^m \stackrel{\text{def}}{=} \hat{e}\,,$$

where the entries of $D^{-1}L$ and $D^{-1}U$ are given by (3) and

$$\hat{e} = (\beta_1 e_2^m, \beta_2 e_3^m, \ldots, \beta_{n-1}e_n^m, 0)^T.$$

Divide the system into two parts

$$\begin{pmatrix} B_+ & 0 \\ B_0 & B_- \end{pmatrix}\begin{pmatrix} e_+^{m+1/2} \\ e_-^{m+1/2} \end{pmatrix} = \begin{pmatrix} \hat{e}^+ \\ \hat{e}^- \end{pmatrix}$$

where B_+ is an $l \times l$ matrix and B_- is an $(n-l)\times(n-l)$ matrix. Then

$$B_+ e_+^{m+1/2} = \hat{e}^+ \quad (11)$$

$$B_+ e_-^{m+1/2} = \hat{e}^- - B_0 e_+^{m+1/2} \stackrel{\text{def}}{=} \bar{e} \quad (12)$$

where $B_0 e_+^{m+1/2} = (-\alpha_{l+1}e_l^{m+1/2}, 0, \ldots, 0)^T$. The analysis for (11) follows as in [6] [7] and Theorem 1, except $\check{p}$ is replaced by μh. Thus for $\epsilon \leq .8\hat{p}h$,

$$||e_+^{m+1/2}|| \leq \frac{2\epsilon}{\mu h^3}||e^m||. \quad (13)$$

For (12), writing $B_- = I - D_- L_-$ and letting D_0 and L_0 be as in Theorem 1 we have

$$||D_-^{-1}L_-|| \leq ||D_0^{-1}L_0|| = \frac{\delta}{2\delta + \check{p}} < 1.$$

Hence

$$\begin{aligned} ||(I - D_-^{-1}L_-)^{-1}|| &\leq ||(I - D_0^{-1}L_0)^{-1}|| = 1/||(I - D_0^{-1}L_0)|| \\ &= \frac{2\delta + \check{p}}{\delta + \check{p}} \leq 2\,. \end{aligned} \quad (14)$$

Now, using (13) and $\epsilon \leq \mu h^3/2$,

$$\bar{e}_{l+1} = \frac{\delta + |p_{l+1}|}{2\delta + |p_{l+1}| + r_{l+1}h}e_{l+2}^m + \frac{\delta}{2\delta + |p_{l+1}| + r_{l+1}h}e_l^{m+1/2}\,,$$

$$|\bar{e}_{l+1}| \leq \frac{2\delta + |p_{l+1}|}{2\delta + |p_{l+1}| + r_{l+1}h}||e^m|| \leq ||e^m||\,, \quad (15)$$

$$|\bar{e}_i| \leq \frac{\delta + |p_i|}{2\delta + |p_i| + r_i h}|e_{i+1}^m| \leq ||e^m||\,. \quad (16)$$

Thus combining (14), (15), (16) we get

$$||e_-^{m+1/2}|| \leq 2||e^m||\,. \quad (17)$$

Combining (13) and (17) with similar estimates for the backward sweep gives

$$||e^{m+1}|| \leq \frac{4\epsilon}{\mu h^3}||e^m||\,. \quad \square$$

Figure 3 and Table 3 illustrate the results for a source problem:

$$-\epsilon u''(x) + (x - 1/2)u'(x) + u(x) = 0, \quad 0 < x < 1, \quad (18)$$

$$u(0) = -1, \quad u(1) = 1.$$

In sink and source cases it is clear that for FGS and BGS the

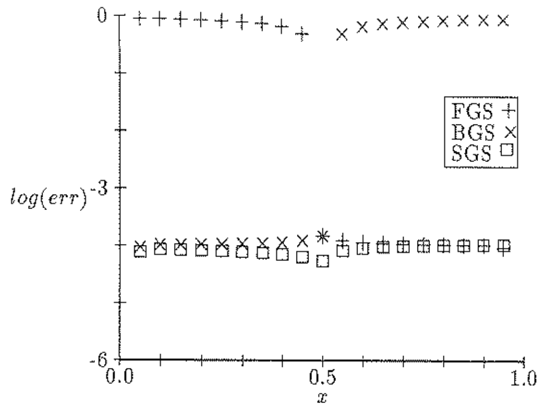

Figure 3: Log of error after first iteration for problem (18)

	Method		
1/h	FGS	BGS	SGS
20	11	11	2
100	52	52	3

Table 3: Problem (18), Iterations to achieve error 10^{-5}

error is reduced rapidly where we are solving in the correct direction whereas we reduce the error significantly at only one point where we are solving in the wrong direction. SGS achieves its effect by reducing the error significantly at each point in either the forward or backward sweep.

Figure 4 and Table 4 illustrate the results for a problem with sources and sinks:

$$-\epsilon u''(x) - (x-1/4)(x-1/2)(x-3/4)u'(x) + u(x) = 0, \ 0 < x < 1, \quad (19)$$

$$u(0) = -1, \quad u(1) = 1.$$

	Method		
1/h	FGS	BGS	SGS
20	6	7	2
100	26	28	3

Table 4: Problem (19), Iterations to achieve error 10^{-5}

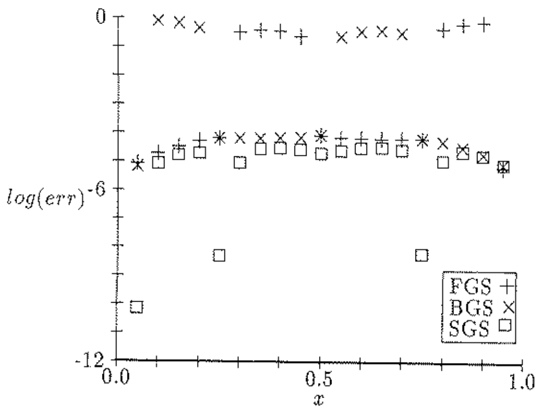

Figure 4: Log of error after first iteration for problem (19)

3. Two Dimensional Problems

We consider the model two-dimensional convection-diffusion problem on the unit square Ω:

$$-\epsilon\Delta u+p(x,y)u_x+q(x,y)u_y+r(x,y)u=f(x,y),\ (x,y)\in\Omega \quad (20)$$
$$u(x,y)=0,\ \ (x,y)\in\delta\Omega.$$

We consider five-point upwinded difference schemes for this problem.

3.1 Gauss-Seidel on Admissible Partitions

In [6], it is proved that, if the mesh-points are re-ordered to satisfy an *admissibility* condition, which is related to the direction of flow at each point, then rapid convergence of the type discussed in the previous section can be achieved by a block Gauss-Seidel method. The admissibility condition requires that the mesh points be divided into an *ordered partition* $\mathcal{N}_1,\ldots,\mathcal{N}_m$, such that if $P\in\mathcal{N}_k$ and $Q\in\mathcal{N}_l$, $k<l$, are two neighbouring points then the coefficient of u_Q in the difference scheme for u_P is $-\epsilon$. This is a two-dimensional generalization of the one-dimensional ordering criteria above. Details are also given there of a method of finding an admissible partition. To be practical the diagonal matrices D_i of such an iterative method must be easily inverted. Provided the coefficients $p(x,y)$ and $q(x,y)$ are such that their curves of zeros are a positive distance δ apart, and $h<\delta/2$, then an admissible partition can be found where each of the matrices D_i are either 2×2 or 1×1.

3.2 Symmetric Line Gauss-Seidel

These methods, while giving an excellent rate of convergence, are not easy to implement. Also asymptotic error results (c.f. [5, Theorem 3.15, p199]) suggest that block methods, such as line Gauss-Seidel should achieve better asymptotic rates of convergence. If the flow is complex, however, a re-ordering to satisfy the admissibility condition is not possible for a line Gauss-Seidel method. This is because the flow at each point of a line is not necessarily in the same direction. In practice as is shown in Table 5, the forward (backward) line Gauss-Seidel FLGS (BLGS) also show the disadvantages of FGS (BGS). Motivated by these facts, we propose a Symmetric Line Gauss-Seidel (SLGS) method which is defined analogously to SGS. The matrices D_i in this case are tri-diagonal and the systems are thus easily solved using exact methods. The matrix of the difference scheme, A, has the form:

$$\begin{pmatrix} D_1 & U_1 & 0 & \ldots & 0 \\ L_2 & D_2 & U_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & L_n & D_n \end{pmatrix}$$

where D_i is a square tri-diagonal matrix of order n, the number of mesh points on each line in the x direction, and L_i and U_i are diagonal matrices of order n. Consider the splitting of A into a block tri-diagonal $D=\text{diag}\,(D_1,D_2,\ldots,D_n)$, a block lower triangular L, and an block upper triangular U matrix, $A=D-L-U$. The SLGS is then defined as :

$$M_f u^{k+1/2}=N_f u^k+f,\quad \text{where } M_f=D-L, N_f=U,$$
$$M_b u^{k+1}=N_b u^{k+1/2}+f,\quad \text{where } M_f=D-U, N_f=L.$$

In practice, under certain restrictions on the coefficients, this method achieves rapid convergence results of the type seen for SGS. These restrictions are essentially similar to the ones above. They exclude simultaneous zeros of $p(x,y)$ and $q(x,y)$ and hence spiral and radially symmetric flows. Again, to prevent the cost of this method from being double that of a FLGS or BLGS method, it is necessary to save intermediate results for use in the next half iteration. Results for this method are illustrated in Table 5. For problems 4 and 5,

$$w=[(x-.5)^2+(y-.5)^2]^{1/2}.$$

The characteristics of problems 4 and 5 are expanding spirals and contracting spirals respectively. These are considerably more complex flows than have been treated theoretically.

Problem	Coefficients	1/h	LGS Method F	B	S
1	$p(x)=3x-y-1$	20	2	20	2
	$q(x)=1$	40	2	40	2
2	$p(x)=3x-y-1$	20	23	23	4
	$q(x)=4x+2y-3$	40	40	40	5
3	$p(x)=3x-y-1$	20	13	13	3
	$q(x)=-x-3y+2$	40	26	26	3
4	$p(x)=2(x-.5)-(y-.5)w$	20	13	13	5
	$q(x)=(x-.5)w+2(x-.5)$	40	23	23	5
5	$p(x)=-2(x-.5)-(y-.5)w$	20	11	11	2
	$q(x)=(x-.5)w-2(x-.5)$	40	21	21	3

Table 5: Iterations to achieve error 10^{-5} for problem (20) with $r(x)=.5$ and $f(x)=0$.

3.3 Alternative Flow Conforming Methods

Han et. al [6],[7] also propose a number of alternative methods, which are simpler to implement than those based on the admissible partition. One of these, is the natural generalization of FDPI to two dimensions, in which the mesh points are divided into four sets, one for each direction of flow NE, NW, SE and SW. These are then swept in the natural flow conforming directions.

Another, Flow Directed Horizontal Iteration (FDHI) is a variant of the line Gauss-Seidel method in which only the horizontal component of the flow $p(x, y)$ is considered in choosing the solution direction. The points on each horizontal line are divided into two sets, one where the flow is to the right, which is swept left to right, and one where the flow is to the left, which is swept right to left. This is essentially an improvement on SLGS since only the part part of SLGS which gives the greatest reduction in error is performed. If intermediate results are not stored in SLGS, each iteration would involve just half the work of a SLGS iteration. An analogous Flow Directed Vertical Iteration (FDVI) can be defined similarly. These can be combined to give FDHVI, which consists of an iteration of FDHI followed by an iteration of FDVI.

In general, the admissible partition based method of [6] [7], performs better than the other alternatives but at a cost of greater implementational complexity. In all case these methods are significantly better than the FLGS or BLGS used in either the x or y directions, and illustrate well the importance of conforming to the natural flow direction in the problem not only in formulating the difference equations but also in solving them iteratively.

References

[1] E.P.Doolan, J.J.H.Miller, W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers (Boole Press, Dublin, 1980).

[2] R.B.Kellogg, A.Tsan, Analysis of some Finite Difference Approximations for a Singular Perturbation Problem without Turning points, Math. Comp., 32, (1978) pp 1025-1039.

[3] P. A. Farrell, Sufficient conditions for the uniform convergence of a difference schemes for a singularly perturbed turning point problem, SIAM J. Numer. Anal., 25, (1988) pp618-643.

[4] P.A.Farrell, E.C.Gartland Jr., A Uniformly Convergent Method for a Turning Point Problem, to appear in: J.J.H.Miller (ed.) , Proceedings of BAIL V (Boole Press, Dublin, 1988).

[5] R.S.Varga, Matrix Iterative Analysis, (Prentice-Hall, Englewood Cliffs, NJ, 1962).

[6] H.Han, V.P.Il'in, R.B.Kellogg, W.Yuan, Flow Directed Iteration for Convection Dominated Flow, to appear in: J.J.H.Miller (ed.) , Proceedings of BAIL V (Boole Press, Dublin, 1988).

[7] H.Han, V.P.Il'in, R.B.Kellogg, W.Yuan, Analysis of Flow Directed Iterations, to appear.

[8] W.Niethammer, Relaxation bei Komplexen Matrizen, Math. Zeit., 86, (1964) pp 33-40.