

LIMITING DISTRIBUTIONS OF SOME CIRCUIT  
SWITCHED NONHIERARCHICAL NETWORKS

by

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## ABSTRACT

Among the two major types of networks, packet switched and circuit switched networks, we discuss the second type of networks which are often used in telephone traffic. By providing simple examples we show how one can obtain the exact limiting distribution of the network. This can be used to obtain the exact end to end blocking probability of an incoming call along with the system utilization statistics. We provide some results for three node symmetric and nonsymmetric networks. The generalizations to higher numbers of nodes is analogous. Some examples of the system utilization and blocking probability are provided under various input and holding time parameters.

## INTRODUCTION

A communication network consists of a set of nodes which are pairwise connected by links. Requests arrive randomly into a node to send information (voice, data, image etc) to another node through the use of a link or more than one link should the need arise. The telephone networks are mainly for sending voice data between nodes. These types of networks reserve a channel on the link during the conversation and release it after the conversation ends. The other types of networks do not reserve the channel for a time interval. It simply sends packets of information in any way it can. The first type of such networks are called circuit switch network and the latter type are called packet switch networks. We will refer to the input traffic as calls. These networks are explained in elaborate detail by [6].

The packet switching networks came into existence much later. The earliest major packet switching network in the U.S., which is still in use is known as ARPANET. Some other major networks are TELENET, TYMNET, DNA, SNA. These and a few others are explained in detail by [4].

Each network has associated with it a topology. The topology of the network consists of the set of nodes, the set of links, and the routing paths that the nodes are to use for sending the information. In some networks some nodes are capable of redirecting the traffic while others are not. In such cases, the network is termed as an hierarchical network. When all nodes are capable of performing the redirecting independently, the network is called a nonhierarchical network. Some recent issues for hierarchical networks are discussed by [5].

Obviously, we assume that there is no node which is isolated from the rest of the graph. If the node in which the request arrives, called the source node, has a direct link to the node to which the information is to be sent, called the destination node, then the source node will try to send the information through the direct link. If the direct link is full, the node may or may not reject the request depending upon its protocol. A direct routing network rejects a request if the direct route is busy. An alternate routing

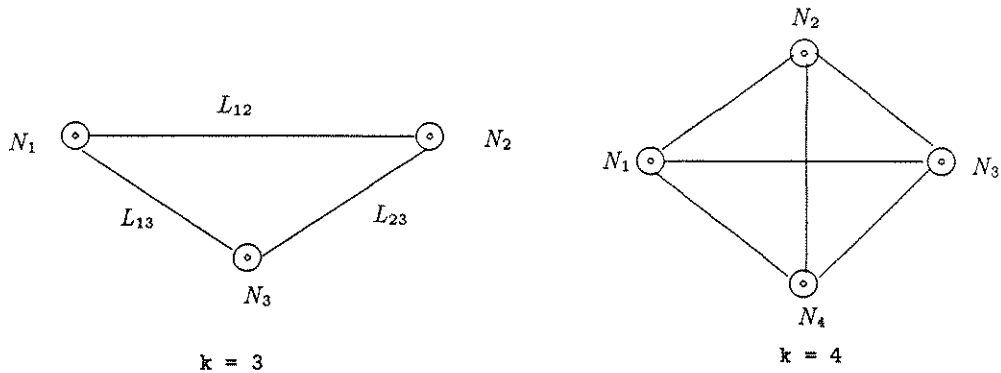
protocol could search for another path leading to the destination node to try to send the information in case the direct path is busy. The hope is that by having this option, the system may be utilized fully. However, the trade off of this apparent gain is that the longer the route is the more system resources it will use and hence if there are too many requests, the alternate traffic may detract from the direct traffic opportunities unduly.

In this paper, we study the nonhierarchical circuit switching networks. The aim is to find the probabilities of the system being in a state where calls are lost or where the system is idle. The next section presents the statistical model and the distributional assumptions that we make. Our aim is to find the limiting stationary distribution of the network. The performances of two routing protocols are discussed. Our aim is to determine the performance of the network when alternate routing is permitted as compared to when direct routing is the only option available. We provide a few limiting distributions for a simple three node network. Higher numbers of nodes does not cause much mathematical problem. This paper contains the main idea and the preliminary results. The more detailed analysis of different protocols will be presented else where.

## STATISTICAL MODEL AND ASSUMPTIONS

For the sake of exposition we will assume that the network is fully connected, however, the procedure is general for nonhierarchical networks. We will assume that the transmission bandwidth of each link is the same and is capable of transmitting in both directions. The input traffic may or may not be of the same intensity at different nodes.

More precisely, let  $N_1, \dots, N_k$  denote the switching nodes and  $L_{ij}$  be the link which directly connects the nodes  $N_i, N_j$ . Since, we assume that the network is fully connected, there are  $k(k - 1)/2$  links in the system. Simple case of  $k = 3$  and  $k = 4$  are presented below.



To understand the complexity of the situation, even a three node network would be enough. Therefore, most of the discussion will be restricted to this network. However, we should emphasize that the method is general.

The main assumptions for the traffic are that the arrivals at each node in each direction are independent Poisson processes and that holding times (service times) are independent and identically distributed exponential random variables with parameter  $\mu$ . These are standard assumptions in telephone traffic modelling. We will assume that the redirected traffic takes negligible amount of time for finding an alternate connection, or declaring that the call is lost due to lack of availability of an end to end connection.

The two routing protocols that we will compare are the direct routing protocol (in which no alternate route is allowed) and the alternate routing protocol (in which every effort is made to find an end to end connection by searching other links).

Of special interest are the probabilities that the system will be full in the steady state. That is to say, should a call arrive during that time it will be considered lost. The other question of interest is to find out how often the system is idle and hence the links are not utilized. These and other similar questions can be answered as soon as we have the limiting distribution of the process.

## DIRECT ROUTING PROTOCOL

In this section we have independent arrivals for the use of link  $L_{ij}$  where  $i < j, i, j = 1, 2, \dots, k$ . At any time  $t$ , if a call arrives (either at node  $N_i$ , or  $N_j$  to use the link  $L_{ij}$ ) then the request will be granted if there is a free channel (out of  $n$  total channels) on the link. If there is no free channel, the call is declared lost. Thus the nodes in this protocol perform no redirection work. If the call is granted a channel, it holds the channel for a random duration which is exponentially distributed with parameter  $\mu$ . We assume that the arrival rate of the Poisson process for the use of link  $L_{ij}$  is  $\lambda_{ij}$ . Clearly, the limit distribution of the number of busy channels at each link must exist. Let  $P_{ij}(c)$  denote the probability that in the limit the link  $L_{ij}$  has  $c$  number of channels busy. Thus,  $P_{ij}(n)$  is the probability that the link is full.

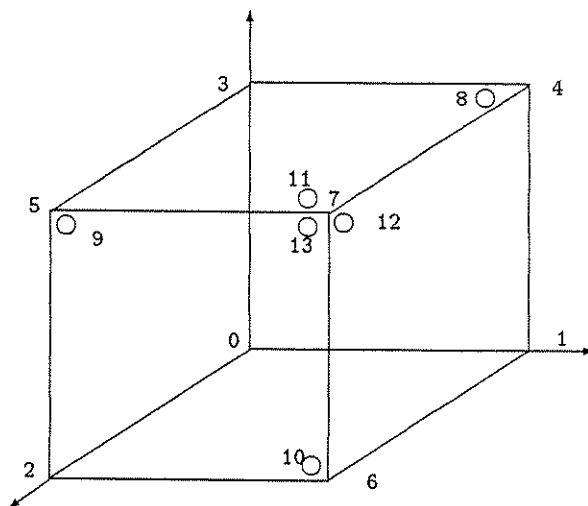
Since the arrival processes are mutually independent, the joint limiting distribution is the product of the marginal limiting distribution of each link. Thus by the usual Erlang formula the limiting probability that the link  $L_{ij}$  has  $c_{ij}$  channels busy is:

$$\prod_{i,j=1, i < j}^k \frac{\rho_{ij}^{c_{ij}}}{c_{ij}! (\sum_{l=0}^n \rho_{ij}^l / l!)}$$

where  $c_{ij} \in \{0, 1, \dots, n\}$  and  $\rho_{ij} = \lambda_{ij} / \mu$ . We will compare this distribution with the limiting distribution of the alternate routing protocol later.

## ALTERNATE ROUTING PROTOCOL

The independence of the link status is no longer valid due to the alternately routed traffic. The situation is easily explained by considering the three node network. Further simplify the situation by assuming  $n = 1$ . The state space of the system can be represented by the corners of a unit cube. Here the x-axis is the state of the link  $L_{12}$ , the y-axis is the state of link  $L_{13}$  and the z-axis is the state of link  $L_{23}$ . These states are numbered as depicted in the following figure.



The corners represent the original eight states and the circles represent extra states that we will define in the following. First, note that the system is a Markov process [3] due to the fact that holding times and interarrivals times are memory less. Second, if the network is in state 1 at time  $t$ , then at time  $t + \Delta t$ , the network could go into state 0 if the only busy channel is freed. It could go into state 6 if another call arrives asking for the direct link  $L_{13}$ . Similarly, it could go into the state 4. It could also go into what we call the state 12 if another call arrives to use the link  $L_{12}$  which is already busy and since the links  $L_{13}$ ,  $L_{23}$  are free, the alternate route could be established to take the network into a fully loaded state. We denote it differently from the state 7 in which we have three different independent calls being served simultaneously while in state 12 only 2 independent calls are being served. Another way of seeing this difference between the two states 7 and 12 is to note that from state 12 we can go either to state 1 or state 9 while from state 7 we can go to either the states 4,5 or 6 and neither to state 1 nor to state 9. Thus the circled states represent different redirected traffic while the corners represent the direct traffic states. Using the symmetry of the graph other states can be understood similarly.

The Kolmogorov equations of this network are given below.

$$\begin{aligned}
(\lambda_{12} + \lambda_{13} + \lambda_{23})P(0) &= \mu(P(1) + P(2) + P(3) + P(8) + P(9) + P(10)) \\
(\lambda_{12} + \lambda_{13} + \lambda_{23} + \mu)P(1) &= \mu(P(4) + P(6) + P(12)) + \lambda_{12}P(0) \\
(\lambda_{12} + \lambda_{13} + \lambda_{23} + \mu)P(2) &= \mu(P(5) + P(6) + P(13)) + \lambda_{13}P(0) \\
(\lambda_{12} + \lambda_{13} + \lambda_{23} + \mu)P(3) &= \mu(P(4) + P(5) + P(11)) + \lambda_{23}P(0) \\
(\lambda_{13} + 2\mu)P(4) &= \mu P(7) + \lambda_{23}P(1) + \lambda_{12}P(3) \\
(\lambda_{12} + 2\mu)P(5) &= \mu P(7) + \lambda_{23}P(2) + \lambda_{13}P(3) \\
(\lambda_{23} + 2\mu)P(6) &= \mu P(7) + \lambda_{13}P(1) + \lambda_{12}P(2) \\
3\mu P(7) &= \lambda_{13}P(4) + \lambda_{12}P(5) + \lambda_{23}P(6) \\
(\lambda_{13} + \mu)P(8) &= \mu P(13) \\
(\lambda_{12} + \mu)P(9) &= \mu P(12) \\
(\lambda_{23} + \mu)P(10) &= \mu P(11) \\
2\mu P(11) &= \lambda_{23}(P(10) + P(3)) \\
2\mu P(12) &= \lambda_{12}(P(9) + P(1)) \\
2\mu P(13) &= \lambda_{13}(P(8) + P(2)) \\
\sum_{i=0}^{13} P(i) &= 1.
\end{aligned}$$

For the case when  $\lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$  the solution of these equations is particularly simple,

$$\begin{aligned}
P(0) &= C2(\rho + 1) \\
P(1) = P(2) = P(3) &= C\rho(\rho + 2) \\
P(4) = P(5) = P(6) &= C\rho^2(\rho + 2) \\
P(7) &= C\rho^3(\rho + 2) \\
P(8) = P(9) = P(10) &= C\rho^2 \\
P(11) = P(12) = P(13) &= C\rho^2(\rho + 1)
\end{aligned}$$

where

$$C = (2 + 8\rho + 15\rho^2 + 8\rho^3 + \rho^4)^{-1}$$

and  $\rho = \lambda/\mu$ . For instance, when  $\rho = 1$ , we get  $P(0) = 4/37, P(j) = 3/34, 1 \leq j \leq 7$ . And the redirected states have probabilities  $P(8) = P(9) = P(10) = 1/34$  and  $P(11) =$



$P(12) = P(13) = 2/34$ . Thus the probability that the network is full (in the state of blocking calls) is  $9/34$ . And the probability that the network is idle is  $4/37$  while the probability that the network rerouting utilized empty channels without blocking is  $P(8) + P(9) + P(10) = 3/34$ .

The explicit solution for the case when two nodes have the same input rate and the third has a different input rate is given below.

$$\begin{aligned}
P(0) = & (24 \mu^7 + (44 \lambda_{13} + 52 \lambda) \mu^6 \\
& + (24 \lambda_{13}^2 + 66 \lambda \lambda_{13} + 36 \lambda^2) \mu^5 \\
& + (4 \lambda_{13}^3 + 24 \lambda \lambda_{13}^2 + 30 \lambda^2 \lambda_{13} + 8 \lambda^3) \mu^4 \\
& + (2 \lambda \lambda_{13}^3 + 6 \lambda^2 \lambda_{13}^2 + 4 \lambda^3 \lambda_{13}) \mu^3) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(1) = & (24 \lambda \mu^6 + (44 \lambda \lambda_{13} + 40 \lambda^2) \mu^5 \\
& + (18 \lambda \lambda_{13}^2 + 50 \lambda^2 \lambda_{13} + 22 \lambda^3) \mu^4 \\
& + (2 \lambda \lambda_{13}^3 + 15 \lambda^2 \lambda_{13}^2 + 18 \lambda^3 \lambda_{13} + 4 \lambda^4) \mu^3 \\
& + (\lambda^2 \lambda_{13}^3 + 3 \lambda^3 \lambda_{13}^2 + 2 \lambda^4 \lambda_{13}) \mu^2) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(2) = & (24 \lambda_{13} \mu^6 + (32 \lambda_{13}^2 + 52 \lambda \lambda_{13}) \mu^5 \\
& + (14 \lambda_{13}^3 + 52 \lambda \lambda_{13}^2 + 24 \lambda^2 \lambda_{13}) \mu^4 \\
& + (2 \lambda_{13}^4 + 15 \lambda \lambda_{13}^3 + 18 \lambda^2 \lambda_{13}^2 + 4 \lambda^3 \lambda_{13}) \mu^3 \\
& + (\lambda \lambda_{13}^4 + 3 \lambda^2 \lambda_{13}^3 + 2 \lambda^3 \lambda_{13}^2) \mu^2) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(3) = & (24 \lambda \mu^6 + (44 \lambda \lambda_{13} + 40 \lambda^2) \mu^5 \\
& + (18 \lambda \lambda_{13}^2 + 50 \lambda^2 \lambda_{13} + 22 \lambda^3) \mu^4 \\
& + (2 \lambda \lambda_{13}^3 + 15 \lambda^2 \lambda_{13}^2 + 18 \lambda^3 \lambda_{13} + 4 \lambda^4) \mu^3 \\
& + (\lambda^2 \lambda_{13}^3 + 3 \lambda^3 \lambda_{13}^2 + 2 \lambda^4 \lambda_{13}) \mu^2) / \text{quot}
\end{aligned}$$

$$P(4) = (24 \lambda^2 \mu^5 + (44 \lambda^2 \lambda_{13} + 40 \lambda^3) \mu^4$$

$$\begin{aligned}
&+(16 \lambda^2 \lambda_{13}^2 + 52 \lambda^3 \lambda_{13} + 22 \lambda^4) \mu^3 \\
&+(2 \lambda^2 \lambda_{13}^3 + 15 \lambda^3 \lambda_{13}^2 + 18 \lambda^4 \lambda_{13} + 4 \lambda^5) \mu^2 \\
&+(\lambda^3 \lambda_{13}^3 + 3 \lambda^4 \lambda_{13}^2 + 2 \lambda^5 \lambda_{13}) \mu) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(5) = & (24 \lambda \lambda_{13} \mu^5 + (38 \lambda \lambda_{13}^2 + 46 \lambda^2 \lambda_{13}) \mu^4 \\
&+(16 \lambda \lambda_{13}^3 + 52 \lambda^2 \lambda_{13}^2 + 22 \lambda^3 \lambda_{13}) \mu^3 \\
&+(2 \lambda \lambda_{13}^4 + 15 \lambda^2 \lambda_{13}^3 + 18 \lambda^3 \lambda_{13}^2 + 4 \lambda^4 \lambda_{13}) \mu^2 \\
&+(\lambda^2 \lambda_{13}^4 + 3 \lambda^3 \lambda_{13}^3 + 2 \lambda^4 \lambda_{13}^2) \mu) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(6) = & (24 \lambda \lambda_{13} \mu^5 + (38 \lambda \lambda_{13}^2 + 46 \lambda^2 \lambda_{13}) \mu^4 \\
&+(16 \lambda \lambda_{13}^3 + 52 \lambda^2 \lambda_{13}^2 + 22 \lambda^3 \lambda_{13}) \mu^3 \\
&+(2 \lambda \lambda_{13}^4 + 15 \lambda^2 \lambda_{13}^3 + 18 \lambda^3 \lambda_{13}^2 + 4 \lambda^4 \lambda_{13}) \mu^2 \\
&+(\lambda^2 \lambda_{13}^4 + 3 \lambda^3 \lambda_{13}^3 + 2 \lambda^4 \lambda_{13}^2) \mu) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(7) = & (24 \lambda^2 \lambda_{13} \mu^4 + (40 \lambda^2 \lambda_{13}^2 + 44 \lambda^3 \lambda_{13}) \mu^3 \\
&+(16 \lambda^2 \lambda_{13}^3 + 52 \lambda^3 \lambda_{13}^2 + 22 \lambda^4 \lambda_{13}) \mu^2 \\
&+(2 \lambda^2 \lambda_{13}^4 + 15 \lambda^3 \lambda_{13}^3 + 18 \lambda^4 \lambda_{13}^2 + 4 \lambda^5 \lambda_{13}) \\
&+\lambda^3 \lambda_{13}^4 + 3 \lambda^4 \lambda_{13}^3 + 2 \lambda^5 \lambda_{13}^2) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(8) = & (12 \lambda_{13}^2 \mu^5 + (10 \lambda_{13}^3 + 26 \lambda \lambda_{13}^2) \mu^4 \\
&+(2 \lambda_{13}^4 + 13 \lambda \lambda_{13}^3 + 12 \lambda^2 \lambda_{13}^2) \mu^3 \\
&+(\lambda \lambda_{13}^4 + 3 \lambda^2 \lambda_{13}^3 + 2 \lambda^3 \lambda_{13}^2) \mu^2) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(9) = & (12 \lambda^2 \mu^5 + (22 \lambda^2 \lambda_{13} + 14 \lambda^3) \mu^4 \\
&+(9 \lambda^2 \lambda_{13}^2 + 14 \lambda^3 \lambda_{13} + 4 \lambda^4) \mu^3 \\
&+(\lambda^2 \lambda_{13}^3 + 3 \lambda^3 \lambda_{13}^2 + 2 \lambda^4 \lambda_{13}) \mu^2) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
p(10) = & (12 \lambda^2 \mu^5 + (22 \lambda^2 \lambda_{13} + 14 \lambda^3) \mu^4 \\
& + (9 \lambda^2 \lambda_{13}^2 + 14 \lambda^3 \lambda_{13} + 4 \lambda^4) \mu^3 \\
& + (\lambda^2 \lambda_{13}^3 + 3 \lambda^3 \lambda_{13}^2 + 2 \lambda^4 \lambda_{13}) \mu^2) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(11) = & (12 \lambda^2 \mu^5 (22 \lambda^2 \lambda_{13} + 26 \lambda^3) \mu^4 \\
& + (9 \lambda^2 \lambda_{13}^2 + 36 \lambda^3 \lambda_{13} + 18 \lambda^4) \mu^3 \\
& + (\lambda^2 \lambda_{13}^3 + 12 \lambda^3 \lambda_{13}^2 + 16 \lambda^4 \lambda_{13} + 4 \lambda^5) \mu^2 \\
& + (\lambda^3 \lambda_{13}^3 + 3 \lambda^4 \lambda_{13}^2 + 2 \lambda^5 \lambda_{13}) \mu) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(12) = & (12 \lambda^2 \mu^5 + (22 \lambda^2 \lambda_{13} + 26 \lambda^3) \mu^4 \\
& + (9 \lambda^2 \lambda_{13}^2 + 36 \lambda^3 \lambda_{13} + 18 \lambda^4) \mu^3 \\
& + (\lambda^2 \lambda_{13}^3 + 12 \lambda^3 \lambda_{13}^2 + 16 \lambda^4 \lambda_{13} + 4 \lambda^5) \mu^2 \\
& + (\lambda^3 \lambda_{13}^3 + 3 \lambda^4 \lambda_{13}^2 + 2 \lambda^5 \lambda_{13}) \mu) / \text{quot}
\end{aligned}$$

$$\begin{aligned}
P(13) = & (12 \lambda_{13}^2 \mu^5 + (22 \lambda_{13}^3 + 26 \lambda \lambda_{13}^2) \mu^4 \\
& + (12 \lambda_{13}^4 + 39 \lambda \lambda_{13}^3 + 12 \lambda^2 \lambda_{13}^2) \mu^3 \\
& + (2 \lambda_{13}^5 + 14 \lambda \lambda_{13}^4 + 15 \lambda^2 \lambda_{13}^3 + 2 \lambda^3 \lambda_{13}^2) \mu^2 \\
& + (\lambda \lambda_{13}^5 + 3 \lambda^2 \lambda_{13}^4 + 2 \lambda^3 \lambda_{13}^3) \mu) / \text{quot}
\end{aligned}$$

where

$$\begin{aligned}
\text{quot} = & 24 \mu^7 + (68 \lambda_{13} + 100 \lambda) \mu^6 + (80 \lambda_{13}^2 + 254 \lambda \lambda_{13} + 188 \lambda^2) \mu^5 \\
& + (50 \lambda_{13}^3 + 240 \lambda \lambda_{13}^2 + 402 \lambda^2 \lambda_{13} + 172 \lambda^3) \mu^4 \\
& + (16 \lambda_{13}^4 + 105 \lambda \lambda_{13}^3 + 274 \lambda^2 \lambda_{13}^2 + 284 \lambda^3 \lambda_{13} + 74 \lambda^4) \mu^3 \\
& + (2 \lambda_{13}^5 + 20 \lambda \lambda_{13}^4 + 75 \lambda^2 \lambda_{13}^3 + 145 \lambda^3 \lambda_{13}^2 + 88 \lambda^4 \lambda_{13} + 12 \lambda^5) \mu^2 \\
& + (\lambda \lambda_{13}^5 + 7 \lambda^2 \lambda_{13}^4 + 26 \lambda^3 \lambda_{13}^3 + 31 \lambda^4 \lambda_{13}^2 + 10 \lambda^5 \lambda_{13}) \mu \\
& + \lambda^3 \lambda_{13}^4 + 3 \lambda^4 \lambda_{13}^3 + 2 \lambda^5 \lambda_{13}^2
\end{aligned}$$

The extension of these results to higher number of channels per link and more nodes is only longer and mathematically not more complicated.

## COMPARISON OF THE PROTOCOLS

The probability that the system has all links fully utilized is given in as the blocking probability comparison. The dotted line is the probability that the alternate routing network is in state 7 while the solid line is the probability that the alternate routing network is in either state 7 or states 11, 12 or 13. The dashes are the probability that the direct routing network is in state 7.

The idle probability (that is the network is in state 0) are given in the next figure. The dashes are the probability that the direct routing network is idle. The solid line is the probability that the alternate routing network is idle.

Finally, the probabilities that the alternate routing network utilized the empty channels without taking the system to blocking states ( that is it is in states 8, 9, 10) is given next.

In summary, we see that even for this simplistic network some important observations can be made. The situation becomes more interesting when the number of channels is increased per link.

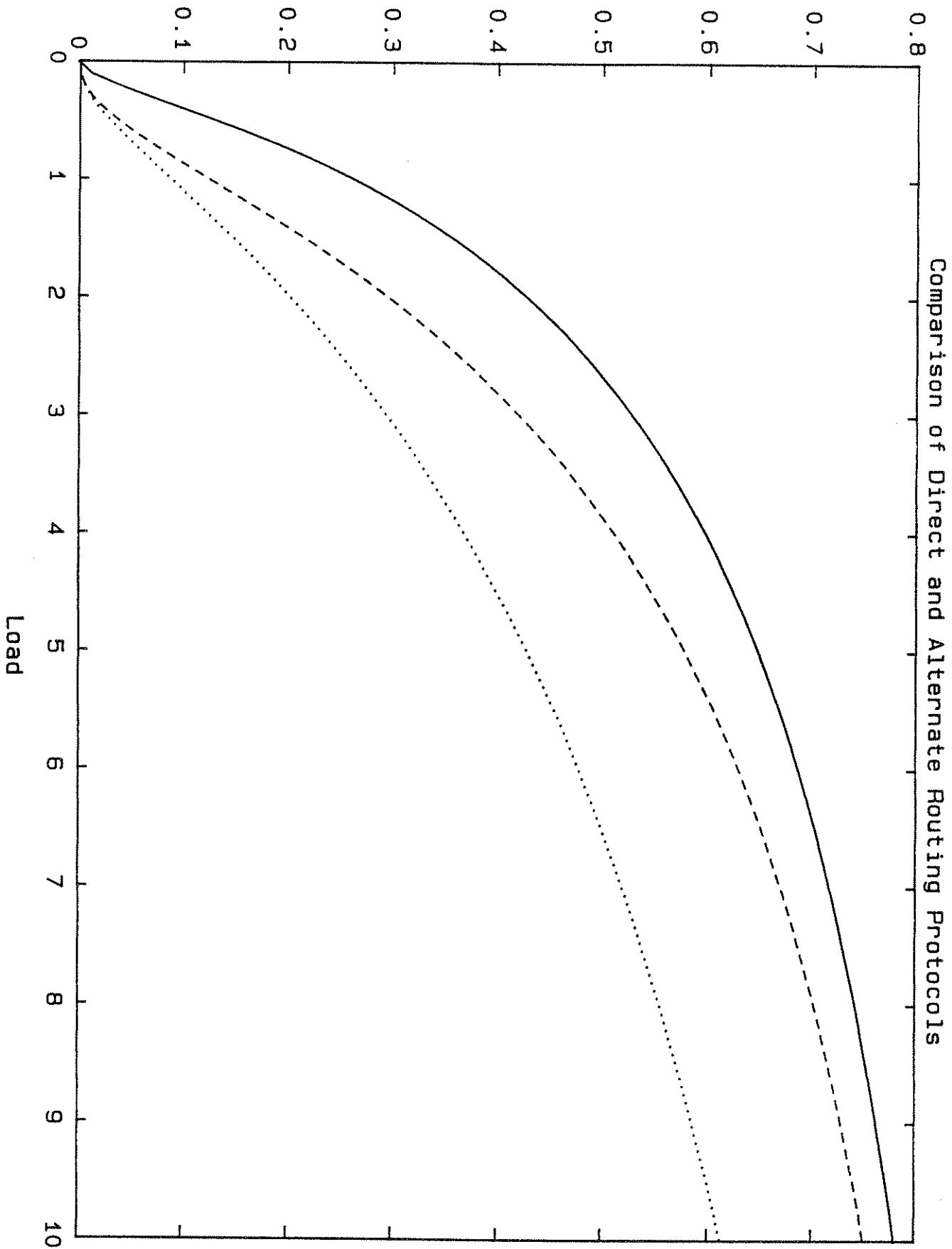
A lengthy comparison of these protocols have been performed by [2] and [1]. Our goal here was essentially to show that the problem can be studied analytically.

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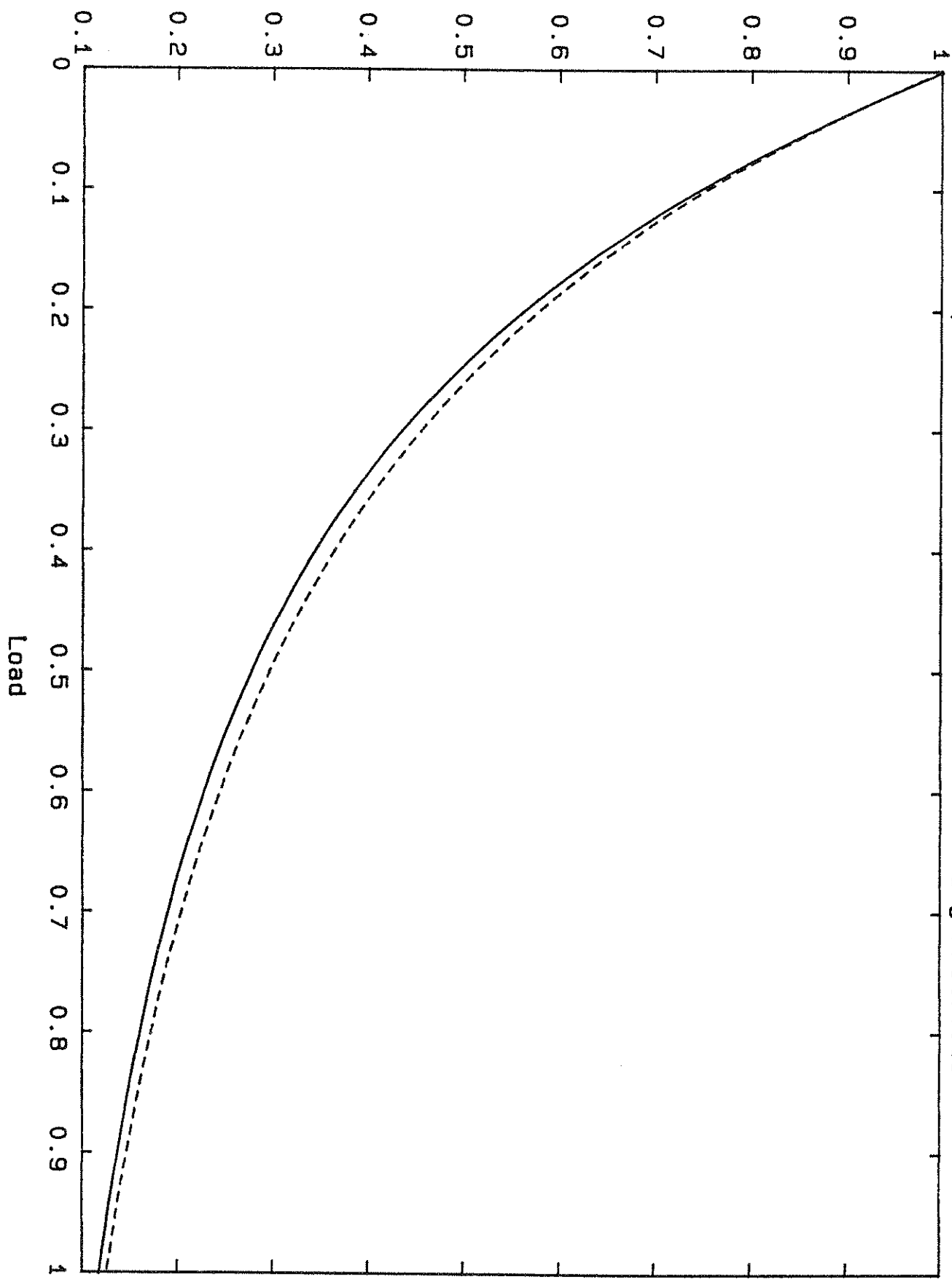
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# Blocking Probability



Idle probability

Comparison of Direct and Alternate Routing Protocols



Alternate Routing Protocol

