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ON THE DETERMINATION OF THE ORDER OF UNIFORM CONVERGENCE

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Abstract:- We shall discuss a number of methods used in the literature to calculate rates of uniform convergence. We mention some anomalies concerning interpretation of the resulting tables and rates, which lead to the determination of experimental rates of uniform convergence lower than the correct rates.

Introduction

The determination of the order of uniform convergence is not always a straightforward task. A number of approaches exist in the literature, the two major variants being that appearing in [6] and [1] and that in [2].

The former approach involves solving numerically a singularly perturbed differential equation on $[x_L, x_R]$ for which the analytic solution is known. The difference equation is solved for decreasing values of h and the rate of convergence calculated from

$$p_{e\epsilon}^h = (\ln e_{e\epsilon}^{2h} - \ln e_{e\epsilon}^h)/\ln(2) \qquad (1)$$

where

$$e_{e\epsilon}^h = \max_{0 \le i \le N} |u_i^h - u_\epsilon(x_i)|, \quad h = [x_L - x_R]/N. \qquad (2)$$

The equation solved is chosen so that the solution and its derivatives exhibit exactly the analytic behaviour hypothesized in the proof of the error estimates. In practise this is achieved by choosing a solution and then determining a differential equation of the correct form which this satisfies.

The uniform rate is determined by inspecting a table of values of $p_{e\epsilon}^h$, for varying h and ϵ, constructed by setting $\epsilon = h^s$ for various values of s. Results of this form are given in [6] for a non-turning point problem and in [1] for a turning-point problem of the type considered in [3, 4]. In the case of the turning-point problem the choice of solution involves making the boundary values functions of ϵ and hence by the choice of $\epsilon = h^s$ functions of h. A disadvantage of this method is the requirement of prior knowledge of the solution since this limits the ease with which it can be applied to other problems. An alternative is, of course, to use an accurate approximation on a *fine* mesh, if this is determinable.

The Double Mesh Method

The method proposed in [2] is based on a consequence of the General Convergence Principle which states

Theorem ([2, Theorem I.5.1])
Let u_ϵ be the solution of a differential equation and u_i^h a difference approximation. Let $p > 0$ and C_1, C_2 be positive constants independent of h. Then, for all $i \ge 0$, all $0 < h < h_0$, and all $\epsilon > 0$

$$|u_\epsilon(x_i) - u_i^h| \le C_1 h^p$$

iff

(i) $\lim_{h \to 0} |u_\epsilon(x_i) - u_i^h| = 0$

(ii) $|u_i^{2h} - u_i^h| \le C_2 h^p$.

Furthermore C_1 is independent of ϵ iff C_2 is.

Essentially the method involves calculating the quantity given in (ii) which we shall call the double mesh error $e_{d\epsilon}^h$ and determining a rate of convergence

$$p_{d\epsilon}^h = (\ln e_{d\epsilon}^{2h} - \ln e_{d\epsilon}^h)/\ln(2) \qquad (3)$$

where

$$e_{d\epsilon}^h = \max_{0 \le i \le N} |u_i^{2h} - u_i^h|, \quad h = [x_L - x_R]/N. \qquad (4)$$

The experimental uniform rate of convergence is then determined as

$$p = \min_\epsilon p_{d\epsilon} \quad \text{where} \quad p_{d\epsilon} = \text{average}_h p_{d\epsilon}^h. \qquad (5)$$

Doolan, Miller and Schilders remark that the choice of the range of h values permissible is limited, since if the mesh is too coarse the solution of the difference scheme is not sufficiently representative of the solution of the differential equation to permit meaningful discussion of convergence, that is h is not *sufficiently small*, whereas if it is too fine then rounding error predominates. The method has the advantage that it requires no a priori knowledge of the nature of the solution of the equation and may be easily programmed to determine an experimental rate of convergence for a wide variety of problems.

Anomalies

In either of these methods however, great care must be taken in interpreting the table of values of $p_{e\epsilon}^h$ or $p_{d\epsilon}^h$ for the reasons which we will outline below. To simplify the arguments we shall consider only the rate of convergence, for a non-turning point problem, as considered in [6], that is :

$$\epsilon u'' + a(x)u' - b(x)u = f(x), \quad 0 < x < 1$$
$$u(0) = A, \qquad u(1) = B.$$

where $a(x) \ge \underline{a} > \alpha > 0$. The determination of the rate of convergence depends on the assumption that

$$e_{e\epsilon}^h < C h^p \qquad (6)$$

where C is independent of h and ϵ. This is not necessarily the strongest bound available and in fact the following one, (*cf.* [6]) , is a more accurate estimate :

$$e_{e\epsilon}^h < C(\frac{h^2}{h + \epsilon} + \frac{h}{\epsilon} e^{-\alpha h/\epsilon}) = Ch(\frac{\rho}{1 + \rho} + \rho\, e^{-\alpha\rho}), \qquad (7)$$

where $\rho = h/\epsilon$ and $\alpha > 0$. So more accurately

$$e_{e\epsilon}^h < C(\rho, \alpha)h,$$

which is not inconsistent with (6) since by considering the limits as $\rho \to 0$ and $\rho \to \infty$, for α fixed , we can see that $C(\rho, \alpha)$ is bounded. Let us assume that equality holds in (8) and determine the rate of convergence $p_{e\epsilon}^h$. Thus

$$p_e^\rho \equiv p_{e\epsilon}^h(\rho) = \ln\left[4\frac{(2\rho + 1)^{-1} + e^{-2\alpha\rho}}{(\rho + 1)^{-1} + e^{-\alpha\rho}}\right]/\ln(2) \qquad (8)$$

and considering this for $\epsilon \to \infty(\rho \to 0)$ and $\epsilon \to 0(\rho \to \infty)$ we get

$$\lim_{\rho \to 0} p_e^\rho = 2 \quad , \quad \lim_{\rho \to \infty} p_e^\rho = 1.$$

Thus, if we determine the rate of convergence p_e^ρ, for fixed h, and ϵ varying from ∞ to 0, it will vary from 2 to 1 as expected. The assumption we implicitly make in evaluating computational orders of convergence is that p_e^ρ is monotonic and hence that the minimum value is 1. This is not in fact the case as the function p_e^ρ given by (9) may attain a minimum less than 1. This may be seen in Fig. 1 , which is a graph of p_e^ρ for $\alpha = 0.25$. This lack of monotonicity is, in fact, most apparent when α is small. Similar results are also given in Fig. 1 for p_d^ρ,
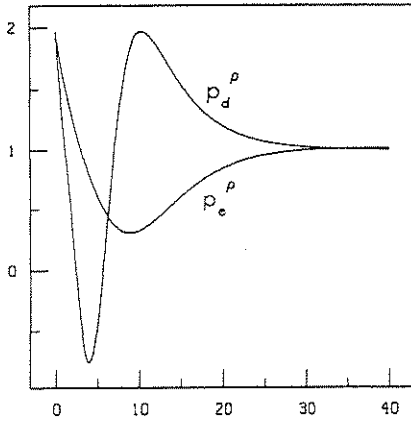
Figure 1: Double Mesh, p_d^ρ, and Exact, p_e^ρ, Rates of Convergence

ϵ	N	8	16	32	64	128	256	$p_{d\epsilon}$
1/	2	1.89	1.94	1.97	1.98	1.99	2.00	1.96
1/	4	1.81	1.89	1.94	1.97	1.98	1.99	1.93
1/	8	1.68	1.81	1.89	1.94	1.97	1.98	1.88
1/	16	1.48	1.68	1.81	1.89	1.94	1.97	1.79
1/	32	1.10	1.48	1.68	1.81	1.89	1.94	1.65
1/	64	0.33	1.10	1.48	1.68	1.81	1.89	1.38
1/	128	-0.75	0.03	1.10	1.48	1.468	1.81	0.94
1/	256	1.55	-0.75	0.33	1.10	1.48	1.68	0.90
1/	512	1.44	1.55	-0.75	0.33	1.10	1.48	0.86
1/	1024	1.02	1.44	1.55	-0.75	0.33	1.10	0.78
1/	2048	1.00	1.02	1.44	1.55	-0.75	0.33	0.76
1/	4096	1.00	1.00	1.02	1.44	1.55	-0.75	0.88
1/	8192	1.00	1.00	1.00	1.02	1.44	1.55	1.17
1/	16384	1.00	1.00	1.00	1.00	1.02	1.44	1.08
1/	32768	1.00	1.00	1.00	1.00	1.00	1.02	1.00
1/	65536	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1/	131072	1.00	1.00	1.00	1.00	1.00	1.00	1.00
p_d^h		1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 1: Double Mesh Rate of Convergence, p_d^ρ, $\alpha = 0.25$

the rate of convergence calculated using the Doolan, Miller, Schilders double-mesh method. In this case the actual rates are given in Table 1. It can be seen that serious problems can arise here in interpreting the table, particularly if we take the rate of convergence to be the minimum of $p_{d\epsilon}^h(\rho)$ over all ϵ, since this will be less than the actual rate of uniform convergence and for some problems may in fact be negative. This problem is significantly more noticeable for the rates calculated using the double mesh method than for those calculated using an exact or fine mesh approximation. In the case $\alpha = 0.25$, for example, the minima are -0.75 and 0.32 respectively. This is what might be expected from comparison of the two curves in Fig. 1. The experimental rates of uniform convergence, for $\alpha = 0.25$, are $p_d = 0.76$ and $p_e = 0.80$ respectively, both of which are significantly lower than the true rate of 1.00. More examples of these phenomena are given in [5]. Reporting excessively low rates of uniform convergence is thus an expected feature of this method.

In view of these reservations we propose the following estimates for the rate of convergence

$$p_d^+ = \text{average}_h p_d^h \quad \text{and} \quad p_d^* = \min_h p_d^h$$

where

$$p_d^h = \ln(e_d^{2h} - e_d^h)/\ln(2)$$

and

$$e_d^h = \max_\epsilon e_{d\epsilon}^h = \max_\epsilon (\max_{0 \le i \le N} |u_i^{2h} - u_{2i}^h|).$$

It is clear that e_d^h is a function of h alone and thus we may expect p_d^h to be approximately a constant independent of h. This will lead to a better estimate for the uniform rate of convergence. In the case $\alpha = 0.25$, this is $p_d^+ = p_e^+ = p_d^* = p_e^* = 1.00$. The p_d^+-method has been used to determine the rate of convergence in [3, 4] and many later papers. We remark however that it does not give any additional information about the variations in behaviour of the scheme as $h \to 0$ or $\epsilon \to 0$. Therefore it is also useful to include tables of $p_{d\epsilon}^h$ to provide more precise details of this behaviour. This is particularly so for problems having more complicated boundary or interior layers.

We should remark at this point that there remain certain problems. In particular, the restriction that h was sufficiently large is crucial. If h becomes small, the most prevalent effect is for rounding errors to corrupt the results. However, if the calculations were done "exactly", so that rounding error were absent or negligible, then a more serious problem would arise. If we produce tables for arbitrarily small h, but only for *finitely* small ϵ, then most of the rates in the table will be for $h \ll \epsilon$. In this case, we are in the region where classical convergence theory applies and thus the rates will be greater than or equal to 1, for most schemes. These rates will dominate the table, and, if we use p_d^+ or p_e^+ as the calculated rate, cause even non-uniform schemes to be reported as uniformly convergent. This may be viewed as a consequence of the form of the tables, where, in this case in particular, the rate of convergence is a function of $\rho = h/\epsilon$. Thus the rates along the diagonals are equal. To get an accurate reflection of the rate of uniform convergence it is therefore necessary to extend the table at least as far in ϵ as in h. In the one dimensional cases, which we have tested, (cf. [3]), we extend the table until the errors, for given h, stabilizes, which occurs when one is solving, up to rounding error, the reduced equation. The finest mesh used in the calculations is $h = 1/4096$. In practice, using either double or fine mesh methods this has given acceptably accurate rates. In all cases the rate calculated using the fine mesh method proved higher. A more cautious approach might be to use p_d^* or p_e^*, which are less prone to this effect, although these will again report lower than actual rates of uniform convergence.

We remark that there are other circumstances in which these methods will report positive uniform convergence rates where, using the normal definition, the scheme would not be considered uniformly convergent. This is particularly true of the centered difference approximation to a self-adjoint problem and of two or higher dimensional problems exhibiting certain phenomena. These issues are considered further in [5].

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