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ANALYTIC ESTIMATES AND UNIFORM NUMERICAL METHODS FOR MULTIPLE BOUNDARY TURNING POINT PROBLEMS

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Abstract. We present results, which characterize the behaviour of a a singularly perturbed boundary value problem with a multiple turning point at a boundary. A representation of the solution and bounds on the derivatives are derived. Criteria for uniform stability of a class of schemes is given, together with some schemes which are uniformly convergent for these problems.

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1 Introduction

In this paper we shall outline the derivation of estimates for the derivatives and a low order asymptotic expansion for the solution of multiple boundary turning point problems of the form :

$$L^{\pm}u := -\epsilon u'' \pm x^k b(x)u' + c(x)u = f(x),\ x \in I = [0,1], \tag{1.1a}$$

$$Bu := (u(0), u(1)) = (U_0, U_1), \tag{1.1b}$$

where U_0 and U_1 are given numbers, $0 < \epsilon \le \epsilon^* \ll 1$, and throughout the paper we shall assume:

$$k = 2 \text{ or } k \in [3, +\infty), \tag{1.2a}$$

$$b, c, f \in C^3(I), \tag{1.2b}$$

$$b(x) \ge b_* > 0,\ x \in I, \tag{1.2c}$$

$$c(x) \ge c_* > 0,\ x \in I, \tag{1.2d}$$

Note that here k is not necessarily an integer—it is sufficient to assume (1.2a) because what we require is that $a \in C^3(I)$, where

$$a(x) := x^k b(x).$$

A turning-point problem of the form L^-, where the first and second derivative's coefficients are of the same sign is called an *attractive* turning-point. If the coefficients are of opposite sign, as in L^+, it is referred to as a *repulsive* turning-point problem. In addition, we shall present some uniform stability criteria for difference schemes for these problems and give some examples of uniformly convergent schemes. More complete proofs of the results will appear in later publications.

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Simple turning point problems, that is problems of the form (1.1) with $k = 1$, have attracted most of the attention, among turning point problems, both analytically and numerically. There are however, some results in the literature on multiple turning point problems, for example [4] and [5], both papers dealing with a general turning point problem. In [4], the asymptotics of a homogeneous problem was investigated, and in [5], a numerical method based on special discretization meshes was given for a semilinear problem. The criteria for uniform stability and convergence, on a uniform mesh, of certain interior multiple turning point problems were considered in [2].

We shall outline, in section 2, the derivation of the analytic estimates for the derivatives and a low order asymptotic expansion for the solution of the attractive boundary turning point case. In section 3, we shall derive criteria for uniform stability for a class of schemes and give sample schemes which are uniformly convergent for this problem. In section 4 we will give analytic estimates, stability estimates and numerical results for repulsive boundary turning point cases.

Throughout this paper M shall denote any positive constant independent of ϵ. Some of these constants will however be denoted by M_0, M_1 etc.

2 Attractive Turning Point — Analytic Results

We give sketches of the results, which will appear in [7].

LEMMA 2.1. *The problem (1.1) has a unique solution u_ϵ which is bounded uniformly in ϵ:*

$$|u_\epsilon(x)| \leq M, \quad x \in I. \tag{2.1}$$

Proof. The proof follows using the maximum principle and the comparison function $p(x) = M_0(2 - x)$. □

Let

$$q_i(x) = c(x) \pm ia'(x), \quad i = 1, 2, 3$$

Because of

$$q_i(0) = c(0) \geq c_* > 0, \quad i = 1, 2, 3,$$

there exists a point $\theta_0 \in (0, 1)$, independent of ϵ, such that

$$q_i(x) > q_* > 0, \quad x \in [0, \theta_0], \quad i = 1, 2, 3. \tag{2.2}$$

Next, let $g_\epsilon(x) \in C^3(I)$ be a function such that

$$|g_\epsilon^{(i)}(x)| \leq M(1 + \epsilon^{-i/2} \exp(-\mu_* x)), \quad i = 0, 1, 2, 3, \quad x \in I, \quad \mu_* = \sqrt{\frac{q_*}{\epsilon}}.$$

Then, in a manner similar to Lemma 2.1, the problem

$$Ly_\epsilon(x) = g_\epsilon(x), \quad x \in I, \quad By_\epsilon = (U_0, U_1),$$

has a unique solution y_ϵ and

$$|y_\epsilon(x)| \leq M, \quad x \in I.$$

Lemma 2.2. *There exist points $\theta_i \in (0, \theta_0)$, $i = 1, 2, 3$, independent of ϵ and such that*

$$|y_\epsilon^{(i)}(\theta_i)| \leq M, \quad i = 1, 2, 3. \tag{2.3}$$

On the other hand:

$$|y_\epsilon^{(i)}(0)| \leq M\epsilon^{-i/2}, \quad i = 1, 2, 3. \tag{2.4}$$

Proof. The proof of (2.3) follows by choosing the θ_i appropiately. For example, θ_1 is chosen such that $\theta_1 \in (0, \theta_0)$ and $y_\epsilon'(\theta_1) = (y_\epsilon(\theta_0) - y_\epsilon(0))/\theta_0$. To prove (2.4), we rewrite the differential equation in the form:

$$-\epsilon y_\epsilon''(x) - (a(x)y_\epsilon(x))' + (a'(x) + c(x))y_\epsilon(x) = g_\epsilon(x)$$

and integrate from 0 to the point x^* such that $y_\epsilon'(x^*) = (y_\epsilon(\sqrt{\epsilon}) - y_\epsilon(0))/\sqrt{\epsilon}$, $x^* \in (0, \sqrt{\epsilon})$. Now $|y_\epsilon'(x^*)| \leq M\epsilon^{-1/2}$. and on division by ϵ, we get $|y_\epsilon'(0)| \leq M(\epsilon^{-\frac{1}{2}}\epsilon^{\frac{k}{2}-1} + \epsilon^{-1}\epsilon^{\frac{1}{2}}) \leq M\epsilon^{-1/2}$. Then (2.4) follows for $i = 2$, from $Ly_\epsilon(x) = g_\epsilon(x)$ at $x = 0$, and for $i = 3$ after differentiation. □

Lemma 2.3. *Let $y_\epsilon(x)$ be as above, then*

$$|y_\epsilon^{(i)}(x)| \leq M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \quad x \in I, \quad i = 1, 2, 3. \tag{2.5}$$

Proof. This follows in a manner similar to [5]. □

Theorem 2.4. *For the solution u_ϵ to the* attractive *problem of the form (1.1) the following representation holds :*

$$u_\epsilon(x) = \omega v_\epsilon(x) + z_\epsilon(x), \quad x \in I, \tag{2.6a}$$

$$v_\epsilon(x) = \exp(-\mu x), \quad \mu = \sqrt{\tfrac{c(0)}{\epsilon}}, \quad |\omega| \leq M, \tag{2.6b}$$

$$|z_\epsilon^{(i)}(x)| \leq M\left(1 + \epsilon^{\frac{1-i}{2}}\exp(-\sqrt{q_*}x/\sqrt{\epsilon})\right), \quad i = 0, 1, 2, 3, \quad x \in I, \tag{2.6c}$$

Proof. The proof is given in [7]. □

3 Attractive Turning Point - Uniform Schemes

Let I^h be the discretization mesh given by $x_i = ih, i = 0, \ldots, n$, where $h = 1/n$, and let w^h, u^h etc. denote mesh functions on I^h, that is $w^h = [w_0, w_1, \cdots, w_n]^T$, with corresponding norm $\|w^h\| = \max_{0 \leq i \leq n} |w_i|$.

Then the discrete problem corresponding to (1.1) is given by

$$L^h w_i := -\epsilon\sigma_i D_+ D_- w_i - a_i^h D_+ w_i + c_i^h w_i = f_i^h, \quad i = 1(1)n-1, \tag{3.1a}$$

$$w_0 = U_0, w_n = U_1, \tag{3.1b}$$

where σ_i is a fitting factor, and

$$D_+ = (w_{i+1} - w_i)/h, \qquad D_+D_- = (w_{i+1} - 2w_i + w_{i-1})/h^2.$$

A sufficient condition for uniform stability for a problem in this form is given by:

LEMMA 3.1. *Let*

$$\sigma_i \geq 0, \quad a_i^h \geq 0, \quad c_i^h \geq 0 \quad \text{and} \quad a_i^h + c_i^h \geq \mu > 0, \tag{3.2}$$

then w^h*, the solution of (3.1), satisfies a stability result of the form*

$$||w^h|| \leq M \; max(|w_0|, |w_n|, ||f^h||).$$

Proof. It is easy to show, using (3.2), that the matrix is an $M - matrix$ and the stability result follows using the comparison function $M_0(2 - x)$. □

One can show that an appropriate choice for the fitting factor, in the case $a_i^h = a(x_i), c_i^h = c(x_i)$ and $f_i^h = f(x_i)$ is similar to that in [1, Chapter 6] and in [6], that is

$$\sigma_i = \frac{\mu^2 h^2}{\sinh^2(\mu h/2)}, \quad \text{where} \quad \mu = \sqrt{\frac{c(0)}{\epsilon}}.$$

We shall call this scheme the Constant Miller Scheme (*cf.* [1, Chapter 6]). The generalization to the non-equidistant case will appear in [7].

Let r^h be the consistency error, $r_i = L^h u_\epsilon(x_i) - f(x_i), r_0 = r_n = 0$. Then we can show, using the representation (1.3), in a manner similar to [7], that

$$|r_i| \leq Mh.$$

Hence, using Lemma 3.1, we can show that the above scheme is uniformly $O(h)$ convergent that is :

$$||u_\epsilon^h - w_\epsilon^h|| \leq Mh. \tag{3.3}$$

We remark that a number of other schemes including the non-constant fitting factor version of the above scheme, which we shall call the Miller Scheme, Complete Exponential Fitting, and the El-Mistikawy and Werle scheme are also uniformly $O(h)$ convergent for this problem. The rate of uniform convergence on equidistant meshes for the problem

$$-\epsilon u'' - x^k u' + (1 + x^2)u \;=\; 4(3x^2 - 3x + 1)(1 + x)^2, \;\; u(0) = 1, \;\; u(1) = 2, \tag{3.4}$$

with $k = 2$ and $k = 3$, are given in Table 3.1. The rate of uniform convergence given is the average rate of convergence, using the double mesh method, for a range of values of h and ϵ, given by

$$H = \{1/2^j | j = 3, \ldots, 9\}, \qquad E = \{1/2^j | j = 0, \ldots, jred\}$$

where *jred* is chosen so that ϵ is a value at which the rate of convergence stabilizes, which normally occurs when, to machine accuracy one is solving the reduced problem. For more details of these tests see [3]. Other uniformly convergent schemes can be formulated, which resemble the above schemes in the

Scheme	k = 2		k = 3	
	Classical Rate	Uniform Rate	Classical Rate	Uniform Rate
Constant Miller	1.08	.93	1.08	.93
Miller Scheme	1.08	.93	1.08	.93
Complete Fitting	2.00	.85	2.00	.79
El-Mistikawy & Werle	1.83	1.21	1.82	1.17

Table 3.1: Convergence Rates for Attractive Multiple Turning Point Case

boundary layer at $x = 0$, but are merely upwinded outside it. It should be noted that, like the interior simple turning point case and unlike the self-adjoint non-turning point case, the El-Mistikawy & Werle scheme is only $O(h)$ uniformly convergent, as opposed to being a second order uniform scheme.

4 Repulsive Multiple Turning Point — Analytic and Numeric Results

The behaviour, in the repulsive case, is significantly more complicated. It exhibits layers at both boundaries, an exponential layer of non-selfadjoint type at $x = 1$ and a more complicated layer at $x = 0$. We can characterize the behaviour of the derivatives, in a manner similar to Lemma 2.1, as follows:

LEMMA 4.1. *For $x \in I$, there exits a γ, $0 < \gamma < b_*$, such that*

$$|y_\epsilon^{(i)}(x)| \leq M\left(1 + (\sqrt{\epsilon} + x)^{-i} + \epsilon^{-i}\exp(-\gamma(x-1)/\epsilon)\right), \quad i = 1,2,3. \tag{4.1}$$

A more precise analysis leads to a first order asymptotic expansion of the form:

THEOREM 4.2. *The solution u_ϵ of the* repulsive *problem (1.1) satisfies the following :*

$$u_\epsilon(x) = \alpha v_\epsilon(x) + \beta w_\epsilon(x) + z_\epsilon(x), \quad x \in I, \tag{4.2a}$$

$$v_\epsilon(x) = \exp(-\mu x), \quad \mu = \sqrt{\tfrac{c(0)}{\epsilon}}, \quad |\alpha| \leq M, \tag{4.2b}$$

$$w_\epsilon(x) = \exp(-b(1)(x-1)/\epsilon), \quad |\beta| \leq M, \tag{4.2c}$$

$$|z_\epsilon^{(i)}(x)| \leq M\left(1 + (\sqrt{\epsilon} + x)^{1-i} + \epsilon^{1-i}\exp(-\gamma(x-1)/\epsilon)\right), \text{ some } 0 < \gamma < b_*, \; i = 0,1,2,3. \tag{4.2d}$$

Thus the first order boundary layer behaviour is similar to the selfadjoint problem at $x = 0$. The higher order boundary layer functions, at $x = 0$, however show significantly more complicated behaviour similar to that of the *simple* interior turning point problem.

A stability result analogous to that in section 3 can be given for this case also. In fact, if we choose to formulate the discrete problem corresponding to (1.1) in the form

$$L^h w_i := -\epsilon\sigma_i D_+ D_- w_i + a_i^h D_- w_i + c_i^h w_i = f_i^h, \quad i = 1(1)n-1, \tag{4.3a}$$

$$w_0 = U_0, w_n = U_1. \tag{4.3b}$$

then Lemma 3.1 gives a sufficient condition for uniform stability for this problem also.

It is clear that any uniformly convergent scheme on a uniform mesh must model the two boundary layer behaviours given in Theorem 4.2. Thus neither the Il'in-Allen-Southwell scheme nor the Miller scheme above exhibit uniform convergence for this problem. On the other hand, we can construct a composite scheme which is Miller-like near $x = 0$ and resembles Il'in-Allen-Southwell near $x = 1$. This will exhibit uniform convergence, as will the Complete Fitting and El-Mistikawy & Werle schemes, which automatically adapt in this way. Table 4.1 gives some numerical results for this problem.

Scheme	k = 2		k = 3	
	Classical Rate	Uniform Rate	Classical Rate	Uniform Rate
Miller Scheme	1.00	-.01	1.02	-.01
Il'in-Allen-Southwell	2.00	.29	1.99	.04
Complete Fitting	2.00	.92	1.99	.93
El-Mistikawy & Werle	1.86	1.14	1.87	1.15

Table 4.1: Convergence Rates for Repulsive Multiple Turning Point Case

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