RADOVI MATEMATIČKI
Vol. 8 (1992), 105–113.

Analysis of Multiple Turning Point Problems

Relja Vulanović (Novi Sad, Yugoslavia)
Paul A. Farrell (Kent, Ohio)

Abstract. Four types of linear singularly perturbed boundary value problems with a multiple interior turning point are considered. Representations of their solutions are given in terms of boundary layer functions and remainders which are bounded.

0. Introduction

We shall consider the following singularly perturbed two-point boundary value problems:

$$\begin{aligned} -\varepsilon u'' \pm x^k b(x)u' + c(x)u = f(x), \quad & x \in I = [-1,1], \\ u(-1) = U^-, \quad & u(1) = U^+, \end{aligned} \qquad (P_k^\pm)$$

with a small positive parameter ε, $k \in \mathbf{N} \setminus \{1\}$, given numbers $U^\pm$ and sufficiently smooth functions b, c, $f \in C^s(I)$, $s \in \mathbf{N} \cup \{0\}$. We shall assume throughout that these functions and their derivatives are bounded uniformly in ε, and that

$$b(x) \geq b_* > 0, \qquad (1)$$

$$c(x) \geq 0,\ x \in I, \quad c(0) > 0. \qquad (2)$$

The coefficient of the first derivative:

$$a(x) := x^k b(x), \qquad (3)$$

vanishes at $x = 0$ only, and this is an isolated interior turning point of multiplicity k. A problem with a boundary multiple turning point is considered in [7] (the problem is the same as (P_k^-) but on $[0,1]$). Here, not only the "+" and "-" signs of $(P_k^\pm)$ are important, but also even and odd values of

---

AMS (1980) Subject Classification: 34 E 20.

Key words and phrases: singular perturbation, boundary value problem, turning point.

k. Thus we shall consider four possible cases. Three of them have solutions with boundary layers, and in one case there are no layers at all. This means that layers do not occur at the turning point. For the three cases with layers we shall give representations of their solutions containing exponential boundary layer terms and remainders which we shall estimate. Such representations are important because they describe the behaviour of the solutions, and they can be used in construction of numerical methods which are accurate uniformly in ε. This has been done for some other types of singularly perturbed boundary value problems, e.g.: in [4] for non-turning point problems, in [1] and [2] for single turning point problems, and in [7] for the boundary multiple turning point problem. Paper [6] deals with a single turning point problem also, but only derivative estimates of the solution are obtained and used in the appropriate numerical procedure. The same is true for [5] which deals with a general turning point problem, including $(P_k^{\pm})$. Our paper differs from [5] not only by giving the solution representations, but by a weaker assumption on the function c as well. Besides [5] and [7] there are not many papers dealing with multiple turning points. We can mention [3] where asymptotics of general homogeneous turning point problems was considered, but the asymptotic expansions derived are not quite suitable for numerical methods.

In sections that follow we shall consider the four cases of $(P_k^{\pm})$. In essence, our technique is a combination of the techniques from [4] and [5], see [7] as well. We shall use inverse monotonicity of linear second order differential operators. We say that an operator A is inverse monotone (i.m.) on $[x_1, x_2] \subseteq I$ if the following implication holds for any $C^2(I)$-functions y_1 and y_2:

$$\begin{aligned} Ay_1(x) \geq Ay_2(x),\ x \in [x_1, x_2],\ \text{ and }\ y_1(x_j) \geq y_2(x_j),\ j = 1, 2 \\ \Rightarrow\ y_1(x) \geq y_2(x),\ x \in [x_1, x_2]. \end{aligned}$$

It is well known that because of (2) the operator corresponding to $(P_k^{\pm})$ is inverse monotone on I. From this it follows that $(P_k^{\pm})$ has a unique solution $u \in C^{s+2}(I)$.

Throughout this paper M shall denote any (in the sense of $O(1)$) positive constant independent of ε. Some particular constants of this type will be denoted by M_0, M_1, etc. Also, y will denote an arbitrary $C^2(I)$-function.

1. (P_k^+) with k even

Let us introduce the following operators:

$$L_j y = -\varepsilon y'' + a(x)y' + q_j(x)y, \quad j = 0, 1, \ldots, s,$$

where

$$q_j(x) = c(x) + ja(x),$$

and a is given by (3) with k even. In this section, we consider the problem

$$L_0 u = f(x), \quad x \in I, \quad u(\pm 1) = U^{\pm}, \tag{4}$$

but we shall analyze first the auxilary problem:

$$L_0 v = r(x), \quad x \in I, \quad v(\pm 1) = U^{\pm}, \tag{5}$$

where r satisfies:

$$|r^{(i)}(x)| \le M_0[1 + \varepsilon^{-i-1} \exp(\beta(x-1)/\varepsilon)], \; i = 0, 1, \ldots, s, \; x \in I, \tag{6}$$

with $0 < \beta < b_*$, b_* given in (1). We have

Lemma 1. *For the solution v to the problem (5) we have*

$$|v^{(i)}(x)| \le M[1 + \varepsilon^{-i} \exp(m(x-1)/\varepsilon)], \tag{7}$$

where $i = 0, 1, \ldots, s$, $x \in I$ and m is some constant from $(0, \beta)$.

Proof. Noting that $q_j(0) = c(0) > 0$, we conclude that there exists an interval $D = [-\delta, \delta]$, with $\delta \in (0, 1)$, such that

$$q_j(x) \ge q_* > 0, \quad x \in D.$$

Then all the operators L_j are i.m. on D. Moreover, L_0 is i.m. on the whole interval I.

Let us prove (7) for $i = 0$ and $x \in I$. Let

$$g(x) = M_1(2 + x) + M_2 \exp(\beta(x-1)/\varepsilon),$$

where M_1 and M_2 are to be chosen so that

$$L_0 g(x) \ge \pm r(x) = L_0(\pm v(x)), \quad x \in I. \tag{8}$$

Therefore

$$L_0 g(x) \ge M_1[a(x) + (2+x)c(x)] + M_2 \frac{\beta}{\varepsilon}[a(x) - \beta] \exp(\beta(x-1)/\varepsilon).$$

There exists a point $\theta \in (0, 1)$ such that

$$a(x) - \beta \ge b_* \theta^k - \beta =: \alpha > 0, \quad x \in [\theta, 1],$$

and we can choose M_2 so that

$$M_2\beta\alpha \geq M_0.$$

Then, because of

$$a(x) + (2+x)c(x) \geq b_* x^k + c(x) \geq \gamma > 0, \quad x \in I,$$

M_1 can be chosen so that (8) holds (note that all the exponential terms are bounded uniformly in ε outside of $[\theta, 1]$). Moreover, M_1 should be taken so that

$$g(\pm 1) \geq |U^{\pm}|,$$

which together with (8) implies

$$|v(x)| \leq g(x), \quad x \in I,$$

and (7) is proved for $i = 0$, $x \in I$.

Next we shall prove that (7) holds for $i = 1$ on $[\alpha_1, \beta_1] \subset D$, where $\alpha_1 \in (-\delta, 0)$ is such a point that

$$|v'(\alpha_1)| = |\delta^{-1}[v(0) - v(-\delta)]| \leq M,$$

and $\beta_1 \in (0, \delta)$ is chosen similarly, so that

$$|v'(\beta_1)| \leq M.$$

The last two inequalities together with the inverse monotonicity of L_1 on $[\alpha_1, \beta_1]$ complete this part of the proof. Indeed, on $[\alpha_1, \beta_1]$ we have

$$L_1(\pm v'(x)) = \pm[r'(x) - c'(x)v(x)] \leq M \leq L_1 M_3,$$

with some appropriate M_3.

Similarly, we use $L_i v^{(i)}$, $i = 2, 3, \ldots, s$, on suitable subintervals $[\alpha_i, \beta_i]$ of D. For instance, $\beta_2 \in (0, \delta)$ is taken as follows:

$$|v''(\beta_2)| = |\delta^{-2}[v(\delta) - 2v(\frac{\delta}{2}) + v(0)]| \leq M.$$

Thus we conclude that (7) holds for $i = 1, 2, \ldots, s$ and $x \in [\alpha_*, \beta_*] \subset D$:

$$\alpha_* = \max_{1 \leq i \leq s} \alpha_i < 0, \quad \beta_* = \min_{1 \leq i \leq s} \beta_i > 0.$$

Now let $x \in [-1, \alpha_*]$ and let

$$\varphi(x) = \int_x^{\alpha_*} a(t)dt.$$

Then

$$-\varepsilon[\exp(\varphi(x)/\varepsilon)v'(x)]' = [r(x) - c(x)v(x)]\exp(\varphi(x)/\varepsilon)$$

and after integration we get

$$|v'(x)| \le M[\varepsilon^{-1}\int_x^{\alpha_*}\exp(\varphi(t)/\varepsilon)dt + |v'(\alpha_*)|]\exp(-\varphi(x)/\varepsilon) \le$$
$$\le M[1+\varepsilon^{-1}\int_x^{\alpha_*}\exp[(\varphi(t)-\varphi(x))/\varepsilon]dt]. \tag{9}$$

Since for $x \le t \le \alpha_*$ we have

$$\varphi(t) - \varphi(x) \le b_*\alpha_*^k(x-t),$$

from (9) we get $|v'(x)| \le M$, i.e. (7) for $i = 1$ and $x \in [-1, \alpha_*]$. Then we can prove (7) for $i = 2, 3, \ldots, s$ on the same interval, by differentiating (5).

It remains to prove (7) for $i = 1, 2, \ldots, s$, $x \in [\beta_*, 1]$. First we show that

$$|v^{(i)}(1)| \le M\varepsilon^{-i}, \quad i = 1, 2, \ldots, s. \tag{10}$$

For $i = 1$ this follows from integrating (5) from ξ to 1, where $\xi \in (1-\varepsilon, 1)$ is given by

$$|v'(\xi)| = |\varepsilon^{-1}[v(1) - v(1-\varepsilon)]| \le M\varepsilon^{-1}.$$

Then (10) follows for $i = 2, 3, \ldots, s$ directly from (5).

Let $x \in [\beta_*, 1]$ and let

$$\psi(x) = \int_x^1 a(t)dt.$$

We apply the same procedure as in (9), noting that in this case r is not bounded uniformly in ε, (6). We get:

$$|v'(x)| \le M(P+Q),$$

$$P = \varepsilon^{-1}\int_x^1 [1 + \varepsilon^{-1}\exp(\beta(t-1)/\varepsilon)]\exp[(\psi(t)-\psi(x))/\varepsilon]dt,$$

$$Q = |v'(1)|\exp(-\psi(x)/\varepsilon).$$

Setting $\mu = \min\{\beta, b_*\beta_*^k\}$, we obtain:

$$P \le \varepsilon^{-1}\int_x^1 [1+\varepsilon^{-1}\exp(\mu(t-1)/\varepsilon)]\exp(\mu(x-t)/\varepsilon)dt$$
$$\le M\varepsilon^{-2}(1-x)\exp(\mu(x-1)/\varepsilon) \le M[1+\varepsilon^{-1}\exp(m_1(x-1)/\varepsilon)],$$

where $m_1 \in (0, \mu)$. Similarly we get

$$Q \le M\varepsilon^{-1} \exp(m_1(x-1)/\varepsilon),$$

which completes the proof of (7) for $i = 1$, $x \in [\beta_*, 1]$ (with $m \le m_1$). For $i = 2, 3, \dots, s$ we apply the same method to

$$-\varepsilon v^{(i+1)}(x) + a(x)v^{(i)}(x) = h_{i-1}(x),$$

where h_{i-1} contains derivatives of v and r up to the order $i-1$, and thus can be estimated. Then it follows that:

$$|v^{(i)}(x)| \le M[1 + \varepsilon^{-i} \exp(m_i(x-1)/\varepsilon)],$$

with $m_1 > m_2 > \dots > m_s > 0$. This means that (7) holds with $m \le m_s$.

We are ready now to give the representation of the solution.

Theorem 1. *For the solution u to the problem (4) satisfies*

$$u(x) = p\exp[b(1)(x-1)/\varepsilon] + z(x), \quad x \in I,$$

where $|p| \le M$ and z is a differentiable function satisfying

$$|z^{(i)}| \le M[1 + \varepsilon^{1-i} \exp(q(x-1)\varepsilon)], \quad x \in I, \tag{11}$$

for $i = 0, 1, \dots, s$ and some $q \in (0, b_*)$. Thus u has a boundary layer at $x = 1$.

Proof. Let $p = \varepsilon u'(1)/b(1)$. Then we have $|p| \le M$ since u also satisfies the estimate (7). Let

$$w(x) = p\exp[b(1)(x-1)/\varepsilon].$$

Then $z(x) = u(x) - w(x)$ and thus

$$|z'(-1)| \le M, \quad z'(1) = 0.$$

Furthermore,

$$L_0 z'(x) = f'(x) - [L_0 w(x)]' - a'(x)z'(x) - c'(x)z(x) =: h(x),$$

and we shall show that

$$|h^{(i)}(x)| \le M[1 + \varepsilon^{-i-1} \exp(\beta(x-1)/\varepsilon)], \quad x \in I, \tag{12}$$

for $i = 0, 1, \dots, s-1$. This will mean that z' satisfies an equation of type (5) and thus the derivatives of z' satisfy (7), which gives (11).

Let us prove (12). For $x \in I$ we have

$$|z'(x)| \le |u'(x)| + |w'(x)| \le M[1 + \varepsilon^{-1}\exp(\beta(x-1)/\varepsilon)]$$

and

$$|[L_0 w(x)]'| \le M[1 + \varepsilon^{-1} w(x) + \varepsilon^{-2}(1-x)w(x)] \le M[1 + \varepsilon^{-1}\exp(\beta(x-1)/\varepsilon)].$$

Then it follows that (12) holds for $i = 0$, and for $i = 1, 2, \ldots, s-1$ the proof is similar.

2. (P_k^-) with k even

This case reduces to the previous one by changing x to $-x$. This means that the solution to this problem has a boundary layer at $x = -1$.

3. (P_k^-) with k odd

This case is not so interesting since the solution u has no layers:

$$|u^{(i)}(x)| \le M, \quad x \in I, \quad i = 0, 1, \ldots, s.$$

The proof can be found in [5]. It uses the same technique as that in the proof of Lemma 1.

4. (P_k^+) with k odd

In this case we may allow $k = 1$ as well. We shall denote some quantities in this section by the same notation as in Section 1, even though the quantities are not necessarily the same (however, their role is similar). We shall make use of the same operators L_i as in Section 1, except that now k is odd. Let the function r satisfy

$$|r^{(i)}(x)| \le M[1 + \varepsilon^{-i-1}\exp(-\beta_-(x+1)/\varepsilon) + \varepsilon^{-i-1}\exp(\beta_+(x-1)/\varepsilon)], \qquad (13)$$

for $i = 0, 1, \ldots, s$, $x \in I$, and $0 < \beta_\pm < b_*$. We shall consider the problem (5) with such a function r.

Lemma 2. *For the solution v to the problem* (5) *with k odd and r satisfying* (13) *we have*

$$|v^{(i)}(x)| \le M[1 + \varepsilon^{-i}\exp(-m(x+1)/\varepsilon) + \varepsilon^{-i}\exp(m(x-1)/\varepsilon)],$$

for $i = 0, 1, \ldots, s,\ x \in I$ *and* $0 < m < \beta_{\pm}$.

Proof. The proof is analogous to that of Lemma 1. We distinguish again between the three cases: $x \in [-1, \alpha_*]$, $x \in [\alpha_*, \beta_*]$ and $x \in [\beta_*, 1]$, with appropriate α_* and β_* corresponding to those of the proof of Lemma 1. In the case when $x \in [\alpha_*, \beta_*]$ we use inverse monotonicity of L_i on this interval. The case $x \in [\beta_*, 1]$ can be treated in the same way as in Lemma 1, since in this interval

$$\varepsilon^{-i-1} \exp(-\beta_-(x+1)/\varepsilon) \leq M.$$

Finally, the case $x \in [-1, \alpha_*]$ is analogous to $x \in [\beta_*, 1]$.

Theorem 2. *The solution* u *to the problem* (4) *with* k *odd satisfies*

$$u(x) = p_- \exp(-b(-1)(x+1)/\varepsilon) + p_+ \exp(b(1)(x-1)/\varepsilon) + z(x),\ x \in I,$$

where $|p_{\pm}| \leq M$ *and* z *is a differentiable function satisfying*

$$|z^{(i)}(x)| \leq M[1 + \varepsilon^{1-i} \exp(-q(x+1)/\varepsilon) + \varepsilon^{1-i} \exp(q(x-1)/\varepsilon)],$$

for $i = 0, 1, \ldots, s,\ x \in I$ *and* $q \in (0, b_*)$. *Thus* u *has two boundary layers at* $x = \pm 1$.

Proof. Again, we use the same technique as in the proof of Theorem 1. The only interesting difference is that $p_{\pm}$ are determined from the system $z(\pm 1) = 0$. Then using $|u'(\pm 1)| \leq M\varepsilon^{-1}$ we can get $|p_{\pm}| \leq M$.

Acknowledgment. Part of the first author's work was done during his visit to the Department of Mathematical Sciences of Kent State University, Kent, Ohio, in November, 1989. The work and the stay were supported in part by the NSF and and the Fund for Science of Vojvodina through funds made available to the U.S.–Yugoslav Joint Board on Scientific and Technological Cooperation, Project No. JF 799.

REFERENCES

[1] **P. A. Farrell**, *Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem*, SIAM J. Numer. Anal. 25(1988), 618-643.

[2] **P. A. Farrell and E. C. Gartland, Jr.**, *A uniform convergence result for a turning point problem*, in Proc. BAIL V Conference, Shanghai, 1988 (Guo Ben-Yu, J. J. H. Miller and Shi Zhong-Ci, Eds.), Boole Press, Dublin, 1988, pp. 127-132.

[3] **J. Furu**, *Boundary value problems for ordinary differential equations with several multiple turning points*, in Proc. BAIL V Conference, Shanghai, 1988 (Guo Ben-Yu, J. J. H. Miller and Shi Zhong-Ci, Eds.), Boole Press, Dublin, 1988, pp. 172-178.

[4] **R. B. Kellogg and A. Tsan**, *Analysis of some difference approximations for a singular perturbation problem without turning points*, Math. Comput. 32(1978), 1025-1039.

[5] **V. D. Liseikin and V. E. Petrenko**, *Numerical solution of non-linear singularly perturbed problems* (Russian), Preprint 687, USSR Acad. of Sci., Siberian Branch, Comput. Center, Novosibirsk, 1987.

[6] **R. Vulanović**, *On numerical solution of a mildly nonlinear turning point problem*, RAIRO Math. Model. Numer. Anal., 24 (1990), 765-784.

[7] **R. Vulanović and P. A. Farrell**, *Continuous and numerical analysis of a multiple boundary turning point problem*, (submitted).

(Received April 2 1991)
(Revised July 22, 1991)

Institute of Mathematics
Faculty of Science
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad, Yugoslavia

Department of Mathematics
& Computer Science
Kent State University
Kent, Ohio 44242, U. S. A.

Analiza problema sa višestrukim povratnim tačkama

Relja Vulanović (Novi Sad, Yugoslavia)
Paul A. Farrell (Kent, Ohio)

Sadržaj

Posmatraju se četiri tipa linearnih singularno perturbovanih konturnih problema sa višestrukom povratnom tačkom. Reprezentacije njihovih rešenja su date preko funkcija graničnog sloja i ostataka koji su ograničeni.