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LIMITING DISTRIBUTION OF THE BLOCKING PROBABILITY FOR CIRCUIT-SWITCHED NETWORKS

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Abstract—The *end-to-end* blocking probability has been used as a performance criterion to quantify network reliability. Yum and Schwartz have shown that different types of routing schemes for circuit-switched traffic in a nonhierarchical network, with uniform traffic intensity, lead to different performance trade-offs. It is also assumed that the network is symmetric and the redirected traffic on the links is Poisson. It is, however, not clear how the trade-offs would be effected by using different traffic intensities on the links, i.e. if the nodes generate calls with different rates. This paper analyzes the traffic behavior without assuming that the redirected traffic is necessarily Poisson or that the network is symmetric. We examine a statistical model to obtain the *exact limiting distribution* and the *end-to-end blocking* probability of a single specified link, a group of links and the entire network for relatively small networks.

Keywords: Circuit-switching, Limiting distribution, Networks, Nonhierarchical, Routing.

1. INTRODUCTION

Alternate routing schemes have been widely used in hierarchical and nonhierarchical circuit-switched networks [1–3]. The main performance criterion used for the circuit-switched application of a network is the *end-to-end* blocking probability. To improve the performance, different routing algorithms from fixed control (nonalternate) to dynamic control (alternate) have been proposed. For nonhierarchical networks, it was shown by Yum and Schwartz [2] that there is a *cut-off point* for the traffic intensity (ρ), below which alternative routing gives lower blocking probability than direct routing for a given *source–destination* pair in a symmetric network [2]. Their analysis is based on the assumption that calls are generated with the same arrival rate (λ) and the same traffic intensity for each link. The call arrival process was assumed to be Poisson and the call holding time to be exponentially distributed, with the same parameters. Krupp [3] developed a mathematical model for symmetric and uniformly loaded networks with an alternate routing scheme. In his model, the blocking probability was considered as a function of the external traffic load and a bistable blocking behavior was observed during overloads. Later, Akinpelu [1] extended the analysis to nonsymmetric networks and similar instability results were observed. Yum and Schwartz [2] compared different types of routing procedures for fully-connected networks. Their results showed that alternate routing performs better than nonalternate (or direct) routing, if the network traffic is light. For heavy traffic, direct routing performs better as far as the end-to-end blocking probability of a particular link is concerned.

Three parameters determine the performance of a particular routing algorithm. The run-time of the algorithm in which the appropriate path is computed and established. The probability that the algorithm fails to establish a connection due to the network congestion, network failure, etc. Finally, the amount of randomness used by the algorithm. Peleg and Upfa [4] studied the trade-off between these parameters for the problem of routing on a bounded degree network. In this paper, we present an analytical model to compute the exact limiting distribution for direct and alternate routing schemes, under various traffic load conditions, for both symmetric and asymmetric, fully and nonfully connected networks. This technique is different from the technique discussed in Ref. [2]. The network does not have to be symmetric and/or fully connected. The method works for any arbitrary network. We shall show that for a nonsymmetric network there exists a *cut-off surface*, analogous to the *cut-off point* in Ref. [2], for the blocking probability of a single link. Below this cut-off surface, alternate routing has lower blocking probability, whereas above it direct routing has lower blocking probability. A few natural questions arise when investigating direct and alternate routing for general networks:

- Do such trade-offs always exist?
- How sensitive is this surface, when the traffic intensity is not the same for all nodes, i.e. nonuniform traffic is generated by different nodes?
- One would suspect the cut-off surface, when it exists, to be a more complicated function of traffic intensities, especially when the network is not fully connected or is asymmetric. Also, one might suspect that, for different topologies, one could observe trade-off results which are different from those described in [2].
- What is the blocking probability and the cut-off surface for one specific link, and more generally, what is the behavior of the *joint blocking probability* of a group of links or the entire network?
- What does the cut-off surface look like when the links have different capacities?

Section 2 presents an overview of nonhierarchical networks and their control mechanisms. The analytical model and statistical assumptions are discussed in Section 3. Comparisons of direct and indirect routing schemes and their performance for a specific example network are given in Section 4. Conclusions and remarks are covered in Section 5.

2. NONHIERARCHICAL NETWORKS

In most operational networks, the network routing algorithm selects one route, from a set of possible routes, in setting up the network connections. The routing algorithms should meet certain criteria such as efficient use of resources, less blocking probability and minimum cost. The question of interest, in the routing context, is how the routes used affect the overall performance of the network. Certain tools have been developed to calculate the minimum-cost route for any source–destination pair [5].

A network consists of a set of nodes, $N = \{n_1, n_2, \ldots, n_k\}$ and a set of links $L = \{l_{ij}; 1 \leqslant i, j \leqslant k, i < j\}$. Link l_{ij} connecting node i to node j carries c_{ij} number of channels. Path $p_{n_{i_1,i_2,\ldots,i_m}}$ is defined to be a path between nodes n_{i_1} and n_{i_m} via nodes $n_{i_2}, n_{i_3}, \ldots, n_{m-1}$. Without loss of generality, we assume undirected graphs in this work. If the network is fully connected then there would be $k(k-1)/2$ links in its graph. Otherwise, some of the links may be missing. Figure 1 illustrates a four-node fully-connected graph.

A route R is an alternating sequence of nodes and links, beginning and ending with nodes. In a typical and relatively small network, N and L might be measured in tens and the set R could be much larger. Major related issues such as network topology, network connectivity, node and link capacities determine the routing policy. A route can be computed before the call is established, as in *source routing* [6] and the minimum *spanning tree* [7] algorithms, or it can be computed, in a highly distributed fashion, during the call establishment process on a node-by-node basis. In either case, adaptice capability and alternate routing can be added to the controller algorithms. In alternate (indirect) routing, a series of links and paths might be tried before the connection is established. In nonalternate routing, the node at which the request is generated, the source node, uses a fixed path which has already been determined by the shortest path algorithm or any other minimum cost procedure. Studies have shown that alternate routing algorithms reduce the blocking probability [2,8]. There is a trade-off for this apparent gain. Longer routes use more system

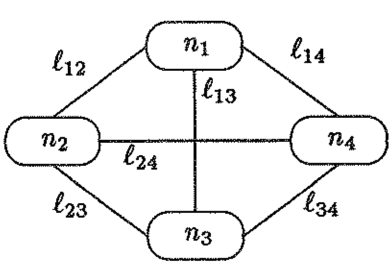

Fig. 1. The configuration of a four-node fully-connected network.

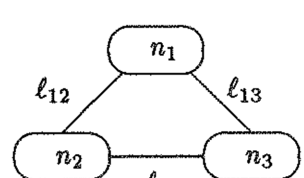

Fig. 2. A configuration of a fully-connected three-node network.

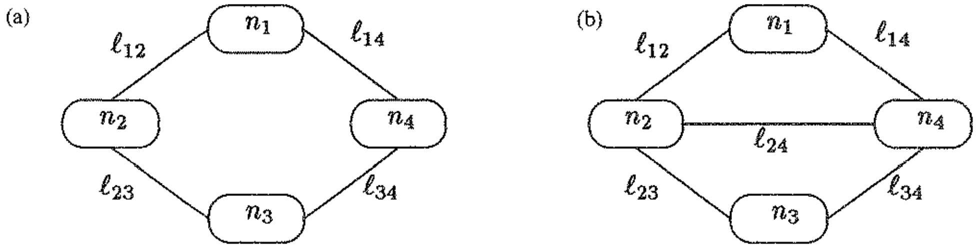

Fig. 3. A configuration of a four-node network.

resources. If there are too many requests, the congestion due to alternately routed traffic may detract unduly from the bandwidth available for the direct route [9].

3. NETWORK MODEL AND STATISTICAL ASSUMPTIONS

Consider the three-node network of Fig. 2. For each source–destination pair, there is a primary path and a secondary (alternate) path in the network. For an arbitrary network, we may have a nonuniform number of paths between any given pair of nodes. These paths are not necessarily disjoint. There are six direct and alternate paths in this network; p_{12}, p_{13}, p_{23}, p_{123}, p_{132} and p_{213}, where for example p_{12} is a path directly connecting node 1 and 2 via link l_{12}, and p_{132} is a path indirectly connecting node 1 and 2 via links l_{13} and l_{32}.

A call is a basic unit of circuit traffic. Each call is originated from a source node i and it is connected to a destination node j. The call arrival process at each node is assumed to be Poisson with rate λ_{ij} and the call holding time is exponentially distributed with mean $1/\mu$ for an Erlang rate of $\rho_{ij} = \lambda_{ij}/\mu$. The call arrival distributions for different nodes are not necessarily uniform. The call setup and disconnection times are very short compared to the call holding time and they are assumed to be zero in this paper.

First, we study the same network as in Ref. [2] but with a different approach. This approach is based on the *limiting joint* distribution of the underlying Markov process. We compare the direct and alternate routing in terms of blocking probability for different types of traffic. The main assumption here is that the call arrival distributions are independent Poisson processes and the call holding times are exponential.

For large number of nodes, the problem may become intractable, when each link has many channels. Even for a small network, the number of possible states could be extremely large, depending on the topology of the network. We developed a set of algorithms to compute the set of all possible valid states a network could attain, the state transitions and the balance equations. Then we used a linear system solver, such as those in the LINPACK routines, to solve these equations, for the probabilities of the various states. Using these probabilities, we can then compute the limiting blocking probability over a specified group of nodes under different traffic patterns for both direct and alternate routing.

The first step to derive the Markov process is to compute the set of valid states. In this paper we shall restrict ourselves to the case where each link has only one channel and hence can carry at most one call at a time. This reduces the complexity of the analysis. Computational results indicate that qualitatively similar results hold for a larger number of channels per link. There are 64 possible states for the network of Fig. 2 and only 14 of them are valid. The network cannot attain the other states due to limited available bandwidth on some links. For example, the state in which a direct call is connected via p_{12} and an indirect call is connected via p_{123} is not a valid state. If we increase the number of channels on each link to 2, then we would have $3^6 = 729$ states and only 85 of them are valid. The four-node fully-connected network of Fig. 1 consists of 30 paths. There are 2^{30} possible states and only 534 of them are valid.

For the four-node network of Fig. 3a, there are 14 paths between nodes and 2^{14} states, only 42 of them valid. For the network of Fig. 3b, there are 20 paths, and 144 valid states out of 2^{20} possible states. The network of Fig. 4 is a more realistic network. It consists of 36 paths and 2^{36} states. Only 306 of them are valid states.

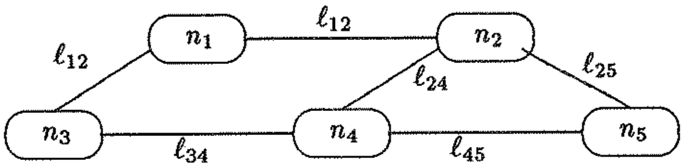

Fig. 4. A configuration of a five-node network.

We developed a parallel algorithm to compute the set of all possible valid states of a network. A bit pattern of size m is assumed for the state representations of the network, where m is the number of different paths in the network. Let Q be an array representing the number of occupied channels on different paths and let Q_k be a path which utilizes link(s) l_{ij} for some i and j, $1 \leqslant i$, $j \leqslant k$ and $i < j$. The following conditions can be checked, in parallel, to determine the validity of a particular state. These conditions determine whether a state can be embedded in the network or not:

1. For every link l_{ij}, $0 \leqslant flow(l_{ij}) \leqslant c_{ij}$.
2. $\Sigma_{k=1}^{l} flow(Q_k) \leqslant c_{ij}$ where Q_k is a path utilizing link(s) l_{ij}, $1 < i, j \leqslant k$ and $i < j$, and l is the number of such paths.

The next step is to compute the state transitions. State transitions were computed by applying events such as call arrival (Add) and departure (Del) to each state of the network. Table 1 shows

Table 1. State transitions

State No.	Current state	Events	Path	Next state
1	000000	Add	p_{12}	100000
		Add	p_{13}	010000
		Add	p_{23}	001000
2	100000	Add	p_{132}	100100
		Add	p_{13}	110000
		Add	p_{23}	101000
		Del	p_{12}	000000
3	010000	Add	p_{12}	110000
		Add	p_{213}	010010
		Add	p_{23}	011000
		Del	p_{13}	000000
4	001000	Add	p_{12}	101000
		Add	p_{13}	011000
		Add	p_{123}	001001
		Del	p_{23}	000000
5	100100	Del	p_{12}	000100
		Del	p_{132}	100000
6	110000	Add	p_{23}	111000
		Del	p_{12}	001000
		Del	p_{13}	100000
7	101000	Add	p_{13}	111000
		Del	p_{12}	001000
		Del	p_{23}	100000
8	010010	Del	p_{13}	000010
		Del	p_{13}	010000
9	011000	Add	p_{12}	111000
		Del	p_{13}	001000
		Del	p_{23}	010000
10	001001	Del	p_{23}	000001
		Del	p_{123}	001000
11	000100	Add	p_{12}	100100
		Del	p_{132}	000000
12	111000	Del	p_{12}	011000
		Del	p_{13}	101000
		Del	p_{23}	110000
13	000010	Add	p_{13}	010010
		Del	p_{13}	000000
14	000001	Add	p_{23}	001001
		Del	p_{123}	000000

the state and state transitions for the network of Fig. 2. The state vector corresponds to paths p_{12}, p_{13}, p_{23}, p_{132}, p_{213} and p_{123}. For example, state 1 corresponds to the network when it is idle. State 8 represents the network when two calls are in progress; a direct call on p_{12} and an indirect call on p_{213}.

Having the states and their transitions, one can generate the corresponding balance equations. The balance equations for the above state transitions are:

$$(\lambda_{12}+\lambda_{13}+\lambda_{23})S_1=\mu_{12}S_2+\mu_{13}S_3+\mu_{23}S_4+\mu_{12}S_{11}+\mu_{13}S_{13}+\mu_{23}S_{14},$$
$$(\lambda_{12}+\lambda_{13}+\lambda_{23}+\mu_{12})S_2=\lambda_{12}S_1+\mu_{12}S_5+\mu_{13}S_6+\mu_{23}S_7,$$
$$(\lambda_{12}+\lambda_{13}+\lambda_{23}+\mu_{13})S_3=\lambda_{13}S_1+\mu_{12}S_6+\mu_{13}S_8+\mu_{23}S_9,$$
$$(\lambda_{12}+\lambda_{13}+\lambda_{23}+\mu_{23})S_4=\lambda_{23}S_1+\mu_{12}S_7+\mu_{13}S_9+\mu_{23}S_{10},$$
$$(\mu_{12}+\mu_{12})S_5=\lambda_{12}S_2+\lambda_{12}S_{11},$$
$$(\lambda_{23}+\mu_{12}+\mu_{13})S_6=\lambda_{13}S_2+\lambda_{12}S_3+\mu_{23}S_{12},$$
$$(\lambda_{13}+\mu_{12}+\mu_{23})S_7=\lambda_{23}S_2+\lambda_{12}S_4+\mu_{13}S_{12},$$
$$(\mu_{13}+\mu_{13})S_8=\lambda_{13}S_3+\lambda_{13}S_{13},$$
$$(\lambda_{12}+\mu_{13}+\mu_{23})S_9=\lambda_{23}S_3+\lambda_{13}S_4+\mu_{12}S_{12},$$
$$(\mu_{23}+\mu_{23})S_{10}=\lambda_{23}S_4+\lambda_{23}S_{14},$$
$$(\lambda_{12}+\mu_{12})S_{11}=\mu_{12}S_5,$$
$$(\mu_{12}+\mu_{13}+\mu_{23})S_{12}=\lambda_{23}S_6+\lambda_{13}S_7+\lambda_{12}S_9,$$
$$(\lambda_{13}+\mu_{13})S_{13}=\mu_{13}S_8,$$
$$(\lambda_{23}+\mu_{23})S_{14}=\mu_{23}S_{10}.$$

Where λ_{ij} is the arrival rate between node i and node j for the Poisson process, μ_{ij} is the departure rate for the same link and P_i is the probability that the network is at state i. The call holding times are assumed to be exponentially distributed with the same parameter μ. Hence $\mu_{ij}=\mu$, for all i, j, $i\neq j$. By dividing both sides of the equations by μ, we have:

$$(\rho_{12}+\rho_{13}+\rho_{23})S_1=S_2+S_3+S_4+S_{11}+S_{13}+S_{14},$$
$$(\rho_{12}+\rho_{13}+\rho_{23}+1)S_2=\rho_{12}S_1+S_5+S_6+S_7,$$
$$(\rho_{12}+\rho_{13}+\rho_{23}+1)S_3=\rho_{13}S_1+S_6+S_8+S_9,$$
$$(\rho_{12}+\rho_{13}+\rho_{23}+1)S_4=\rho_{23}S_1+S_7+S_9+S_{10},$$
$$2S_5=\rho_{12}S_2+\rho_{12}S_{11},$$
$$(\rho_{23}+2)S_6=\rho_{13}S_2+\rho_{12}S_3+S_{12},$$
$$(\rho_{13}+2)S_7=\rho_{23}S_2+\rho_{12}S_4+S_{12},$$
$$2S_8=\rho_{13}S_3+\rho_{13}S_{13},$$
$$(\rho_{12}+2)S_9=\rho_{23}S_3+\rho_{13}S_4+S_{12},$$
$$2S_{10}=\rho_{23}S_4+\rho_{23}S_{14},$$
$$(\rho_{12}+1)S_{11}=S_5,$$
$$3S_{12}=\rho_{23}S_6+\rho_{13}S_7+\rho_{12}S_9,$$
$$(\rho_{13}+1)S_{13}=S_8,$$
$$(\rho_{23}+1)S_{14}=S_{10},$$
$$\sum_{i=1}^{14}S_i=1,$$

where $\rho_{ij} = \lambda_{ij}/\mu$. Using MACSYMA or LINPACK software packages the system can be solved for the S_is, $1 \leqslant i \leqslant 14$. The above technique can be used to compute the end-to-end blocking probability between two nodes, a set of nodes or the entire network, as a function of traffic intensities ρ_{ij} on different links [9]. We solved the above system symbolically using MACSYMA for small networks such as the networks of Figs 2 and 3. For large network the system can be solved numerically using LINPACK software. The results are discussed in Section 3.

The main assumptions for the traffic is that the arrivals at each node and in each direction are independent Poisson processes and the call holding times (service times) are independent and identically exponentially distributed with parameter μ. These are standard assumptions in telephone traffic modeling. We will assume that the redirected traffic takes negligible amount of time to find an alternate connection, or to declare the call is lost due to lack of availability of an end-to-end connection.

3.1. Direct routing scheme

We assume that we have independent arrivals for the use of link l_{ij} where $i, j = 1, 2, \ldots, k$ and $i < j$. At any time t, if a call arrives (either at node n_i, or n_j to use the link l_{ij}) then the request will be granted if there is a free channel on the link. If there is no free channel, the call is declared lost. Thus the nodes in this approach perform no redirection of the traffic. If the call is granted a channel, it holds the channel for a random duration which is exponentially distributed with paramater μ. We assume that the arrival rate for the Poisson process to use the link l_{ij} is λ_{ij}. Clearly, the limiting distribution of the number of busy channels on each link must exist. Let $P_{ij}(c)$ denote the probaiblity that in the limit the link l_{ij} has c channels busy. Thus, $P_{ij}(c_{ij})$ is the probability that the link is full. Since the arrival processes are mutually independent, the joint limiting distribution is the product of the marginal limiting distributions of each link. Therefore, by the usual Erlang formula, the limiting probability, that the link l_{ij} has c_{ij} channels busy, is:

$$\prod_{\substack{i,j=1 \\ i<j}}^{k} \frac{\rho_{ij}^{c_{ij}}}{c_{ij}!\left(\sum_{l=0}^{n} \rho_{ij}^{l}/l!\right)},$$

where $c_{ij} \in \{0, 1, \ldots, n\}$ and $\rho_{ij} = \lambda_{ij}/\mu$. We will compare this distribution with the limiting distribution of the alternate routing scheme.

3.2. Alternate routing scheme

In this scheme, the indepndence of the link status is no longer valid due to the alternately routed traffic. The situation can be easily explained by considering the three-node network. Further, we can simplify the model by assuming $c_{ij} = 1$, for $1 \leqslant j, j \leqslant k$, $i < j$. The state space of the system can be represented by the corners of a unit cube. Here the x-axis is the state of the link l_{12}, the y-axis is the state of link ell_{13} and the z-axis is the state of link l_{23}. These states are numbered as depicted in Fig. 5.

The corners represent the original eight states and the circles represent extra states representing the redirected traffic. This is a Markov process [10] as described in Ref. [9]. The comparison of these schemes is presented in the next section.

4. COMPARISON OF ROUTING SCHEMES

To avoid triviality, we assume that the network has no isolated node. The node, in which the call is originated, is called the source node and the node, which the cell is connected to, is called the destination node. If the direct link connecting the source node to the destination is available then the source node will try to send the information through the direct link. However, if the direct link is not available (either due to congestion or failure) then the source node may or may not reject the request depending upon its routing scheme. In direct routing, the local routing algorithm rejects any call once the direct route is not available. In alternate routing, in the absence of direct link availability, the local routing algorithm tries to find the next available path, leading to the destination, satisfying some performance criterion such as minimum cost, least congested path,

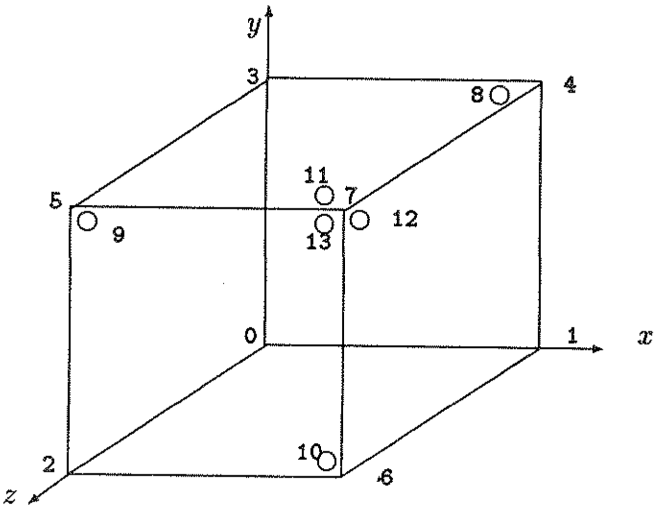

Fig. 5. Illustration of states and their numbering.

least blocking probability, least signalling time, etc. By having this option, the system may utilize available resources better with less overall blocking probability. However, the trade-off of this apparent gain is that the longer the alternate route, the more system resources it will use, and hence, if there are too many requests, the alternate traffic may occupy sufficient bandwidth to detract significantly from the direct traffic opportunities. The direct routing scheme can be analyzed without difficulty and the results are well known [9]. In the following, we provide a method to compute the trade-off using the alternate routing scheme and then compare the results with direct routing.

This model provides a mechanism to compute the end-to-end blocking probability for a given source–destination pair, a set of source–destination pairs and the whole network. We study the performance of the routing schemes for networks with small numbers of channels per link. One of the encouraging observations is that the performance pattern of the two schemes did not change when more channels were added to each link. This gives analytic evidence that one could study a small prototype network of the same topology, rather than the actual network, having a large number of channels per link. The prototype network would have a smaller number of channels per link, and hence a smaller number of balance equations. This approach is fairly general and the algorithms developed here can, in principle, be used for any network.

For the sake of clarity of exposition, we present the analysis for a three-node network. Therefore, most of the discussion will be restricted to this network. The Kolmogorov equations of this network were given earlier. However, we should emphasize that the method is general and can be applied to larger networks.

The system of equations can be solved by hand with some effort and the solution was presented in Ref. [9]. In this paper, we look at the case where the links have two channels. Now it can be shown that there are 85 different states. It is not easy to solve these equations by hand especially when the traffic rates ρ_{12}, ρ_{13} and ρ_{23} are different. We solved the system symbolically using the MACSYMA symbolic algebra program developed at MIT, for the case $\rho_{12} = \rho_{13} = \rho_{23} = \rho$. Using alternate routing, the following blocking probability for link l_{12} is obtained for the case where each link has only one channel:

$$P_{\text{alt}}(l_{12} \text{ traffic is blocked}) = \frac{\rho^4 + 7\rho^3 + 9\rho^2}{\rho^4 + 8\rho^3 + 15\rho^2 + 8\rho + 2}.$$

It is trivial to get the corresponding probability for the direct routing scheme:

$$P_{\text{dir}}(l_{12} \text{ traffic is blocked}) = \frac{\rho}{1 + \rho}.$$

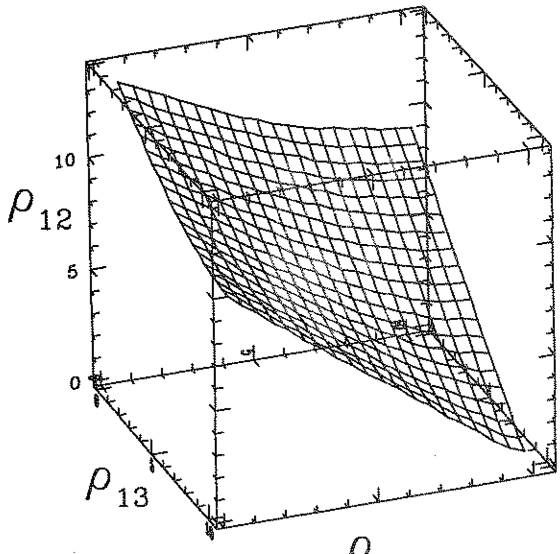

Fig. 6. Cut-off surface for the three-node network with one channel per link.

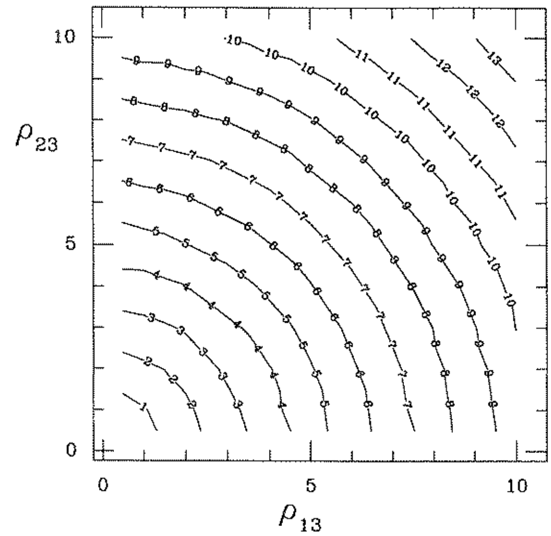

Fig. 7. Contour plot of cut-off surface for the three-node network with one channel per link.

The inequality $P_{alt} \leqslant P_{dir}$ is equivalent to solving $\rho^2 + \rho - 2 \leqslant 0$ which implies that the alternate routing scheme performs better than the direct routing scheme if and only if $\rho \leqslant 1$. In fact, we can find the cut-off surface in the more general case when the input traffic rates are different as well. In this case we obtain the surface given in Fig. 6.

The values of ρ_{12}, ρ_{13} and ρ_{23}, which lie below the surface are the ones for which the alternate routing scheme performs better (i.e. has smaller blocking probability) than the direct routing scheme and the situation is reversed above the surface. For purposes of comparison with the more complicated case considered later, the contour plot of this surface is depicted in Fig. 7.

Similarly, if we consider the probability that both links l_{12} and l_{13} traffic is blocked, then for the equal traffic rates, we get

$$P_{alt}(l_{12} \text{ and } l_{13} \text{ are blocked}) = \frac{\rho^4 + 6\rho^3 + 6\rho^2}{\rho^4 + 8\rho^3 + 15\rho^2 + 8\rho + 2}.$$

It is again trivial to get the corresponding probability for the direct routing scheme:

$$P_{dir}(l_{12} \text{ and } l_{13} \text{ are blocked}) = \left(\frac{\rho}{1+\rho}\right)^2.$$

For this case we can easily show that for all values of ρ we have $P_{dir} \leqslant P_{alt}$. This should not be surprising since we intuitively suspect that the alternate routing scheme will use up the empty links quickly by allowing indirect traffic to utilize them and hence will provide poorer availability for direct traffic on the two links l_{12} and l_{13}. This is illustrated in Fig. 8. In fact, our numerical results show that this observation remains true even when the traffic input rates are different. The behavior is, however, qualitatively different as illustrated in Fig. 9.

If we consider the situation that no call is getting through (i.e. each link in the network is blocked), then for equal arrival rates, we have:

$$P_{alt}(\text{all traffic is blocked}) = \frac{\rho^4 + 5\rho^3 + 3\rho^2}{\rho^4 + 8\rho^3 + 15\rho^2 + 8\rho + 2}.$$

It is trivial to obtain the corresponding probability for the direct routing scheme:

$$P_{dir}(\text{all traffic is blocked}) = \left(\frac{\rho}{1+\rho}\right)^3.$$

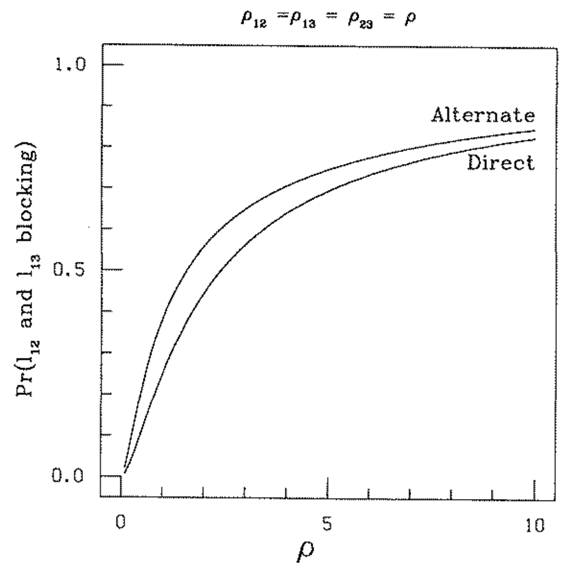

Fig. 8. Blocking probability for l_{12} and l_{13}.

Again we have $P_{\text{dir}} \leqslant P_{\text{alt}}$ for all values of ρ as one would suspect. We have also shown numerically that this phenomenon remains true even for the case when input traffic rates are not necessarily equal.

For the event that some traffic is being blocked somewhere in the network we have the probability:

$$P_{\text{alt}}(\text{some traffic is blocked}) = \frac{\rho^4 + 8\rho^3 + 12\rho^2}{\rho^4 + 8\rho^3 + 15\rho^2 + 8\rho + 2}.$$

One can easily compute the corresponding probability for the direct routing scheme:

$$P_{\text{dir}}(\text{some traffic is blocked}) = 1 - \left(\frac{1}{1+\rho}\right)^3.$$

One would suspect that, in this case, the alternate routing scheme should outperform the direct routing scheme. This is indeed the case as one can easily verify that $P_{\text{alt}} \leqslant P_{\text{dir}}$ for all values of ρ. The result is illustrated in Fig. 10. In fact, numerical results show that this remains true for different input traffic rates ρ_{12}, ρ_{13} and ρ_{23} as shown in Fig. 11.

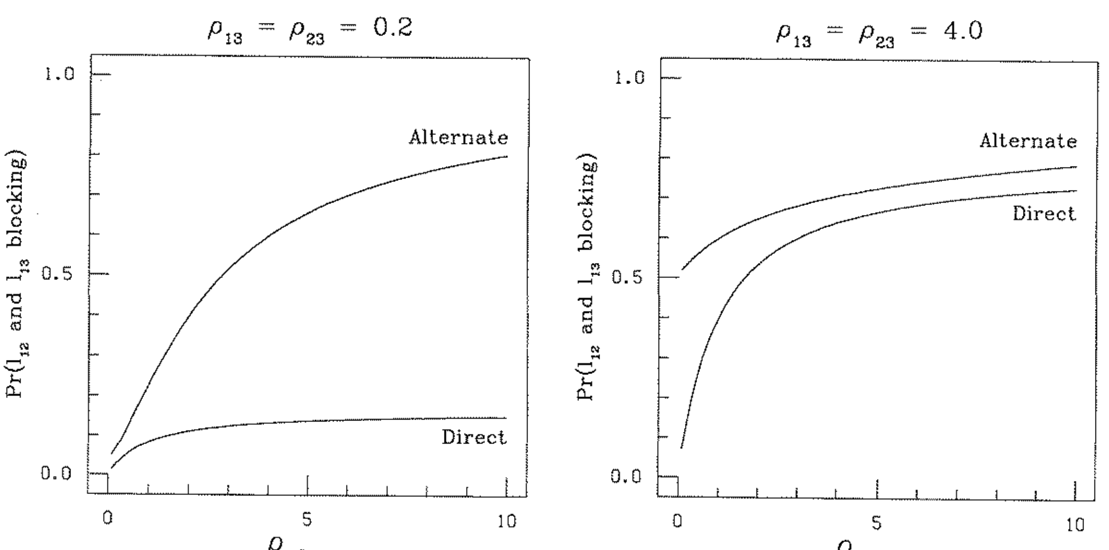

Fig. 9. Blocking probability for l_{12} and l_{13}.

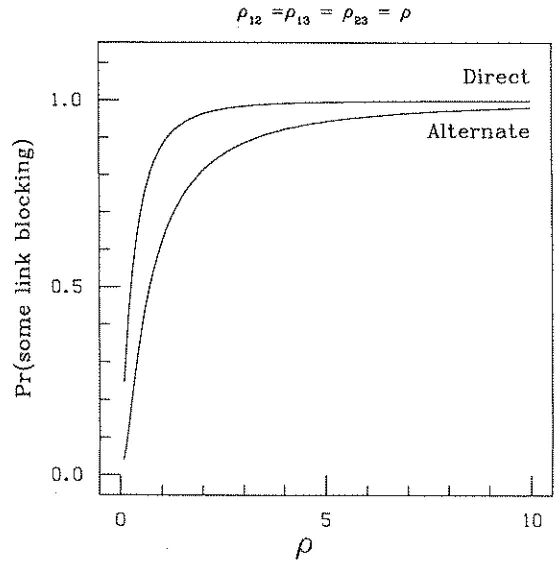

Fig. 10. Blocking probability for some link.

Now we examine the three-node network, where each link has two channels. As stated earlier, there are 85 different states. We again get simple closed form formulae for the above probabilities in the case of equal input traffic rates. We list these probabilities in Appendix A.

For the case, when the traffic arrival rates are different, we can calculate the cut-off surface and contour plot shown in Figs 12 and 13, respectively.

These figures show that the direct routing scheme does not perform as well as the alternate routing scheme for all those arrival rates which lie below the surface and the converse holds above the surface (as far as blocking is concerned for a particular link).

One should note that all the conclusions drawn from the network having one channel per link remain valid for the network having two channels per link. The differences are only in terms of magnitude. In fact, as can be seen from the contour plots for the two cases, they are almost visually indistinguishable. Results, similar to the one channel per link case, are obtained for the blocking probability of l_{12} and l_{13} and of some link being blocked. Therefore, we do not present the results for the larger network in detail. This is a useful observation which gives some credence to assuming

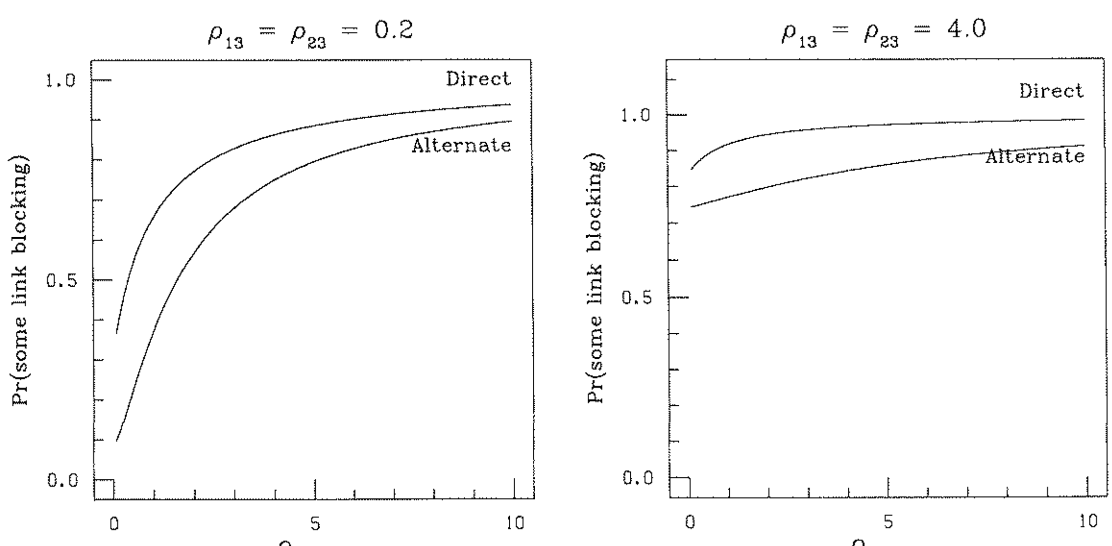

Fig. 11. Blocking probability for some link.

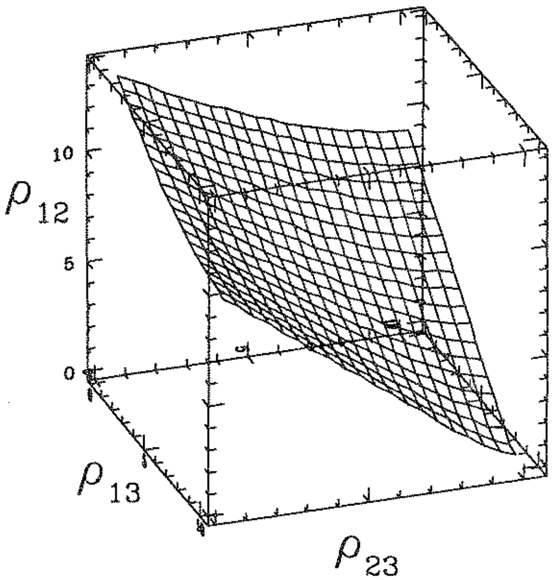

Fig. 12. Cut-off surface for the three-node network, with two channels per link.

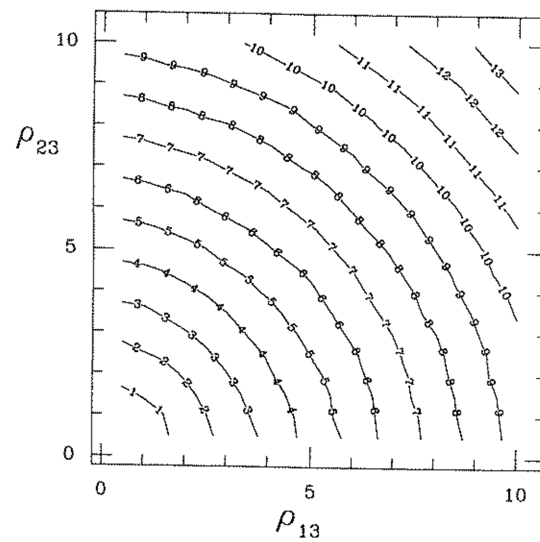

Fig. 13. Contour plot of cut-off surface for the three-node network with two channels per link.

that when a network has a large number of channels per link, we could reduce this number, for analysis purposes and the conclusions could be validly projected back to the original network.

The second observation is that the findings for a network with small number of channels per link should also be visible in the same topology with a larger number of channels. This is due to the fact that as soom as the extra channels are occupied, the two networks behave identically for a short time period. It could be argued that the larger network may have more complicated performance attributes which may not be achievable for the same network with smaller channel capacities. We leave the exhaustive analysis of this topic for further research.

In addition, it appears that, under certain assumptions, larger ring networks behave in a very similar manner to that of the three-node networks considered above. Justification of this conjecture is provided by consideration of the four-node ring network of Fig. 3a, with one channel per link. We remark that, in this case, we must make additional hypotheses in order to guarantee that a comparison between direct and alternate routing is reasonable. In particular, since, unlike the three-node cases considered, this network is not fully connected, we must not permit traffic between nodes 1 and 3 or between 2 and 4, that is we must assume that $\rho_{13} = \rho_{24} = 0$. Otherwise, we permit traffic which cannot possibly be routed under the direct algorithm. This will lead to anomalous results in any comparison of the two methods. If we make this hypothesis, we can show that the network of Fig. 3a behaves in a similar manner to the three-node networks considered earlier. Again the differences in blocking probabilities are only quantitative. For the purpose of illustration we considered the two cases, where the traffic on two adjacent sides (l_{14} and l_{34}) were equal, and the case where traffic on two opposite sides (l_{14} and l_{23}) were equal. For the former case a contour plot of the cut-off surface for the blocking probability of l_{12} is shown in Fig. 14. The surface for the latter case is visually indistinguishable. Also, as in the three-node case, direct routing proves better for the blocking probability of l_{12} and l_{13}, for all loads, whereas alternate routing is always better, if we consider the probability of some link being blocked.

5. CONCLUSIONS

In this paper, we have introduced a new technique to compute the exact end-to-end blocking probability of small nonhierarchical circuit switched networks such as those which arise in private networks. We have analyzed the traffic behavior without assuming that the redirected traffic is necessarily Poisson or the network symmetric. In conclusion, we surmise that:

- The exact limiting distribution (for the alternate routing scheme) of the underlying Markov process for networks with sparse connectivity can be obtained, provided one has sufficient

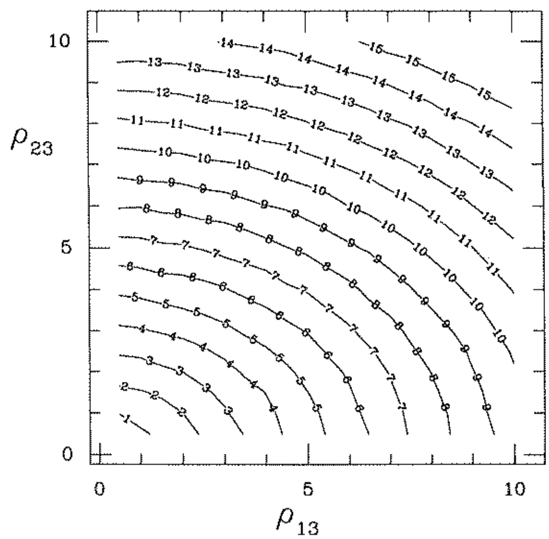

Fig. 14. Contour plot of cut-off surface for the four-node network with one channel per link.

computing facilities. For a few nodes and links this may be obtained in closed form. However for larger networks due to the complexity of the closed form solutions numerical techniques should be used.

- The cut-off surface between the two schemes may not always exist if one wants to study the blocking probability behavior over a group of links. For instance, for the networks studied in this paper, the direct routing scheme performs better (has lower blocking probability) than the alternate routing scheme when we are interested in the traffic of both links l_{12} and l_{13} simultaneously. The same conclusion holds when we want to see the traffic flow over all links. Only when we restrict our attention to one link does there exist a cut-off surface.
- When such a trade-off surface exists, it is a function of the traffic intensities on different links. For the examples presented here, this surface does not seem to be a very complicated function.
- Under certain reasonable hypotheses, the alternate routing scheme always performs better (has lower blocking probability) than the direct routing scheme, when we are interested in minimizing the blocking probability of any link, that is, when we consider the probability that some (arbitrary) link is blocked.
- The channel capacity of links does not seem to play a major role in the analysis of network performance as far as the blocking probabilities are concerned (at least for the networks we studied). It seems as if one can retain the topology of the network, reduce the channel capacity and study the smaller network with little loss in generality.
- There seems evidence to suggest that for ring networks, under the assumptions given, the number of nodes and links does not have a qualitative effect on the relative behavior of the blocking probabilities of direct and alternate routing.

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APPENDIX

Closed Form Solutions for Three-node Network with Two Channels Per Link

We give below the closed form solutions for some blocking probabilities, for the three-node network, where each link has two channels:

$$P_{\text{alt}}(l_{12}\text{ is blocked}) = (4374\rho^{21} + 196{,}101\rho^{20} + 3{,}888{,}648\rho^{19} + 45{,}788{,}490\rho^{18} + 361{,}018{,}998\rho^{17}$$
$$+ 2{,}031{,}625{,}506\rho^{16} + 8{,}475{,}935{,}664\rho^{15} + 26{,}845{,}417{,}248\rho^{14} + 65{,}514{,}779{,}952\rho^{13}$$
$$+ 124{,}249{,}225{,}840\rho^{12} + 183{,}692{,}060{,}688\rho^{11} + 211{,}174{,}905{,}184\rho^{10}$$
$$+ 187{,}013{,}165{,}248\rho^{9} + 125{,}196{,}776{,}576\rho^{8} + 61{,}277{,}812{,}992\rho^{7}$$
$$+ 20{,}671{,}133{,}184\rho^{6} + 4{,}286{,}969{,}640\rho^{5} + 410{,}572{,}800\rho^{4})/\mathit{denom},$$

where

$$\mathit{denom} = 4374\rho^{21} + 204{,}849\rho^{20} + 4{,}272{,}102\rho^{19} + 53{,}331{,}048\rho^{18} + 450{,}282{,}294\rho^{17} + 2{,}748{,}261{,}126\rho^{16}$$
$$+ 12{,}640{,}038{,}840\rho^{15} + 45{,}070{,}169{,}844\rho^{14} + 127{,}214{,}108{,}984\rho^{13} + 288{,}890{,}718{,}880\rho^{12}$$
$$+ 534{,}772{,}537{,}264\rho^{11} + 815{,}500{,}339{,}936\rho^{10} + 1{,}032{,}697{,}868{,}416\rho^{9} + 1{,}091{,}325{,}434{,}240\rho^{8}$$
$$+ 963{,}194{,}629{,}504\rho^{7} + 706{,}529{,}780{,}224\rho^{6} + 425{,}317{,}519{,}872\rho^{5} + 205{,}296{,}832{,}512\rho^{14}$$
$$+ 76{,}459{,}769{,}856\rho^{3} + 20{,}610{,}588{,}672\rho^{2} + 3{,}573{,}227{,}520\rho + 298{,}598{,}400.$$

and

$$P_{\text{dir}}(l_{12}\text{ is blocked}) = \frac{\rho^2}{2 + 2\rho + \rho^2}$$

$$P_{\text{alt}}(l_{12}\text{ and }l_{13}\text{ are blocked}) = (4374\rho^{21} + 187{,}353\rho^{20} + 3{,}557{,}682\rho^{19} + 40{,}196{,}736\rho^{18}$$
$$+ 304{,}618{,}050\rho^{17} + 1{,}649{,}974{,}458\rho^{16} + 6{,}634{,}270{,}536\rho^{15}$$
$$+ 20{,}276{,}793{,}780\rho^{14} + 47{,}812{,}703{,}936\rho^{13} + 87{,}720{,}931{,}600\rho^{12}$$
$$+ 125{,}594{,}399{,}840\rho^{11} + 139{,}939{,}871{,}648\rho^{10} + 120{,}165{,}926{,}656\rho^{9}$$
$$+ 78{,}003{,}245{,}440\rho^{8} + 37{,}001{,}445{,}120\rho^{7} + 12{,}084{,}585{,}984\rho^{6}$$
$$+ 2{,}422{,}794{,}240\rho^{5} + 223{,}948{,}800\rho^{4})/\mathit{denom},$$

and

$$P_{\text{dir}}(l_{12}\text{ and }l_{13}\text{ are blocked}) = \left(\frac{\rho^2}{2 + 2\rho + \rho^2}\right)^2.$$

$$P_{\text{alt}}(\text{all links are blocked}) = (4374\rho^{21} + 178{,}605\rho^{20} + 3{,}226{,}716\rho^{19} + 34{,}604{,}982\rho^{18} + 248{,}217{,}102\rho^{17}$$
$$+ 1{,}268{,}323{,}410\rho^{16} + 4{,}792{,}605{,}408\rho^{15} + 13{,}708{,}170{,}312\rho^{14} + 30{,}110{,}627{,}920\rho^{13}$$
$$+ 51{,}192{,}637{,}360\rho^{12} + 67{,}496{,}738{,}992\rho^{11} + 68{,}704{,}838{,}112\rho^{10}$$
$$+ 53{,}318{,}688{,}064\rho^{9} + 30{,}809{,}714{,}304\rho^{8} + 12{,}725{,}077{,}248\rho^{7}$$
$$+ 3{,}498{,}038{,}784\rho^{6} + 558{,}627{,}840\rho^{5} + 37{,}324{,}800\rho^{4})/\mathit{denom},$$

and

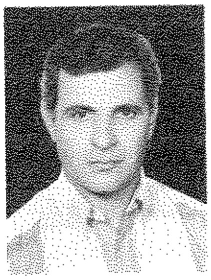
$$P_{\text{dir}}(\text{all links are blocked}) = \left(\frac{\rho^2}{2 + 2\rho + \rho^2}\right)^3.$$

$$P_{\text{alt}}(\text{some link is blocked}) = (4374\rho^{21} + 204{,}849\rho^{20} + 4{,}219{,}614\rho^{19} + 51{,}380{,}244\rho^{18}$$
$$+ 417{,}419{,}946\rho^{17} + 2{,}413{,}276{,}554\rho^{16} + 10{,}317{,}600{,}792\rho^{15}$$
$$+ 3{,}414{,}040{,}716\rho^{14} + 83{,}216{,}855{,}968\rho^{13} + 160{,}777{,}520{,}080\rho^{12}$$
$$+ 241{,}798{,}721{,}526\rho^{11} + 282{,}409{,}938{,}720\rho^{10} + 253{,}860{,}403{,}840\rho^{9}$$
$$+ 172{,}390{,}307{,}712\rho^{8} + 85{,}554{,}180{,}864\rho^{7} + 29{,}257{,}680{,}384\rho^{6}$$
$$+ 6{,}161{,}127{,}040\rho^{5} + 597{,}196{,}800\rho^{4})/\mathit{denom}$$

and

$$P_{\text{dir}}(\text{some link is blocked}) = 1 - \left(\frac{2 + 2\rho}{2 + 2\rho + \rho^2}\right)^3.$$

AUTHORS' BIOGRAPHIES



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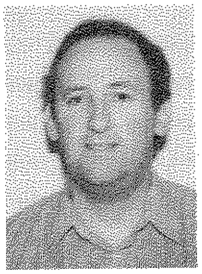

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