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010

CONTINUOUS AND NUMERICAL ANALYSIS OF A MULTIPLE BOUNDARY TURNING POINT PROBLEM*

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Abstract. A singularly perturbed boundary-value problem with a multiple turning point at a boundary is considered. A representation of the solution is given, and it is used in the construction of a uniform finite-difference scheme. The scheme is a first-order exponentially fitted one. An improved modification on a special discretization mesh is given.

Key words. boundary-value problem, singular perturbation, turning point, finite-difference scheme, exponential fitting

AMS subject classifications. 34E10, 65L10

1. Introduction.

Let us consider the following singularly perturbed two-point boundary-value problems:

$$P_k^{\pm}[s,1] : -\epsilon u'' \pm x^k b(x)u' + c(x)u = f(x), \qquad x \in [s,1], \quad u(s) \text{ and } u(1) \text{ given},$$

with a small positive parameter ϵ, $k \in \mathbb{N}$, $s = 0$ or $s = -1$, and

$$b(x) > 0, \qquad c(x) \geq 0, \qquad x \in [s,1].$$

Since the coefficient of the first derivative vanishes at $x = 0$ and at that point only, these problems have an isolated turning point at $x = 0$. The problems $P_k^{\pm}[s,1]$ contain all possible cases modeling turning point behaviour. When $s = 0$ the turning point is at the boundary and we call it a boundary turning point. For $s = -1$ we have interior turning points. If $k = 1$, the turning point is simple, and if $k \geq 2$, it is called a multiple turning point (not only $x^k b(x)$ but its first derivative as well vanishes at $x = 0$). In the plus-sign case it is called a repulsive turning point, and in the minus-sign case it is called an attractive turning point.

Simple turning point problems have attracted the most attention of all turning point problems, both analytically and numerically. $P_1^{\pm}[-1,1]$ (with $c(x) > 0$, $x \in [-1,1]$) is considered in [2]. The numerical method applied there is a (modified) El-Mistikawy–Werle scheme. The result from [2] for $P_1^-[-1,1]$ is improved on in [6] by using a scheme involving parabolic cylinder functions. The same problem is considered in [5], where sufficient conditions for the uniform convergence (i.e., convergence uniform in ϵ of the numerical solution to the exact solution) are investigated. All of these papers are based on continuous analysis of the problems, and they use equidistant discretization meshes. The other approach, discretization on special nonequidistant meshes, also requires analysis of the continuous problem. It is used in [12] and [18] in the case of $P_1^-[0,1]$, but the method can be extended to $P_1^-[-1,1]$ as well.

On the other hand, there are few results in the literature on multiple turning point problems. Examples are [8] and [13], both of which deal with a general turning point problem. In [8] the asymptotics of a homogeneous problem are investigated,

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and in [13] a numerical method based on special discretization meshes is given for a semilinear problem. In the case of linear problems the main assumption from [13] is that $c(x) > 0$ on the whole interval considered.

Singular perturbation problems with turning points arise as mathematical models for various physical phenomena. The problems with interior turning points represent one-dimensional versions of stationary convection-diffusion problems with a dominant convective term and a speed field that changes its sign in the catchbasin. Boundary turning point problems, on the other hand, arise in geophysics [9] and in modeling thermal boundary layers in laminar flow [14, Chap. 12]. The problem from [9] models heat flow and mass transport near an oceanic rise. It is a single boundary turning point problem because of the assumption that the velocity distribution is linear. If one allows for higher orders of velocity distribution, then the boundary turning point becomes multiple. The problems from [14] are multiple (second-order) boundary turning point problems.

In this paper we shall give a continuous and numerical analysis of the attractive multiple boundary turning point, $P_k^-[0,1]$, $k \geq 2$. We consider the problem

$$Lu := -\epsilon u'' - x^k b(x) u' + c(x) u = f(x), \qquad x \in I = [0,1], \tag{1.1a}$$

$$Bu := (u(0), u(1)) = (U_0, U_1), \tag{1.1b}$$

where U_0 and U_1 are given numbers, $0 < \epsilon \leq \epsilon^* \ll 1$, and

$$k = 2 \quad \text{or} \quad k \in [3, +\infty), \tag{1.2a}$$

$$b, c, f \in C^3(I), \tag{1.2b}$$

$$b(x) \geq b_* > 0, \qquad x \in I, \tag{1.2c}$$

$$c(x) \geq 0, \qquad x \in I, \tag{1.2d}$$

$$c(0) > 0. \tag{1.2e}$$

Note that here k is not necessarily an integer—it is sufficient to assume (1.2a) because what we require is that $a \in C^3(I)$, where

$$a(x) := x^k b(x).$$

In §2 we shall show that problem (1.1) has a unique solution $u_\epsilon \in C^5(I)$ bounded uniformly in ϵ. We shall give estimates of the derivatives of u_ϵ by using a technique that is similar to that in [13]. However, our estimates are somewhat different and our assumptions (1.2d), (1.2e) on $c(x)$ are weaker. Moreover, we shall combine the estimates with techniques from [10] and obtain the following, more precise representation of the solution, which is required for the numerical error estimates:

$$u_\epsilon(x) = \omega v_\epsilon(x) + z_\epsilon(x), \qquad x \in I, \tag{1.3a}$$

$$v_\epsilon(x) = \exp(-\mu x), \qquad \mu = \sqrt{\frac{c(0)}{\epsilon}}, \quad |\omega| \leq M, \tag{1.3b}$$

$$|z_\epsilon^{(i)}(x)| \leq M\left(1 + \epsilon^{(1-i)/2} \exp(-mx/\sqrt{\epsilon})\right), \qquad i = 0, 1, 2, 3, \quad x \in I, \tag{1.3c}$$

where m and M are positive constants independent of ϵ. This representation shows that u_ϵ has an $O(\sqrt{\epsilon})$ boundary layer at $x = 0$. Thus the layer is similar to that of the selfadjoint problem ($b(x) \equiv 0,\ c(x) > 0,\ x \in I$), [4, Chap. 6], [16], [17], as opposed to the boundary layers in the nonselfadjoint nonturning point or repulsive interior turning point problems. However, the solution to the selfadjoint problem has two boundary layers and here we have one layer only, because the solution u_0 of the reduced problem, being that of a first-order differential equation, can satisfy the right-hand boundary condition

$$-a(x)u_0' + c(x)u_0 = f(x), \qquad x \in I, \quad u_0(1) = U_1.$$

Such behaviour of u_ϵ might be expected from the asymptotic treatment in [11, p. 65] of the model problem of type (1.1).

In §3 we shall give a first-order uniform finite-difference scheme for problem (1.1). The scheme is an exponentially fitted one with a fitting factor that is, naturally, of a form similar to that of the selfadjoint problem [4, Chap. 6], [16], [17]. An arbitrary discretization mesh will be used. In the case of selfadjoint problems the fitted scheme is improved when a special discretization mesh that is dense in the layers is used [17]. Here we shall show that this is not the case for (1.1). This is due to the upwind discretization of the first-derivative term, which has the same accuracy on any mesh. Still, we are interested in improving the first-order scheme by using a special mesh. One possibility is to apply the second-order scheme from [19] (see [18] as well). However, to prove second-order uniform convergence we would need stronger smoothness assumptions; thus (1.2a) and (1.2b) should be changed to $k \in \{2,3\} \bigcup [4, +\infty)$ and $b, c, f \in C^4(I)$, respectively. To avoid this we choose another approach—we modify the first-order scheme and obtain the error

$$O(\sqrt{\epsilon}n^{-1} + n^{-2})$$

on a special mesh with n mesh steps. This is the result of §4. Obviously, the modified scheme has second-order accuracy as long as $\sqrt{\epsilon} \leq n^{-1}$, and it gives an improvement (on the special mesh only) over the standard fitted scheme.

Both schemes will be investigated by the following well-known principle:

$$\text{uniform stability} \ + \ \text{uniform consistency} \ \Rightarrow \ \text{uniform convergence}.$$

Uniform stability will be proved by using M-matrix theory [15]. As for the uniform consistency, it will follow simply for the first-order scheme, whereas for the improved scheme a technique similar to that in [19] will be used. We shall give numerical results for the schemes in §§3 and 4.

To summarize, the main purpose of this paper is to show that certain numerical methods, designed for particular singular perturbation problems, can be extended to new types of problems that are of interest in the modeling of physical processes. In this paper we focus on problems with a multiple boundary turning point of the attractive type, whereas other types of multiple turning point problems are considered in [20]. The numerical methods examined are exponential fitting and a priori mesh construction. Another purpose of the paper is to draw attention to these mesh construction techniques, which were considered as early as 1969 in [1] but which have not received significant exposure in the English-language literature. The discretization mesh is generated a priori by some suitable function. The mesh-generating function automatically redistributes mesh points as ϵ changes, keeping the same percentage of them in the layer. In order to construct such a function, the behaviour of the continuous solution must be known. This is a shortcoming of the method in comparison to a posteriori mesh-generation techniques. However, an advantage is that for all values of ϵ the same number of mesh points is needed in order to preserve the same accuracy. This is not so with other mesh-generation methods, such as that described in [3]. Thus, rigorously speaking, these methods are *not* uniform in ϵ in the sense we describe here. Finally, our intention here is to show that a combination of exponential fitting and special discretization meshes can improve numerical results significantly.

One must remark, however, that a priori methods have certain limitations. In particular, exponential fitting can be applied only to problems for which the leading-order term in the asymptotic expansion is known. This is not always the case for more complicated nonlinear problems. However, a possible application of our approach to nonlinear problems arises when a linearization technique is used and a sequence of linear problems is obtained. Our approach can then be applied to each of the linear problems in turn. On the other hand, a priori methods have noticeable advantages over a posteriori methods when implementation on parallel computers is considered. This is because the adaption process inherent in a posteriori methods introduces sequentiality to the solution process, which is absent in the a priori case.

2. Continuous analysis. Recall that $a(x) = x^k b(x)$. By M we shall denote any positive constant independent of ϵ. Some of these constants will, however, be denoted by M_0, M_1, etc.

LEMMA 2.1. *Problem* (1.1) *has a unique solution* u_ϵ *that is bounded uniformly in* ϵ*:*

$$|u_\epsilon(x)| \le M, \qquad x \in I. \tag{2.1}$$

Proof. Because of (1.2d) the operator (L, B) is inverse monotone, and existence and uniqueness follow easily. From (1.2e) it follows that there exists a number $\theta \in (0,1)$ independent of ϵ such that

$$c(x) \ge c_* > 0 \quad \text{for } x \in [0, \theta].$$

Letting

$$p(x) = M_0(2 - x),$$

we have

$$Lp(x) = a(x)M_0 + c(x)p(x)$$

and

$$Lp(x) \ge c_* M_0 \quad \text{for } x \in [0, \theta],$$
$$Lp(x) \ge \theta^k b_* M_0 \quad \text{for } x \in [\theta, 1].$$

Let

$$F = \max_{x \in I} |f(x)|,$$

and choose M_0 sufficiently large that

$$c_* M_0 \ge F, \qquad \theta^k b_* M_0 \ge F,$$

and

$$2M_0 \geq |U_0|, \qquad M_0 \geq |U_1|.$$

Thus we get

$$Lp(x) \geq \pm Lu_\epsilon(x), \qquad x \in I,$$
$$Bp \geq \pm Bu_\epsilon \qquad \text{(componentwise)},$$

and by inverse monotonicity we obtain (2.1). □

Next, let $g_\epsilon(x) \in C^3(I)$ be a function such that

$$|g_\epsilon^{(i)}(x)| \leq M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0,1,2,3, \quad x \in I, \quad \mu_* = \sqrt{\frac{q_*}{\epsilon}}.$$

Then according to Lemma 2.1 the problem

$$Lu(x) = g_\epsilon(x), \qquad x \in I, \qquad Bu = (U_0, U_1)$$

has a unique solution y_ϵ and

$$|y_\epsilon(x)| \leq M, \qquad x \in I.$$

Now let us define $q_i(x)$ by

$$q_i(x) = c(x) - ia'(x), \qquad i = 1,2,3.$$

Since

$$q_i(0) = c(0) > 0, \qquad i = 1,2,3,$$

there exists a point $\theta_0 \in (0,1)$, independent of ϵ, such that

$$q_i(x) > q_* > 0, \qquad x \in [0, \theta_0], \quad i = 1,2,3. \tag{2.2}$$

LEMMA 2.2. *There exist points $\theta_i \in (0, \theta_0)$, $i = 1,2,3$, independent of ϵ and such that*

$$|y_\epsilon^{(i)}(\theta_i)| \leq M, \qquad i = 1,2,3. \tag{2.3}$$

On the other hand,

$$|y_\epsilon^{(i)}(0)| \leq M\epsilon^{-i/2}, \qquad i = 1,2,3. \tag{2.4}$$

Proof. Let θ_1 be a point such that

$$y_\epsilon'(\theta_1) = \frac{1}{\theta_0}(y_\epsilon(\theta_0) - y_\epsilon(0)), \qquad \theta_1 \in (0, \theta_0).$$

Then (2.3) is obvious for $i = 1$. Similarly, we choose θ_2 from $(0, \theta_0)$ so that

$$y_\epsilon''(\theta_2) = \theta_0^{-2}\left[y_\epsilon(\theta_0) - 2y_\epsilon\left(\frac{\theta_0}{2}\right) + y_\epsilon(0)\right],$$

and θ_3 can be found analogously.

Now let us prove (2.4) for $i = 1$. Rewrite the differential equation in the form

$$-\epsilon y_\epsilon''(x) - (a(x)y_\epsilon(x))' + (a'(x) + c(x))y_\epsilon(x) = g_\epsilon(x),$$

and integrate from zero to the point x^* such that

$$y_\epsilon'(x^*) = \frac{y_\epsilon(\sqrt{\epsilon}) - y_\epsilon(0)}{\sqrt{\epsilon}}, \qquad x^* \in (0, \sqrt{\epsilon}).$$

It follows that

$$\epsilon y_\epsilon'(0) = \epsilon y_\epsilon'(x^*) + (ay_\epsilon)(x^*) + \int_0^{x^*} [g_\epsilon - (a' + c)y_\epsilon](x)\, dx,$$

and, since from the definition of x^*

$$|y_\epsilon'(x^*)| \le M\epsilon^{-1/2},$$

on division by ϵ we obtain

$$|y_\epsilon'(0)| \le M(\epsilon^{-1/2}\epsilon^{k/2-1} + \epsilon^{-1}\epsilon^{1/2}) \le M\epsilon^{-1/2}.$$

Then (2.4) follows for $i = 2$, from $Ly_\epsilon(x) = g_\epsilon(x)$ at $x = 0$, and for $i = 3$ after differentiation. □

LEMMA 2.3. *For $x \in I$ we have*

$$|y_\epsilon^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 1, 2, 3. \tag{2.5}$$

Proof. Let us introduce the operators

$$L_i u := Lu - ia'(x)u = -\epsilon u'' - a(x)u' + q_i(x)u, \qquad i = 1, 2, 3.$$

Because of (2.2) these operators are inverse monotone on $[0, \theta_0]$. We shall use inverse monotonicity of L_i on $[0, \theta_i]$, $i = 1, 2, 3$, where θ_i are the same as in Lemma 2.2. Let p_i be the following barrier functions:

$$p_i(x) = M_i(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 1, 2, 3.$$

It is easy to show (cf. [13]) that

$$L_i p_i(x) \ge M_i[q_i(x) + \epsilon^{-i/2}(q_i(x) - q_*)\exp(-\mu_* x)], \qquad i = 1, 2, 3, \quad x \in I, \tag{2.6}$$

and

$$\pm L_i y_\epsilon^{(i)}(x) \le M\left(1 + \epsilon^{-i/2}\exp(-\mu_* x)\right), \qquad i = 1, 2, 3, \quad x \in I. \tag{2.7}$$

For instance, since

$$L_1 y_\epsilon'(x) = (g_\epsilon' - c'y_\epsilon)(x),$$

we obtain (2.7) for $i = 1$.

From (2.6) and (2.7) we can conclude that M_i, $i = 1, 2, 3$, can be chosen so that

$$L_i p_i(x) \ge \pm L_i y_\epsilon^{(i)}(x), \qquad x \in [0, \theta_i], \quad i = 1, 2, 3,$$

and

$$p_i(x) \geq \pm y_\epsilon^{(i)}(x), \qquad x \in \{0, \theta_i\}, \quad i = 1, 2, 3.$$

Then by inverse monotonicity (2.5) follows for $x \in [0, \theta_*]$, where

$$\theta_* = \min\{\theta_1, \theta_2, \theta_3\}.$$

It remains to prove the results on $[\theta_*, 1]$, that is,

$$|y_\epsilon^{(i)}(x)| \leq M, \qquad i = 1, 2, 3, \quad x \in [\theta_*, 1]. \tag{2.8}$$

First we prove (2.8) for $i = 1$. Let

$$\varphi(x) = \int_{\theta_*}^x t^k b(t) dt.$$

Then we have

$$-\epsilon(e^{\varphi(x)/\epsilon} y_\epsilon'(x))' = (g_\epsilon - c y_\epsilon)(x) e^{\varphi(x)/\epsilon}$$

and

$$y_\epsilon'(x) = \left[\frac{1}{\epsilon}\int_{\theta_*}^x (c y_\epsilon - g_\epsilon)(t) e^{\varphi(t)/\epsilon} dt + y_\epsilon'(\theta_*)\right] e^{-\varphi(x)/\epsilon}.$$

Since

$$|y_\epsilon'(\theta_*)| \leq M,$$

it follows that

$$|y_\epsilon'(x)| \leq M\left(1 + \frac{1}{\epsilon}\int_{\theta_*}^x e^{(\varphi(t)-\varphi(x))/\epsilon} dt\right), \qquad x \in [\theta_*, 1]. \tag{2.9}$$

Now from

$$\varphi(t) - \varphi(x) \leq b_* \frac{t^{k+1} - x^{k+1}}{k+1} \leq \frac{b_*}{k+1} \theta_*^k (t - x)$$

and (2.9) we obtain (2.8) for $i = 1$.

Similarly, after differentiating $L y_\epsilon(x) = g_\epsilon(x)$ once and expressing $y_\epsilon''(x)$ by means of integration, we obtain (2.8) for $i = 2$. In this proof we use

$$|y_\epsilon'(x)| \leq M, \qquad |g_\epsilon'(x)| \leq M, \qquad x \in [\theta_*, 1].$$

The proof for $i = 3$ is analogous. □

THEOREM 2.4. *The solution u_ϵ to problem* (1.1) *has the following representation:*

$$u_\epsilon(x) = \omega v_\epsilon(x) + z_\epsilon(x), \qquad x \in I,$$

$$v_\epsilon(x) = \exp(-\mu x), \qquad \mu = \sqrt{\frac{c(0)}{\epsilon}}, \quad |\omega| \leq M,$$

$$|z_\epsilon^{(i)}(x)| \leq M\left(1 + \epsilon^{(1-i)/2}\exp(-m x/\sqrt{\epsilon})\right), \qquad i = 0, 1, 2, 3, \quad x \in I,$$

where $m = \sqrt{q_*}$.

Proof. Let the constant ω be determined by the condition

$$z_\epsilon'(0) = 0. \tag{2.10a}$$

Then $|\omega| \le M$ follows because from Lemma 2.2 we have

$$|u_\epsilon'(0)| \le M\epsilon^{-1/2}.$$

Thus we obtain (1.3c) for $i = 0$. Also, it is obvious that

$$|z_\epsilon'(1)| \le M. \tag{2.10b}$$

Furthermore,

$$Lz_\epsilon'(x) = s_\epsilon(x), \qquad x \in I,$$

where

$$s_\epsilon(x) = f'(x) - \omega(Lv_\epsilon(x))' + a'(x)z_\epsilon'(x) - c'(x)z_\epsilon(x).$$

If we show that

$$|s_\epsilon^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0, 1, 2, \quad x \in I, \tag{2.11}$$

then by (2.10) and Lemma 2.3 we obtain

$$|(z_\epsilon')^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0, 1, 2, \quad x \in I,$$

which is (1.3c), for $i = 1, 2, 3$.

To illustrate the proof of (2.11) let us consider the case $i = 0$. We have

$$(Lv_\epsilon(x))' = [c'(x) + a'(x)\mu - \mu(c(x) - c(0) + a(x)\mu)]v_\epsilon(x).$$

Then, since

$$c(x) - c(0) = x\,c'(\eta), \qquad \eta \in (0, x),$$

we obtain

$$|(Lv_\epsilon(x))'| \le M(1 + x^{k-1}\epsilon^{-1/2} + x\epsilon^{-1/2} + x^k\epsilon^{-1})v_\epsilon(x) \le M.$$

Similarly,

$$|a'(x)z_\epsilon'(x)| \le Mx^{k-1}(|u_\epsilon'(x)| + |v_\epsilon'(x)|) \le M.$$

Finally, it is obvious that the remaining terms of $s_\epsilon(x)$ are uniformly bounded. The proof of (2.11) for $i = 1, 2$ is analogous (cf. [10]). Note that we use the condition

$$c(0) > q_*$$

when dealing with v_ϵ. □

and

$$p_i(x) \geq \pm y_\epsilon^{(i)}(x), \qquad x \in \{0, \theta_i\}, \quad i = 1, 2, 3.$$

Then by inverse monotonicity (2.5) follows for $x \in [0, \theta_*]$, where

$$\theta_* = \min\{\theta_1, \theta_2, \theta_3\}.$$

It remains to prove the results on $[\theta_*, 1]$, that is,

$$|y_\epsilon^{(i)}(x)| \leq M, \qquad i = 1, 2, 3, \quad x \in [\theta_*, 1]. \tag{2.8}$$

First we prove (2.8) for $i = 1$. Let

$$\varphi(x) = \int_{\theta_*}^x t^k b(t) dt.$$

Then we have

$$-\epsilon(e^{\varphi(x)/\epsilon} y_\epsilon'(x))' = (g_\epsilon - c y_\epsilon)(x) e^{\varphi(x)/\epsilon}$$

and

$$y_\epsilon'(x) = \left[\frac{1}{\epsilon}\int_{\theta_*}^x (c y_\epsilon - g_\epsilon)(t) e^{\varphi(t)/\epsilon} dt + y_\epsilon'(\theta_*)\right] e^{-\varphi(x)/\epsilon}.$$

Since

$$|y_\epsilon'(\theta_*)| \leq M,$$

it follows that

$$|y_\epsilon'(x)| \leq M\left(1 + \frac{1}{\epsilon}\int_{\theta_*}^x e^{(\varphi(t)-\varphi(x))/\epsilon} dt\right), \qquad x \in [\theta_*, 1]. \tag{2.9}$$

Now from

$$\varphi(t) - \varphi(x) \leq b_* \frac{t^{k+1} - x^{k+1}}{k+1} \leq \frac{b_*}{k+1}\theta_*^k (t - x)$$

and (2.9) we obtain (2.8) for $i = 1$.

Similarly, after differentiating $L y_\epsilon(x) = g_\epsilon(x)$ once and expressing $y_\epsilon''(x)$ by means of integration, we obtain (2.8) for $i = 2$. In this proof we use

$$|y_\epsilon'(x)| \leq M, \qquad |g_\epsilon'(x)| \leq M, \qquad x \in [\theta_*, 1].$$

The proof for $i = 3$ is analogous. □

THEOREM 2.4. *The solution u_ϵ to problem* (1.1) *has the following representation:*

$$u_\epsilon(x) = \omega v_\epsilon(x) + z_\epsilon(x), \qquad x \in I,$$

$$v_\epsilon(x) = \exp(-\mu x), \qquad \mu = \sqrt{\frac{c(0)}{\epsilon}}, \quad |\omega| \leq M,$$

$$|z_\epsilon^{(i)}(x)| \leq M\left(1 + \epsilon^{(1-i)/2}\exp(-mx/\sqrt{\epsilon})\right), \qquad i = 0, 1, 2, 3, \quad x \in I,$$

where $m = \sqrt{q_*}$.

Proof. Let the constant ω be determined by the condition

$$z_\epsilon'(0) = 0. \tag{2.10a}$$

Then $|\omega| \le M$ follows because from Lemma 2.2 we have

$$|u_\epsilon'(0)| \le M\epsilon^{-1/2}.$$

Thus we obtain (1.3c) for $i = 0$. Also, it is obvious that

$$|z_\epsilon'(1)| \le M. \tag{2.10b}$$

Furthermore,

$$Lz_\epsilon'(x) = s_\epsilon(x), \qquad x \in I,$$

where

$$s_\epsilon(x) = f'(x) - \omega(Lv_\epsilon(x))' + a'(x)z_\epsilon'(x) - c'(x)z_\epsilon(x).$$

If we show that

$$|s_\epsilon^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0, 1, 2, \quad x \in I, \tag{2.11}$$

then by (2.10) and Lemma 2.3 we obtain

$$|(z_\epsilon')^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0, 1, 2, \quad x \in I,$$

which is (1.3c), for $i = 1, 2, 3$.

To illustrate the proof of (2.11) let us consider the case $i = 0$. We have

$$(Lv_\epsilon(x))' = [c'(x) + a'(x)\mu - \mu(c(x) - c(0) + a(x)\mu)]v_\epsilon(x).$$

Then, since

$$c(x) - c(0) = x\,c'(\eta), \qquad \eta \in (0, x),$$

we obtain

$$|(Lv_\epsilon(x))'| \le M(1 + x^{k-1}\epsilon^{-1/2} + x\epsilon^{-1/2} + x^k\epsilon^{-1})v_\epsilon(x) \le M.$$

Similarly,

$$|a'(x)z_\epsilon'(x)| \le Mx^{k-1}(|u_\epsilon'(x)| + |v_\epsilon'(x)|) \le M.$$

Finally, it is obvious that the remaining terms of $s_\epsilon(x)$ are uniformly bounded. The proof of (2.11) for $i = 1, 2$ is analogous (cf. [10]). Note that we use the condition

$$c(0) > q_*$$

when dealing with v_ϵ. $\square$

and

$$p_i(x) \geq \pm y_\epsilon^{(i)}(x), \qquad x \in \{0, \theta_i\}, \quad i = 1, 2, 3.$$

Then by inverse monotonicity (2.5) follows for $x \in [0, \theta_*]$, where

$$\theta_* = \min\{\theta_1, \theta_2, \theta_3\}.$$

It remains to prove the results on $[\theta_*, 1]$, that is,

$$|y_\epsilon^{(i)}(x)| \leq M, \qquad i = 1, 2, 3, \quad x \in [\theta_*, 1]. \tag{2.8}$$

First we prove (2.8) for $i = 1$. Let

$$\varphi(x) = \int_{\theta_*}^{x} t^k b(t) dt.$$

Then we have

$$-\epsilon(e^{\varphi(x)/\epsilon} y'_\epsilon(x))' = (g_\epsilon - c y_\epsilon)(x) e^{\varphi(x)/\epsilon}$$

and

$$y'_\epsilon(x) = \left[\frac{1}{\epsilon}\int_{\theta_*}^{x} (c y_\epsilon - g_\epsilon)(t) e^{\varphi(t)/\epsilon} dt + y'_\epsilon(\theta_*)\right] e^{-\varphi(x)/\epsilon}.$$

Since

$$|y'_\epsilon(\theta_*)| \leq M,$$

it follows that

$$|y'_\epsilon(x)| \leq M\left(1 + \frac{1}{\epsilon}\int_{\theta_*}^{x} e^{(\varphi(t)-\varphi(x))/\epsilon} dt\right), \qquad x \in [\theta_*, 1]. \tag{2.9}$$

Now from

$$\varphi(t) - \varphi(x) \leq b_* \frac{t^{k+1} - x^{k+1}}{k+1} \leq \frac{b_*}{k+1}\theta_*^k (t - x)$$

and (2.9) we obtain (2.8) for $i = 1$.

Similarly, after differentiating $Ly_\epsilon(x) = g_\epsilon(x)$ once and expressing $y''_\epsilon(x)$ by means of integration, we obtain (2.8) for $i = 2$. In this proof we use

$$|y'_\epsilon(x)| \leq M, \qquad |g'_\epsilon(x)| \leq M, \qquad x \in [\theta_*, 1].$$

The proof for $i = 3$ is analogous. □

THEOREM 2.4. *The solution u_ϵ to problem* (1.1) *has the following representation:*

$$u_\epsilon(x) = \omega v_\epsilon(x) + z_\epsilon(x), \qquad x \in I,$$

$$v_\epsilon(x) = \exp(-\mu x), \qquad \mu = \sqrt{\frac{c(0)}{\epsilon}}, \quad |\omega| \leq M,$$

$$|z_\epsilon^{(i)}(x)| \leq M\left(1 + \epsilon^{(1-i)/2}\exp(-mx/\sqrt{\epsilon})\right), \qquad i = 0, 1, 2, 3, \quad x \in I,$$

where $m = \sqrt{q_*}$.

Proof. Let the constant ω be determined by the condition

$$z_\epsilon'(0) = 0. \tag{2.10a}$$

Then $|\omega| \le M$ follows because from Lemma 2.2 we have

$$|u_\epsilon'(0)| \le M\epsilon^{-1/2}.$$

Thus we obtain (1.3c) for $i = 0$. Also, it is obvious that

$$|z_\epsilon'(1)| \le M. \tag{2.10b}$$

Furthermore,

$$Lz_\epsilon'(x) = s_\epsilon(x), \qquad x \in I,$$

where

$$s_\epsilon(x) = f'(x) - \omega(Lv_\epsilon(x))' + a'(x)z_\epsilon'(x) - c'(x)z_\epsilon(x).$$

If we show that

$$|s_\epsilon^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0, 1, 2, \quad x \in I, \tag{2.11}$$

then by (2.10) and Lemma 2.3 we obtain

$$|(z_\epsilon')^{(i)}(x)| \le M(1 + \epsilon^{-i/2}\exp(-\mu_* x)), \qquad i = 0, 1, 2, \quad x \in I,$$

which is (1.3c), for $i = 1, 2, 3$.

To illustrate the proof of (2.11) let us consider the case $i = 0$. We have

$$(Lv_\epsilon(x))' = [c'(x) + a'(x)\mu - \mu(c(x) - c(0) + a(x)\mu)]v_\epsilon(x).$$

Then, since

$$c(x) - c(0) = x\,c'(\eta), \qquad \eta \in (0, x),$$

we obtain

$$|(Lv_\epsilon(x))'| \le M(1 + x^{k-1}\epsilon^{-1/2} + x\epsilon^{-1/2} + x^k\epsilon^{-1})v_\epsilon(x) \le M.$$

Similarly,

$$|a'(x)z_\epsilon'(x)| \le Mx^{k-1}(|u_\epsilon'(x)| + |v_\epsilon'(x)|) \le M.$$

Finally, it is obvious that the remaining terms of $s_\epsilon(x)$ are uniformly bounded. The proof of (2.11) for $i = 1, 2$ is analogous (cf. [10]). Note that we use the condition

$$c(0) > q_*$$

when dealing with v_ϵ. $\square$

3. First-order scheme. Let I^h be an arbitrary discretization mesh with mesh points

$$0 = x_0 < x_1 < \cdots < x_n = 1, \qquad n \in \mathbb{N}\backslash\{1\}.$$

Let

$$h_i = x_i - x_{i-1}, \qquad i = 1(1)n,$$

$$h = \max_{1\le i\le n} h_i.$$

In this and the next sections the constants M will be independent of I^h as well.

By w^h, u^h, etc., we shall denote mesh functions on I^h. They will be identified with $\mathbb{R}^{n+1}$ vectors:

$$w^h = [w_0, w_1, \ldots, w_n]^T \qquad (w_i := w_i^h).$$

Let

$$\|w^h\| = \max_{0\le i\le n} |w_i|.$$

The corresponding matrix norm will also be denoted by $\|\cdot\|$.

Let us introduce the following finite-difference operators:

$$D''w_i = 2[(w_{i+1} - w_i)/h_{i+1} + (w_{i-1} - w_i)/h_i]/(h_i + h_{i+1}),$$
$$D'w_i = (w_{i+1} - w_i)/h_{i+1},$$

and let

$$\sigma_i = \frac{\mu^2}{2}(h_i + h_{i+1})h_i h_{i+1}[h_i(e^{-\mu h_{i+1}} - 1) + h_{i+1}(e^{\mu h_i} - 1)]^{-1}.$$

Then the discrete problem corresponding to (1.1) is given by

(3.1a) $$w_0 = U_0,$$

(3.1b) $$L^h w_i := -\epsilon\sigma_i D''w_i - a(x_i)D'w_i + c(x_i)w_i = f(x_i), \qquad i = 1(1)n-1,$$

(3.1c) $$w_n = U_1,$$

where σ_i is a fitting factor similar to that used in [16], [17] and, in the equidistant case, in [4, Chap. 6]. It is derived from

(3.2) $$\sigma_i D''v_\epsilon(x_i) = v''_\epsilon(x_i),$$

where $v_\epsilon(x)$ is the boundary-layer function from (1.3b).

Let r^h be the consistency error:

$$r_i = L^h u_\epsilon(x_i) - f(x_i), \qquad i = 1(1)n-1,$$
$$r_0 = r_n = 0.$$

Then we have the following lemma.

LEMMA 3.1. *For the difference scheme* (3.1) *we have*

$$\|r^h\| \le Mh.$$

Proof. For $i = 1(1)n-1$ we have

$$|r_i| \le P_i + Q_i,$$

where

$$\begin{aligned} P_i &= \epsilon|\sigma_i D'' u_\epsilon(x_i) - u_\epsilon''(x_i)|, \\ Q_i &= a(x_i)|D' u_\epsilon(x_i) - u_\epsilon'(x_i)|. \end{aligned}$$

Then, for some $\alpha_i \in (x_i, x_{i+1})$

$$Q_i \le M x_i^k h |u_\epsilon''(\alpha_i)| \le M h x_i^k \left(1 + \frac{1}{\epsilon} e^{-\mu_* x_i}\right) \le Mh.$$

Furthermore, by using the representation (1.3) and (3.2) we obtain

$$\begin{aligned} P_i &= \epsilon|\sigma_i D'' z_\epsilon(x_i) - z_\epsilon''(x_i)| \\ &\le \epsilon|(\sigma_i - 1) D'' z_\epsilon(x_i)| + \epsilon|D'' z_\epsilon(x_i) - z_\epsilon''(x_i)| \\ &\le \epsilon|\sigma_i - 1||z_\epsilon''(\beta_i)| + M\epsilon h|z_\epsilon'''(\gamma_i)|, \end{aligned}$$

where $\beta_i, \gamma_i \in (x_{i-1}, x_{i+1})$. Thus,

$$P_i \le M[\sqrt{\epsilon}|\sigma_i - 1| + h] \le Mh$$

since

$$\sqrt{\epsilon}|\sigma_i - 1| \le M \max(h_i, h_{i+1}) \tag{3.3}$$

(cf. [16]). It follows that

$$|r_i| \le Mh. \qquad \square$$

Let u_ϵ^h be the restriction of the exact solution on the mesh I^h:

$$u_\epsilon^h = [u_\epsilon(x_0),\, u_\epsilon(x_1), \ldots, u_\epsilon(x_n)]^T.$$

THEOREM 3.2. *The discrete problem* (3.1) *has a unique solution* w_ϵ^h *that satisfies*

$$\|u_\epsilon^h - w_\epsilon^h\| \le Mh. \tag{3.4}$$

Proof. Rewrite (3.1) in the matrix form

$$Aw^h = d^h,$$

where $d_0 = U_0$, $d_i = f(x_i)$, $i = 1(1)n-1$, $d_n = U_1$, and A is the corresponding tridiagonal matrix, $A = [a_{ij}] \in \mathbb{R}^{n+1,n+1}$. It is easy to show that

$$\sigma_i > 0$$

(cf. [16]) and that A is an L-matrix ($a_{ii} > 0,\ a_{ij} \le 0,\ i \ne j,\ i,j = 0(1)n$). Let e^h be the vector with the components

$$e_i = 2 - x_i, \qquad i = 0(1)n.$$

Then

$$\begin{aligned} Ae^h &= v^h, \\ v_0 &= 2, \\ v_i &= a(x_i) + c(x_i)(2 - x_i), \qquad i = 1(1)n-1, \\ v_n &= 1. \end{aligned}$$

In a manner similar to the proof of Lemma 2.1 we can show that there exists a positive constant m_0, independent of ϵ and I^h, such that

$$v_i \ge m_0, \qquad i = 0(1)n.$$

It follows (cf. [18]) that A is an inverse monotone matrix (A is nonsingular and $A^{-1} \ge 0$, componentwise) that is an M-matrix (inverse monotone L-matrix; see [15]). Thus w_ϵ^h exists uniquely. Moreover, it holds that

$$\|A^{-1}\| \le \frac{\|e^h\|}{m_0} = \frac{2}{m_0},$$

which means that the discrete problem (3.1) is stable uniformly in ϵ. Hence we have

$$\|u_\epsilon^h - w_\epsilon^h\| \le \frac{2}{m_0}\|r^h\|,$$

and (3.4) follows from Lemma 3.1. □

We shall now confirm the theoretical results by some numerical results. We shall consider two test problems. The first one is

$$(3.5) \qquad -\epsilon u'' - x^k u' + u = f(x), \qquad u(0) = 2, \quad u(1) \approx e,$$

with the exact solution

$$u_\epsilon(x) = e^{-x/\sqrt{\epsilon}} + e^x,$$

from which we determine $f(x)$. The second one, for which we do not have a closed form of the exact solution, is

$$(3.6) \qquad -\epsilon u'' - x^k u' + (\tfrac{1}{2} - x)^2 u = e^x, \qquad u(0) = u(1) = 0.$$

We calculate the rate of uniform convergence on equidistant meshes for problems with $k = 2$ and $k = 3$. Tables 3.1–3.4 present rates of convergence calculated for a range of values of h and ϵ given by

$$H = \left\{ \tfrac{1}{2}^j | j = 3, \dots, 9 \right\}, \qquad E = \left\{ \tfrac{1}{2}^j | j = 0, \dots, jred \right\},$$

where $jred$ is chosen so that ϵ is a value at which the rate of convergence stabilizes, which normally occurs when, to machine accuracy, we are solving the reduced problem.

The final column in each table gives the average rate of convergence for that value of ϵ. The last row gives the uniform rate of convergence for each h, that is,

$$p^h = [\ln(e^{2h}) - \ln(e^h)]/\ln(2),$$

where

$$e^h = \max_{\epsilon \in E}(\max_{0\le i\le n} |u_i^{2h} - u_{2i}^h|).$$

The figure in the lower-right-hand corner is an estimate of the overall rate of uniform convergence

$$p = \text{ mean}_{h\in H}\ p^h.$$

For more details of these tests see [4], [5], and [7]. As can be seen from Tables 3.1–3.4, the scheme is uniformly convergent of order approximately one.

TABLE 3.1
Experimental order of uniform convergence for problem (3.5) *with* $k = 2$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.95	0.98	0.99	0.99	1.00	1.00	0.98
1/4	0.90	0.95	0.97	0.99	0.99	1.00	0.97
1/8	0.86	0.92	0.96	0.98	0.99	1.00	0.95
1/16	0.80	0.91	0.95	0.98	0.99	0.99	0.94
1/32	0.80	0.90	0.95	0.97	0.99	0.99	0.93
1/64	0.80	0.90	0.95	0.97	0.99	0.99	0.93
1/128	0.81	0.90	0.95	0.97	0.99	0.99	0.94
1/256	0.83	0.90	0.95	0.97	0.99	0.99	0.94
1/512	0.84	0.90	0.95	0.97	0.99	0.99	0.94
1/1024	0.85	0.90	0.95	0.97	0.99	0.99	0.94
1/2048	0.86	0.91	0.95	0.97	0.99	0.99	0.95
1/4096	0.86	0.91	0.95	0.97	0.99	0.99	0.95
1/8192	0.86	0.92	0.96	0.98	0.99	0.99	0.95
1/16384	0.85	0.92	0.96	0.98	0.99	0.99	0.95
1/32768	0.85	0.92	0.96	0.98	0.99	0.99	0.95
Aggregate	0.80	0.90	0.95	0.97	0.99	0.99	0.93

Next we shall use a special nonequidistant mesh that is dense near the origin. The mesh points are given by

(3.7a)
$$x_i = \lambda(t_i), \qquad t_i = i/n, \qquad i = 0(1)n,$$

(3.7b)
$$\lambda(t) = \begin{cases} \psi(t) := \sqrt{\epsilon}\frac{t}{\gamma - t}, & t\in[0,\alpha], \\ \Pi(t) := \beta(t-\alpha)^3 + \frac{\psi''(\alpha)}{2}(t-\alpha)^2 + \psi'(\alpha)(t-\alpha) + \psi(\alpha), & t\in[\alpha,1], \end{cases}$$

where

$$\gamma = \alpha + \sqrt[6]{\epsilon},$$

α is an arbitrary number from $(0,1)$, and β is determined by

$$\Pi(1) = 1.$$

TABLE 3.2
Experimental order of uniform convergence for problem (3.5) *with* $k = 3$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.92	0.98	0.99	1.00	1.00	1.00	0.98
1/4	0.86	0.95	0.98	0.99	0.99	1.00	0.96
1/8	0.82	0.92	0.96	0.98	0.99	0.99	0.94
1/16	0.80	0.89	0.95	0.97	0.99	0.99	0.93
1/32	0.78	0.88	0.94	0.97	0.98	0.99	0.92
1/64	0.75	0.87	0.94	0.97	0.98	0.99	0.92
1/128	0.73	0.88	0.94	0.97	0.98	0.99	0.91
1/256	0.73	0.88	0.94	0.97	0.98	0.99	0.92
1/512	0.75	0.88	0.94	0.97	0.98	0.99	0.92
1/1024	0.77	0.89	0.94	0.97	0.98	0.99	0.92
1/2048	0.78	0.89	0.94	0.97	0.98	0.99	0.93
1/4096	0.79	0.90	0.94	0.97	0.98	0.99	0.93
1/8192	0.78	0.90	0.95	0.97	0.98	0.99	0.93
1/16384	0.78	0.90	0.95	0.97	0.99	0.99	0.93
1/32768	0.78	0.90	0.95	0.97	0.99	0.99	0.93
Aggregate	0.78	0.90	0.95	0.97	0.99	0.99	0.93

TABLE 3.3
Experimental order of uniform convergence for problem (3.6) *with* $k = 2$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.80	0.90	0.95	0.98	0.99	0.99	0.94
1/4	0.70	0.87	0.94	0.97	0.98	0.99	0.91
1/8	0.67	0.84	0.92	0.96	0.98	0.99	0.89
1/16	0.61	0.83	0.92	0.96	0.98	0.99	0.88
1/32	0.44	0.74	0.89	0.94	0.97	0.99	0.83
1/64	0.70	0.87	0.92	0.96	0.98	0.99	0.91
1/128	0.84	0.86	0.92	0.96	0.98	0.99	0.93
1/256	0.84	0.88	0.93	0.96	0.98	0.99	0.93
1/512	0.84	0.88	0.93	0.96	0.98	0.99	0.93
1/1024	0.77	0.87	0.94	0.96	0.98	0.99	0.92
1/2048	0.89	0.92	0.95	0.95	0.91	0.96	0.93
1/4096	0.94	0.94	0.95	0.82	0.90	0.95	0.92
1/8192	0.97	0.93	0.91	0.79	0.89	0.94	0.91
1/16384	0.99	0.92	0.91	0.77	0.87	0.94	0.90
1/32768	0.99	0.92	0.88	0.74	0.87	0.93	0.89
1/65536	0.99	0.92	0.83	0.71	0.87	0.93	0.88
1/131072	0.99	0.92	0.77	0.73	0.87	0.93	0.87
1/262144	0.99	0.86	0.76	0.79	0.86	0.93	0.87
1/524288	0.99	0.81	0.74	0.83	0.87	0.93	0.86
1/1048576	0.99	0.80	0.71	0.85	0.88	0.94	0.86
1/2097152	0.99	0.80	0.70	0.85	0.90	0.94	0.86
1/4194304	0.99	0.80	0.69	0.84	0.91	0.95	0.86
1/8388608	0.99	0.80	0.69	0.83	0.91	0.95	0.86
1/16777216	0.99	0.80	0.69	0.82	0.91	0.96	0.86
1/33554432	0.99	0.80	0.69	0.82	0.90	0.96	0.86
Aggregate	0.99	0.80	0.69	0.82	0.90	0.96	0.86

The main part of λ is the function ψ, which gives the mesh points in the layer. Essentially, ψ is a modification of the inverse of the boundary-layer function v_ϵ. On the rest of the interval ψ is extended by a polynomial so that $\lambda \in C^2(I)$. Moreover, λ is strictly monotone.

Let us mention that meshes of similar types were used in [12], [13], [17], and [19].

TABLE 3.4
Experimental order of uniform convergence for problem (3.6) *with* $k = 3$.

ϵ	n: 8	16	32	64	128	256	Average
1/2	0.68	0.86	0.93	0.97	0.98	0.99	0.90
1/4	0.58	0.82	0.91	0.96	0.98	0.99	0.87
1/8	0.46	0.76	0.89	0.95	0.97	0.99	0.84
1/16	0.20	0.71	0.87	0.94	0.97	0.99	0.78
1/32	0.75	0.83	0.92	0.95	0.98	0.99	0.90
1/64	0.83	0.89	0.94	0.97	0.98	0.99	0.93
1/128	0.87	0.87	0.94	0.97	0.98	0.99	0.94
1/256	0.87	0.87	0.94	0.96	0.98	0.99	0.94
1/512	0.89	0.86	0.94	0.97	0.98	0.99	0.94
1/1024	0.91	0.92	0.93	0.96	0.98	0.99	0.95
1/2048	0.94	0.85	0.93	0.96	0.98	0.99	0.94
1/4096	0.79	0.92	0.95	0.96	0.98	0.99	0.93
1/8192	0.72	0.96	0.94	0.97	0.99	0.99	0.93
1/16384	0.70	0.94	0.96	0.98	0.99	0.99	0.93
1/32768	0.69	0.93	0.97	0.98	0.99	1.00	0.93
1/65536	0.69	0.93	0.97	0.98	0.97	0.93	0.91
1/131072	0.69	0.92	0.98	0.98	0.90	0.94	0.90
1/262144	0.69	0.92	0.98	0.97	0.88	0.94	0.90
1/524288	0.69	0.92	0.98	0.94	0.89	0.94	0.89
1/1048576	0.69	0.92	0.98	0.92	0.90	0.94	0.89
1/2097152	0.69	0.92	0.98	0.90	0.92	0.95	0.89
1/4194304	0.69	0.92	0.98	0.88	0.93	0.95	0.89
1/8388608	0.69	0.92	0.98	0.87	0.93	0.96	0.89
1/16777216	0.69	0.92	0.98	0.86	0.93	0.96	0.89
1/33554432	0.69	0.92	0.98	0.86	0.92	0.97	0.89
Aggregate	0.69	0.92	0.98	0.86	0.92	0.97	0.89

It can be proved that the upwind scheme (scheme (3.1) with σ_i replaced by 1) on the mesh (3.7) has first-order uniform convergence. Therefore, the question arises as to whether the fitted scheme (3.1) on the mesh (3.7) gives better results than the upwind scheme. The answer is no, as the results of Table 3.5 show: obviously, first-order uniform convergence is present, and the results are practically the same as the results of the upwind scheme. The reason for this is that both schemes use the same first-order discretization of the $a(x)u'$-term.

TABLE 3.5
Error $\|u_\epsilon^h - w_\epsilon^h\|$ *for problem* (3.5) *with* $k = 2$; *scheme* (3.1) *on the mesh* (3.7) *with* $\alpha = \frac{1}{2}$.

n	ϵ: 10^{-6}	10^{-12}	10^{-18}
50	3.19-2	3.65-2	3.70-2
100	1.55-2	1.83-2	1.86-2
200	7.78-3	9.11-3	9.31-3

The choice $\alpha = \frac{1}{2}$ gives about 25% of the total of mesh steps in the interval $[0, \sqrt{\epsilon}]$ representing the layer. The percentage can be higher if a greater value of α is used.

4. Improved scheme. We would now like to improve scheme (3.1) on the special mesh (3.7). We modify (3.1) to obtain a discretization of problem (1.1) at the midpoints

$$x_{i+1/2} := x_i + \frac{h_{i+1}}{2}, \qquad i = 1(1)n - 1.$$

TABLE 3.2
Experimental order of uniform convergence for problem (3.5) *with* $k = 3$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.92	0.98	0.99	1.00	1.00	1.00	0.98
1/4	0.86	0.95	0.98	0.99	0.99	1.00	0.96
1/8	0.82	0.92	0.96	0.98	0.99	0.99	0.94
1/16	0.80	0.89	0.95	0.97	0.99	0.99	0.93
1/32	0.78	0.88	0.94	0.97	0.98	0.99	0.92
1/64	0.75	0.87	0.94	0.97	0.98	0.99	0.92
1/128	0.73	0.88	0.94	0.97	0.98	0.99	0.91
1/256	0.73	0.88	0.94	0.97	0.98	0.99	0.92
1/512	0.75	0.88	0.94	0.97	0.98	0.99	0.92
1/1024	0.77	0.89	0.94	0.97	0.98	0.99	0.92
1/2048	0.78	0.89	0.94	0.97	0.98	0.99	0.93
1/4096	0.79	0.90	0.94	0.97	0.98	0.99	0.93
1/8192	0.78	0.90	0.95	0.97	0.98	0.99	0.93
1/16384	0.78	0.90	0.95	0.97	0.99	0.99	0.93
1/32768	0.78	0.90	0.95	0.97	0.99	0.99	0.93
Aggregate	0.78	0.90	0.95	0.97	0.99	0.99	0.93

TABLE 3.3
Experimental order of uniform convergence for problem (3.6) *with* $k = 2$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.80	0.90	0.95	0.98	0.99	0.99	0.94
1/4	0.70	0.87	0.94	0.97	0.98	0.99	0.91
1/8	0.67	0.84	0.92	0.96	0.98	0.99	0.89
1/16	0.61	0.83	0.92	0.96	0.98	0.99	0.88
1/32	0.44	0.74	0.89	0.94	0.97	0.99	0.83
1/64	0.70	0.87	0.92	0.96	0.98	0.99	0.91
1/128	0.84	0.86	0.92	0.96	0.98	0.99	0.93
1/256	0.84	0.88	0.93	0.96	0.98	0.99	0.93
1/512	0.84	0.88	0.93	0.96	0.98	0.99	0.93
1/1024	0.77	0.87	0.94	0.96	0.98	0.99	0.92
1/2048	0.89	0.92	0.95	0.95	0.91	0.96	0.93
1/4096	0.94	0.94	0.95	0.82	0.90	0.95	0.92
1/8192	0.97	0.93	0.91	0.79	0.89	0.94	0.91
1/16384	0.99	0.92	0.91	0.77	0.87	0.94	0.90
1/32768	0.99	0.92	0.88	0.74	0.87	0.93	0.89
1/65536	0.99	0.92	0.83	0.71	0.87	0.93	0.88
1/131072	0.99	0.92	0.77	0.73	0.87	0.93	0.87
1/262144	0.99	0.86	0.76	0.79	0.86	0.93	0.87
1/524288	0.99	0.81	0.74	0.83	0.87	0.93	0.86
1/1048576	0.99	0.80	0.71	0.85	0.88	0.94	0.86
1/2097152	0.99	0.80	0.70	0.85	0.90	0.94	0.86
1/4194304	0.99	0.80	0.69	0.84	0.91	0.95	0.86
1/8388608	0.99	0.80	0.69	0.83	0.91	0.95	0.86
1/16777216	0.99	0.80	0.69	0.82	0.91	0.96	0.86
1/33554432	0.99	0.80	0.69	0.82	0.90	0.96	0.86
Aggregate	0.99	0.80	0.69	0.82	0.90	0.96	0.86

The main part of λ is the function ψ, which gives the mesh points in the layer. Essentially, ψ is a modification of the inverse of the boundary-layer function v_ϵ. On the rest of the interval ψ is extended by a polynomial so that $\lambda \in C^2(I)$. Moreover, λ is strictly monotone.

Let us mention that meshes of similar types were used in [12], [13], [17], and [19].

TABLE 3.4
Experimental order of uniform convergence for problem (3.6) with $k = 3$.

ϵ	n 8	16	32	64	128	256	Average
1/2	0.68	0.86	0.93	0.97	0.98	0.99	0.90
1/4	0.58	0.82	0.91	0.96	0.98	0.99	0.87
1/8	0.46	0.76	0.89	0.95	0.97	0.99	0.84
1/16	0.20	0.71	0.87	0.94	0.97	0.99	0.78
1/32	0.75	0.83	0.92	0.95	0.98	0.99	0.90
1/64	0.83	0.89	0.94	0.97	0.98	0.99	0.93
1/128	0.87	0.87	0.94	0.97	0.98	0.99	0.94
1/256	0.87	0.87	0.94	0.96	0.98	0.99	0.94
1/512	0.89	0.86	0.94	0.97	0.98	0.99	0.94
1/1024	0.91	0.92	0.93	0.96	0.98	0.99	0.95
1/2048	0.94	0.85	0.93	0.96	0.98	0.99	0.94
1/4096	0.79	0.92	0.95	0.96	0.98	0.99	0.93
1/8192	0.72	0.96	0.94	0.97	0.99	0.99	0.93
1/16384	0.70	0.94	0.96	0.98	0.99	0.99	0.93
1/32768	0.69	0.93	0.97	0.98	0.99	1.00	0.93
1/65536	0.69	0.93	0.97	0.98	0.97	0.93	0.91
1/131072	0.69	0.92	0.98	0.98	0.90	0.94	0.90
1/262144	0.69	0.92	0.98	0.97	0.88	0.94	0.90
1/524288	0.69	0.92	0.98	0.94	0.89	0.94	0.89
1/1048576	0.69	0.92	0.98	0.92	0.90	0.94	0.89
1/2097152	0.69	0.92	0.98	0.90	0.92	0.95	0.89
1/4194304	0.69	0.92	0.98	0.88	0.93	0.95	0.89
1/8388608	0.69	0.92	0.98	0.87	0.93	0.96	0.89
1/16777216	0.69	0.92	0.98	0.86	0.93	0.96	0.89
1/33554432	0.69	0.92	0.98	0.86	0.92	0.97	0.89
Aggregate	0.69	0.92	0.98	0.86	0.92	0.97	0.89

It can be proved that the upwind scheme (scheme (3.1) with σ_i replaced by 1) on the mesh (3.7) has first-order uniform convergence. Therefore, the question arises as to whether the fitted scheme (3.1) on the mesh (3.7) gives better results than the upwind scheme. The answer is no, as the results of Table 3.5 show: obviously, first-order uniform convergence is present, and the results are practically the same as the results of the upwind scheme. The reason for this is that both schemes use the same first-order discretization of the $a(x)u'$-term.

TABLE 3.5
Error $\|u_\epsilon^h - w_\epsilon^h\|$ for problem (3.5) with $k = 2$; scheme (3.1) on the mesh (3.7) with $\alpha = \frac{1}{2}$.

n	ϵ 10^{-6}	10^{-12}	10^{-18}
50	3.19-2	3.65-2	3.70-2
100	1.55-2	1.83-2	1.86-2
200	7.78-3	9.11-3	9.31-3

The choice $\alpha = \frac{1}{2}$ gives about 25% of the total of mesh steps in the interval $[0, \sqrt{\epsilon}]$ representing the layer. The percentage can be higher if a greater value of α is used.

4. Improved scheme. We would now like to improve scheme (3.1) on the special mesh (3.7). We modify (3.1) to obtain a discretization of problem (1.1) at the midpoints

$$x_{i+1/2} := x_i + \frac{h_{i+1}}{2}, \qquad i = 1(1)n-1.$$

TABLE 3.2
Experimental order of uniform convergence for problem (3.5) *with* $k = 3$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.92	0.98	0.99	1.00	1.00	1.00	0.98
1/4	0.86	0.95	0.98	0.99	0.99	1.00	0.96
1/8	0.82	0.92	0.96	0.98	0.99	0.99	0.94
1/16	0.80	0.89	0.95	0.97	0.99	0.99	0.93
1/32	0.78	0.88	0.94	0.97	0.98	0.99	0.92
1/64	0.75	0.87	0.94	0.97	0.98	0.99	0.92
1/128	0.73	0.88	0.94	0.97	0.98	0.99	0.91
1/256	0.73	0.88	0.94	0.97	0.98	0.99	0.92
1/512	0.75	0.88	0.94	0.97	0.98	0.99	0.92
1/1024	0.77	0.89	0.94	0.97	0.98	0.99	0.92
1/2048	0.78	0.89	0.94	0.97	0.98	0.99	0.93
1/4096	0.79	0.90	0.94	0.97	0.98	0.99	0.93
1/8192	0.78	0.90	0.95	0.97	0.98	0.99	0.93
1/16384	0.78	0.90	0.95	0.97	0.99	0.99	0.93
1/32768	0.78	0.90	0.95	0.97	0.99	0.99	0.93
Aggregate	0.78	0.90	0.95	0.97	0.99	0.99	0.93

TABLE 3.3
Experimental order of uniform convergence for problem (3.6) *with* $k = 2$.

ϵ	n = 8	16	32	64	128	256	Average
1/2	0.80	0.90	0.95	0.98	0.99	0.99	0.94
1/4	0.70	0.87	0.94	0.97	0.98	0.99	0.91
1/8	0.67	0.84	0.92	0.96	0.98	0.99	0.89
1/16	0.61	0.83	0.92	0.96	0.98	0.99	0.88
1/32	0.44	0.74	0.89	0.94	0.97	0.99	0.83
1/64	0.70	0.87	0.92	0.96	0.98	0.99	0.91
1/128	0.84	0.86	0.92	0.96	0.98	0.99	0.93
1/256	0.84	0.88	0.93	0.96	0.98	0.99	0.93
1/512	0.84	0.88	0.93	0.96	0.98	0.99	0.93
1/1024	0.77	0.87	0.94	0.96	0.98	0.99	0.92
1/2048	0.89	0.92	0.95	0.95	0.91	0.96	0.93
1/4096	0.94	0.94	0.95	0.82	0.90	0.95	0.92
1/8192	0.97	0.93	0.91	0.79	0.89	0.94	0.91
1/16384	0.99	0.92	0.91	0.77	0.87	0.94	0.90
1/32768	0.99	0.92	0.88	0.74	0.87	0.93	0.89
1/65536	0.99	0.92	0.83	0.71	0.87	0.93	0.88
1/131072	0.99	0.92	0.77	0.73	0.87	0.93	0.87
1/262144	0.99	0.86	0.76	0.79	0.86	0.93	0.87
1/524288	0.99	0.81	0.74	0.83	0.87	0.93	0.86
1/1048576	0.99	0.80	0.71	0.85	0.88	0.94	0.86
1/2097152	0.99	0.80	0.70	0.85	0.90	0.94	0.86
1/4194304	0.99	0.80	0.69	0.84	0.91	0.95	0.86
1/8388608	0.99	0.80	0.69	0.83	0.91	0.95	0.86
1/16777216	0.99	0.80	0.69	0.82	0.91	0.96	0.86
1/33554432	0.99	0.80	0.69	0.82	0.90	0.96	0.86
Aggregate	0.99	0.80	0.69	0.82	0.90	0.96	0.86

The main part of λ is the function ψ, which gives the mesh points in the layer. Essentially, ψ is a modification of the inverse of the boundary-layer function v_ϵ. On the rest of the interval ψ is extended by a polynomial so that $\lambda \in C^2(I)$. Moreover, λ is strictly monotone.

Let us mention that meshes of similar types were used in [12], [13], [17], and [19].

TABLE 3.4
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	n						
ϵ	8	16	32	64	128	256	Average
1/2	0.68	0.86	0.93	0.97	0.98	0.99	0.90
1/4	0.58	0.82	0.91	0.96	0.98	0.99	0.87
1/8	0.46	0.76	0.89	0.95	0.97	0.99	0.84
1/16	0.20	0.71	0.87	0.94	0.97	0.99	0.78
1/32	0.75	0.83	0.92	0.95	0.98	0.99	0.90
1/64	0.83	0.89	0.94	0.97	0.98	0.99	0.93
1/128	0.87	0.87	0.94	0.97	0.98	0.99	0.94
1/256	0.87	0.87	0.94	0.96	0.98	0.99	0.94
1/512	0.89	0.86	0.94	0.97	0.98	0.99	0.94
1/1024	0.91	0.92	0.93	0.96	0.98	0.99	0.95
1/2048	0.94	0.85	0.93	0.96	0.98	0.99	0.94
1/4096	0.79	0.92	0.95	0.96	0.98	0.99	0.93
1/8192	0.72	0.96	0.94	0.97	0.99	0.99	0.93
1/16384	0.70	0.94	0.96	0.98	0.99	0.99	0.93
1/32768	0.69	0.93	0.97	0.98	0.99	1.00	0.93
1/65536	0.69	0.93	0.97	0.98	0.97	0.93	0.91
1/131072	0.69	0.92	0.98	0.98	0.90	0.94	0.90
1/262144	0.69	0.92	0.98	0.97	0.88	0.94	0.90
1/524288	0.69	0.92	0.98	0.94	0.89	0.94	0.89
1/1048576	0.69	0.92	0.98	0.92	0.90	0.94	0.89
1/2097152	0.69	0.92	0.98	0.90	0.92	0.95	0.89
1/4194304	0.69	0.92	0.98	0.88	0.93	0.95	0.89
1/8388608	0.69	0.92	0.98	0.87	0.93	0.96	0.89
1/16777216	0.69	0.92	0.98	0.86	0.93	0.96	0.89
1/33554432	0.69	0.92	0.98	0.86	0.92	0.97	0.89
Aggregate	0.69	0.92	0.98	0.86	0.92	0.97	0.89

It can be proved that the upwind scheme (scheme (3.1) with σ_i replaced by 1) on the mesh (3.7) has first-order uniform convergence. Therefore, the question arises as to whether the fitted scheme (3.1) on the mesh (3.7) gives better results than the upwind scheme. The answer is no, as the results of Table 3.5 show: obviously, first-order uniform convergence is present, and the results are practically the same as the results of the upwind scheme. The reason for this is that both schemes use the same first-order discretization of the $a(x)u'$-term.

TABLE 3.5
Error $\|u_\epsilon^h - w_\epsilon^h\|$ for problem (3.5) with $k = 2$; scheme (3.1) on the mesh (3.7) with $\alpha = \frac{1}{2}$.

	ϵ		
n	10^{-6}	10^{-12}	10^{-18}
50	3.19-2	3.65-2	3.70-2
100	1.55-2	1.83-2	1.86-2
200	7.78-3	9.11-3	9.31-3

The choice $\alpha = \frac{1}{2}$ gives about 25% of the total of mesh steps in the interval $[0, \sqrt{\epsilon}]$ representing the layer. The percentage can be higher if a greater value of α is used.

4. Improved scheme. We would now like to improve scheme (3.1) on the special mesh (3.7). We modify (3.1) to obtain a discretization of problem (1.1) at the midpoints

$$x_{i+1/2} := x_i + \frac{h_{i+1}}{2}, \qquad i = 1(1)n - 1.$$

The modified scheme is given by

$$w_0 = U_0, \tag{4.1a}$$

(4.1b)
$$\overline{L}^h w_i := -\epsilon \overline{\sigma_i} D'' w_i - a(x_{i+1/2}) D' w_i + c(x_{i+1/2}) D^0 w_i = f(x_{i+1/2}), \quad i = 1(1)n-1,$$

$$w_n = U_1. \tag{4.1c}$$

Here

$$D^0 w_i = \tfrac{3}{2} w_i - \tfrac{1}{2} w_{i-1}$$

and

$$\overline{\sigma_i} = \sigma_i \exp(-\mu h_{i+1}/2)$$

so that

$$\overline{\sigma_i} D'' v_\epsilon(x_{i+1/2}) = v_\epsilon''(x_{i+1/2}).$$

The midpoint scheme (4.1) with 1 instead of $\overline{\sigma_i}$ was used in [18]. Note that the values of a, c, and f at $x_{i+1/2}$ could be replaced by $\frac{1}{2}(a(x_i) + a(x_{i+1}))$, etc.

To analyze the consistency error of the operator $\overline{L}^h$ on the special mesh we need properties of the function λ from (3.7b). First, it is easy to see that

$$\lambda^{(i)}(t) > 0, \qquad i = 0, 1, 2, \quad t \in [0, \alpha],$$

and, since ϵ is small, it follows that $\beta > 0$, i.e.,

$$\lambda'''(t) > 0, \qquad t \in [\alpha, 1].$$

Then we have

$$\lambda^{(i)}(t) \geq \lambda^{(i)}(\alpha) = \psi^{(i)}(\alpha) > 0, \qquad t \in [\alpha, 1],$$

first for $i = 2$ and then for $i = 1, 0$. Thus

$$\lambda^{(i)}(t) > 0, \qquad i = 0, 1, 2, \quad t \in I. \tag{4.2}$$

Also, note that

$$\lambda^{(i)}(t) \leq M, \qquad i = 0, 1, 2, \quad t \in I. \tag{4.3}$$

We shall use the following inequalities as well:

$$\exp(-\psi(t)/\sqrt{\epsilon}) \leq M \exp(-\gamma/(\gamma - t)), \qquad t \in [0, \gamma), \tag{4.4}$$

$$\Pi\left(\alpha + \frac{2}{n}\right) \leq M(n^{-2} + \sqrt[6]{\epsilon} n^{-1} + \sqrt[3]{\epsilon}). \tag{4.5}$$

Let

$$\overline{r_i} = \overline{L}^h u_\epsilon(x_i) - f(x_{i+1/2}), \qquad i = 1(1)n-1,$$
$$\overline{r_0} = \overline{r_n} = 0.$$

Then we have the following lemma.

LEMMA 4.1. *The consistency error of the operator* $\overline{L}^h$ *on the mesh* (3.7) *satisfies*

$$\|\overline{r}^h\| \leq \delta := M\frac{1}{n}\left(\sqrt{\epsilon} + \frac{1}{n}\right).$$

Proof. For $i = 1(1)n-1$ we have

$$|\overline{r_i}| \leq R_i + S_i + T_i,$$

where

$$\begin{aligned} R_i &= \epsilon|\overline{\sigma_i}D''u_\epsilon(x_i) - u_\epsilon''(x_{i+1/2})| \\ &= \epsilon|\overline{\sigma_i}D''z_\epsilon(x_i) - z_\epsilon''(x_{i+1/2})|, \\ S_i &= a(x_{i+1/2})|D'u_\epsilon(x_i) - u_\epsilon'(x_{i+1/2})|, \\ T_i &= c(x_{i+1/2})|D^0u_\epsilon(x_i) - u_\epsilon(x_{i+1/2})|. \end{aligned}$$

Let us consider R_i first. In the same way as in the proof of Lemma 3.1, it follows that

$$R_i \leq R_i^1 + R_i^2,$$

where

$$R_i^1 = M\epsilon|\overline{\sigma_i} - 1|\left(1 + \frac{1}{\sqrt{\epsilon}}e^{-\mu_* x_{i-1}}\right) \leq Mh_{i+1}(\sqrt{\epsilon} + e^{-\mu_* x_{i-1}})$$

(the last inequality follows because of (3.3)) and

$$R_i^2 = \epsilon|D''z_\epsilon(x_i) - z_\epsilon''(x_{i+1/2})| \leq Mh_{i+1}(\epsilon + e^{-\mu_* x_{i-1}}).$$

Let us show that

$$R_i^j \leq \delta, \qquad j = 1, 2. \tag{4.6}$$

Using (4.2) and (4.3), we have

$$h_{i+1} \leq \frac{1}{n}\lambda'(t_{i+1}) \leq \frac{M}{n}, \tag{4.7}$$

and to complete the proof of (4.6) we have only to show that

$$h_{i+1}e^{-\mu_* x_{i-1}} \leq \delta. \tag{4.8}$$

To do this we distinguish three cases:

$$\begin{aligned} &1^0: \qquad t_{i-1} \geq \alpha, \\ &2^0: \qquad t_{i-1} < \alpha \quad \text{and} \quad t_{i-1} \leq \gamma - \frac{3}{n}, \\ &3^0: \qquad \gamma - \frac{3}{n} < t_{i-1} < \alpha. \end{aligned}$$

In case 1^0, by (4.2) and (4.4) we have

$$e^{-\mu^* x_{i-1}} \leq e^{-\mu^*\lambda(\alpha)} \leq Me^{-\sqrt{q_*}\gamma/\sqrt[6]{\epsilon}},$$

which, together with (4.7), gives (4.8). In case 2^0

$$h_{i+1}e^{-\mu_* x_{i-1}} \le M\frac{\sqrt{\epsilon}}{n}(\gamma - t_{i+1})^{-2}e^{-\gamma/(\gamma-t_{i-1})} \le M\frac{\sqrt{\epsilon}}{n}$$

because in this case

$$\gamma - t_{i+1} \ge \tfrac{1}{3}(\gamma - t_{i-1}).$$

Finally, in case 3^0 we use (4.7) and

$$e^{-\mu_* x_{i-1}} \le e^{-\mu_* \psi(\gamma-(3/n))} \le Me^{-\sqrt{q_*}\gamma n/3}$$

(where (4.4) has been used), and (4.8) follows again. Next we have

$$\begin{aligned} S_i &\le Mx_{i+1/2}^k h_{i+1}^2(1+\epsilon^{-2/3}e^{-\mu_* x_i}) \\ &\le M(n^{-2} + x_{i+1}^2 h_{i+1}^2 \epsilon^{-2/3}e^{-\mu_* x_{i-1}}). \end{aligned}$$

Then in cases 1^0 and 2^0 we can prove

$$S_i \le Mn^{-2} \tag{4.9}$$

by the same technique as in the proof of (4.8). Case 3^0 must be treated in a different way. We have

$$S_i \le S_i^1 + S_i^2,$$

where

$$\begin{aligned} S_i^1 &= Mx_{i+1/2}^2\frac{1}{h_{i+1}}|u_\epsilon(x_{i+1}) - u_\epsilon(x_i)| \\ &\le M\left[(x_{i+1}^2 - x_i^2)\frac{1}{h_{i+1}} + x_i^2\left(1 + \frac{1}{\sqrt{\epsilon}}e^{-\mu_* x_i}\right)\right] \\ &\le M(x_{i+1} + \sqrt{\epsilon}) \end{aligned}$$

and

$$S_i^2 = Mx_{i+1/2}^2\left(1 + \frac{1}{\sqrt{\epsilon}}\right)e^{-\mu_* x_{i+1/2}} \le M(x_{i+1} + \sqrt{\epsilon}).$$

Note that case 3^0 is possible only if

$$\sqrt[6]{\epsilon} < \frac{3}{n}. \tag{4.10}$$

Next, because $t_{i+1} < \alpha + \frac{2}{n}$ we have

$$x_{i+1} < \lambda\left(\alpha + \frac{2}{n}\right) = \Pi\left(\alpha + \frac{2}{n}\right),$$

and from (4.5) and (4.10) it follows that

$$x_{i+1} \le Mn^{-2}. \tag{4.11}$$

From this and from (4.10) we conclude that

$$S_i^j \leq Mn^{-2}, \qquad j = 1, 2,$$

and (4.9) is proved.

Finally, for T_i we have

$$T_i \leq M\left[\tfrac{1}{2}(h_{i+1} - h_i)u_\epsilon'(x_{i+1/2}) + h_{i+1}^2 u_\epsilon''(\eta_i)\right],$$

with $\eta_i \in (x_{i-1}, x_{i+1})$. Then, by the same technique, in cases 1^0 and 2^0 we obtain

$$T_i \leq Mn^{-2}. \tag{4.12}$$

Note here that because of (4.2), (4.3)

$$h_{i+1} - h_i \leq n^{-2}\lambda''(t_{i+1}) \leq Mn^{-2}.$$

In case 3^0 we first denote T_i by $T_i(u_\epsilon)$. Then, by Theorem 2.4

$$T_i(u_\epsilon) \leq |\omega| T_i(v_\epsilon) + T_i(z_\epsilon).$$

Now we have

$$T_i(v_\epsilon) \leq Mv_\epsilon(x_{i-1}) \leq Mn^{-2}$$

in the same way as in the proof of (4.8) in case 3^0. Next, for some $\tau_i \in (x_{i-1}, x_{i+1})$

$$T_i(z_\epsilon) \leq Mh_{i+1}|z_\epsilon'(\tau_i)| \leq Mx_{i+1},$$

and by (4.11) we have

$$T_i(z_\epsilon) \leq Mn^{-2}.$$

Thus (4.12) is proved, and the lemma follows. □

THEOREM 4.2. *The discrete problem* (4.1) *is stable uniformly in* ϵ, *and it has a unique solution* $\overline{w}_\epsilon^h$. *Additionally, on the mesh* (3.7) *we have*

$$\|u_\epsilon^h - \overline{w}_\epsilon^h\| \leq \delta.$$

Proof. The stability can be proved in the same way as the stability of (3.1) in Theorem 3.2. The result follows by Lemma 4.1. □

In Tables 4.1 and 4.2 we present some numerical results of scheme (4.1) on the mesh (3.7). Problem (3.5) is treated with $k = 2$ and $k = 3$. Since ϵ is small, the rate of the uniform convergence is 2.

TABLE 4.1
Error $\|u_\epsilon^h - \overline{w}_\epsilon^h\|$ *for problem* (3.5) *with* $k = 2$; *scheme* (4.1) *on the mesh* (3.7) *with* $\alpha = \frac{1}{2}$.

	ϵ		
n	10^{-6}	10^{-12}	10^{-18}
50	7.12-3	7.55-3	7.58-3
100	1.89-3	2.01-3	2.02-3
200	4.85-4	5.19-4	5.27-4

A comparison of Tables 3.5 and 4.1 clearly shows that scheme (4.1) is better than (3.1) on the special mesh. Note that this is not the case on arbitrary meshes; for instance, scheme (4.1) does not converge uniformly in ϵ on equidistant meshes.

TABLE 4.2
Error $\|u_\epsilon^h - \overline{w}_\epsilon^h\|$ *for problem* (3.5) *with* $k = 3$; *scheme* (4.1) *on the mesh* (3.7) *with* $\alpha = \frac{1}{2}$.

	ϵ		
n	10^{-6}	10^{-12}	10^{-18}
50	7.83-3	8.33-3	8.37-3
100	2.09-3	2.23-3	2.24-3
200	5.39-4	5.75-4	5.78-4

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