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A UNIFORMLY CONVERGENT FINITE DIFFERENCE SCHEME FOR A SINGULARLY PERTURBED SEMILINEAR EQUATION

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Abstract. Boundary value problems for singularly perturbed semilinear elliptic equations are considered. Special piecewise-uniform meshes are constructed which yield accurate numerical solutions irrespective of the value of the small parameter. Numerical methods composed of standard monotone finite difference operators and these piecewise-uniform meshes are shown theoretically to be uniformly (with respect to the singular perturbation parameter) convergent. Numerical results are also presented, which indicate that in practice the method is first order accurate.

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Key words. Semilinear boundary value problem, singular perturbation, finite-difference scheme, piecewise uniform mesh.

1. Introduction. Partial differential equations with a small parameter ε multiplying the highest order derivative terms are said to be singularly perturbed. Singularly perturbed differential equations are all pervasive in applications of mathematics to problmes in the sciences and engineering. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number, the drift-diffusion equations of semiconductor device physics [17, 12], the Michaelis-Menten theory for enzyme reactions [16], and mathematical models of liquid crystal materials and of chemical reactions [25].

The use of classical numerical methods for solving such problems may give rise to difficulties when the singular perturbation parameter ε is small. In particular, methods based on centered differences or upwinded differences on uniform meshes yield error bounds, in the maximum norm, which depend on an inverse power of ε. Similarly Brandt and Yavneh [1] demonstrated that anisotropic artificial viscosity in the first-order upwind finite-difference scheme may result in inaccurate solutions, when $\varepsilon/h = O(1)$, where h is the mesh width. Two alternative approaches may be taken to the resolution of this problem. Either additional information about the solution may be used to produce accurate efficient methods, which may involve *a priori* modification of the mesh or operator, or an attempt may be made to produce *a postiori* adaptive methods or black box methods.

The latter approach leads to codes that are designed to handle a wider variety of problems than non-adaptive codes, usually at the expense of greater execution time. Moreover, such methods are less suitable than non-adaptive codes to implementation in a parallel environment. This is because the adaption process inherent in *a posteriori* methods, introduces sequentiality to the solution process, which is absent in the *a priori* case. The

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former approach uses physical or mathematical knowledge about the problem to enhance the solution strategy. Such methods are widespread in the literature. These include fitted finite difference methods, finite element methods using special elements such as exponential elements, and methods which use *a priori* refined or special meshes. Examples of these include methods for convection-diffusion problems devised by the British Central Electricity Generating Board [11], fluid flow in aerodynamics [3], semiconductor device physics [18, 4, 15, 14] and chemical reactions [24].

It is of theoretical and practical interest to consider numerical methods for such problems, which exhibit ε-uniform convergence, that is, numerical methods for which there exists an N_0, independent of ε, such that for all $N \geq N_0$, where N is the number of mesh elements, the error constant and rate of convergence in the maximum norm are independent of ε. Thus a numerical method is said to be ε-uniform of order p on the mesh $\Omega_N = \{x_i, i = 0, 1, \ldots, N\}$ if there exists an N_0 independent of ε such that for all $N \geq N_0$

$$\sup_{0 < \varepsilon \leq 1} \max_{\Omega_N} |u(x) - u_N(x)| \leq C N^{-p},$$

where u is the solution of the differential equation, u_N is the numerical approximation to u, C and $p > 0$ are independent of ε and N.

Singularly perturbed boundary value problems for linear elliptic equations, which reduce for $\varepsilon = 0$ to zero-order equations, were examined in [10, 19, 20, 21, 22]. For such problems ε-uniform methods consisting of exponentially fitted finite difference operators on uniform meshes were thoroughly investigated and applied successfully to ordinary differential equations in [2, 8] and to linear partial differential equations in [10, 19, 20, 21].

In [6, 13] it is shown that it is impossible to find an exponentially fitted finite difference operator on a uniform rectangular mesh which yields an ε-uniform numerical method for a certain class of one-dimensional semilinear differential equations. In this paper ε-uniform numerical methods are constructed for a class of semilinear problems, using classical finite difference operators on special piecewise-uniform meshes. Thus ε-uniform methods can be constructed on special piecewise uniform meshes even if it is not possible on uniform meshes. An added advantage of standard difference methods on *a priori* refined meshes over exponentially fitted methods arises from the non-linear nature of the difference schemes. These must be solved using an iterative technique. In the case of exponentially fitted methods, such as those in [2], assembly of the iteration matrix involves the evaluation of exponential or related hyperbolic functions *in each iteration*, since the fitting factor depends on the solution. Refined mesh methods such as those described here do not suffer from this disadvantage.

In §2 the continuous problem is formulated and bounds on the derivatives of the solution are given. In §3, a nonlinear finite difference method is constructed which is ε-uniform. In §4 this method is linearized by means of continuation, and it is shown that the resulting linear finite difference method is still ε-uniform. Finally, in §5, numerical results are presented which support these theoretical results and show that in practice the methods are ε-uniform and are first order accurate.

2. Problem formulation. Consider the Dirichlet problem for the semilinear elliptic equation

(2.1) $$L u(x) \equiv \varepsilon^2 L_2 u(x) + \varepsilon L_1 u(x) - g\Big(x, u(x)\Big) = 0, x \in \Omega,$$

$$u(x) = \phi(x), \ x \in \partial\Omega.$$

on the n-dimensional infinite strip $\Omega = \Big\{x \in \Re^n, 0 < x_1 < d\Big\}$, where

$$L_2 \equiv \sum_{r,s=1}^{n} a_{rs}(x)\frac{\partial^2}{\partial x_r \partial x_s} + \sum_{r=1}^{n} b_r(x)\frac{\partial}{\partial x_r} - c(x)$$

$$L_1 \equiv \sum_{r=1}^{n} b_r^1(x)\frac{\partial}{\partial x_r} - c^1(x)$$

are linear differential operators. The more general case, where the coefficients of L_1 and L_2 depend on $u(x)$, is dealt with in [13]. The coefficients $a_{rs}(x), b_r(x), c(x), b_r^1(x), c^1(x), \ \ r, s = 1, ..., n$, and the functions $g(x, u), \phi(x)$ are assumed to be sufficiently smooth on the sets $\bar{\Omega}, \bar{\Omega} \times \Re$ and $\partial\Omega$ respectively, and to satisfy the conditions

$$a_0 \sum_{r=1}^{n} \xi_r^2 \leq \sum_{r,s=1}^{n} a_{rs}(x)\xi_r\xi_s \leq a^0 \sum_{r=1}^{n} \xi_r^2, \ x \in \Omega$$

$$a_0 > 0, \ c(x), \ c^1(x) \geq 0, \ x \in \bar{\Omega}$$

The parameter ε takes arbitrary values in the half-open interval $(0, 1]$.
In addition, it is assumed that the function $g(x, u)$ satisfies the condition

(2.2) $$\frac{\partial}{\partial u}g(x, u) \geq g_0 > 0, \ \ (x, u) \in \bar{\Omega} \times [-M_1, M_1]$$

where M_1 is a sufficiently large number. Throughout this paper, M and m (sometimes subscripted) denote positive constants that are independent of ε, and in the case of discrete problems also independent of the mesh. A function $u \in C^2(\Omega)$, continuous and bounded on $\bar{\Omega}$, is a solution of the boundary value problem (2.1) if it satisfies (2.1) on Ω.

We shall find it useful later to use the formal notation,

$$\mathcal{L}\Big(f(x, w)\Big)u(x) = \Big(\varepsilon^2 L_2 + \varepsilon L_1\Big)u(x) - f\Big(x, w(x)\Big).$$

In this notation the operator in (2.1) is given by

$$L u(x) \equiv \mathcal{L}\Big(g(x, u)\Big)u(x).$$

Furthermore

$$\mathcal{L}\Big(g(x,v)\Big)u(x) \equiv \Big\{\varepsilon^2 L_2 + \varepsilon L_1\Big\}u(x) - g\Big(x,v(x)\Big),$$

where $v = v(x)$, $v \in C^1(\Omega)$. The following lemma is proved in [9, Ch. 4, §8.1].

LEMMA 2.1.   *Let $w_i \in C^2(\Omega) \cap C^0(\bar{\Omega})$ for $i = 1,2$ and $x \in \bar{\Omega}$. Suppose that the inequalities*

$$\mathcal{L}\Big(g(x,u)\Big)w_1(x) \geq Lu(x) \geq \mathcal{L}\Big(g(x,u)\Big)w_2(x)$$

hold for all x such that

$$w_1(x) > u(x) > w_2(x).$$

Then, for all $x \in \bar{\Omega}$, the function u satisfies the following inequalities

$$\min\Big\{w_1(x), -\mathrm{osc}[w_1|_\Omega] + \min_{\partial\Omega}\phi\Big\} \leq u(x) \leq \max\Big\{w_2(x), \mathrm{osc}[w_2|_\Omega] + \max_{\partial\Omega}\phi\Big\},$$

where $\mathrm{osc}[v|_\Omega]$ is the oscillation of $v(x)$ on Ω, that is the difference between essential $\max_\Omega v(x)$ and essential $\min_\Omega v(x)$.

Using this lemma it can be shown that

$$(2.3) \qquad\qquad\qquad |u(x)| \leq M_2, \ x \in \bar{\Omega},$$

where

$$M_2 = \max\left[g_0^{-1}\max_{\bar{\Omega}}|g(x,0)|, \max_{\partial\Omega}|\phi(x)|\right].$$

For the solution of a semilinear boundary value problem to be unique, it is necessary that, for each value of ε, $\max_{\bar{\Omega}}|u|$ and $\max_{\bar{\Omega}}|\nabla u|$ are bounded see [9, Ch. 4]. Since the operator containing second order derivatives can be written in divergence form, the estimate of $|\nabla u|$ depends only on $\max_{\bar{\Omega}}|u|$, ε, a_0, a^0 and on the distance to $\partial\Omega$. In a neighbourhood of $\partial\Omega$ it depends also on $|\phi|_\Omega^{(2)}$ [9, Ch. 4, Theorem 4.1].

Assume that the data of Problem (2.1) satisfy

$$(2.4) \qquad\qquad a_{rs}, b_r, c, b_r^1, c^1 \in C^{\ell-1+\lambda}(\bar{\Omega}), \ g \in C^{\ell-1+\lambda}(\bar{\Omega} \times [-M_2, M_2]),$$

$$\phi \in C^{\ell+\lambda}(\partial\Omega), \ \ r, s = 1, ..., n, \ell \geq 2, \lambda > 0$$

Then $u \in C^{\ell+\lambda}(\bar{\Omega})$ (see [9]).

We are now in a position to state the following bounds on $u(x)$ and its derivatives:

THEOREM 2.2.   *[13, Theorem 4.1, pg. 110] Suppose that the data of Problem (2.1) satisfy the bound (2.2) and the smoothness conditions (2.4). Then, for $\varepsilon \in (0,1]$,*

$$|u(x)| \leq M_2, \ x \in \bar{\Omega} \quad and$$

$$(2.5) \qquad \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} u(x) \right| \leq M \varepsilon^{-|\alpha|}, |\alpha| \leq \ell, x \in \bar{\Omega}, |\alpha| = \alpha_1 + \dots \alpha_n,$$

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *and* $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

In what follows, more precise information on the behaviour of the solution of (2.1) is needed. To obtain this, the solution is written in the form

$$(2.6) \qquad u(x) = U(x) + V(x), \; x \in \bar{\Omega}$$

where U and V are the regular and singular parts respectively of the solution u of (2.1). Estimates of U and V are now derived separately.

The function U is the solution of the equation $L\big(U(x)\big) = 0, x \in \Omega$ with boundary values U_0 on $\partial\Omega$. On $\bar{\Omega}$, U_0 is the solution of the equation $g\big(x, U_0(x)\big) = 0$. Under the condition

$$(2.7) \qquad g \in C'^{\ell_1}(\bar{\Omega} \times [-M_3, M_3]), \; \ell_1 \geq 1,$$

where M_3 is a sufficiently large number, U_0 satisfies

$$(2.8) \qquad \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} U_0(x) \right| \leq M, \;\; x \in \bar{\Omega}, \; |\alpha| \leq \ell_1.$$

On $\bar{\Omega}$, U can be written as a sum of functions $U = U_0 + v$, where v is the solution of

$$(2.9) \qquad \mathcal{L}\big(\hat{g}(x, U, U_0)\big) v(x) = F_1\big(x, U_0\big), \; x \in \Omega, \; v(x) = 0, \; x \in \partial\Omega$$

where

$$\hat{g}_u(x, v_1, v_2) \equiv \int_0^1 \frac{\partial}{\partial u} g(x, \nu v_1 + (1 - \nu)v_2) d\nu,$$

and

$$F_1\big(x, v\big) \equiv - \left\{ \varepsilon^2 L_2 + \varepsilon L_1 \right\} v(x).$$

Under condition (2.7), it can be shown that v satisfies

$$(2.10) \qquad \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} v(x) \right| \leq M \varepsilon^{1-|\alpha|}, \; x \in \bar{\Omega}, \; |\alpha| \leq \ell_1 - 2$$

which is similar to (2.5). From inequalities (2.8) and (2.10), the required bounds on the derivatives of U are obtained:

$$(2.11) \qquad \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} U(x) \right| \leq M \varepsilon^{1-|\alpha|}, \ x \in \bar{\Omega}, \ |\alpha| \leq \ell_1 - 2.$$

The function V is the solution of

$$(2.12) \qquad \mathcal{L}\Big(\hat{g}(x,u,U)\Big) V(x) \ x \in \Omega, \ V(x) = \phi(x) - U(x), \ x \in \partial\Omega.$$

Let m_1 be a sufficiently small number satisfying

$$(2.13) \qquad g_0 - a^0(m_1)^2 - \max_{\bar{\Omega}} \Big[|b_1(x)| + |b_1^1(x)| \Big] m_1 \geq m_2 g_0,$$

where $m_2 \in (0,1)$ is arbitrarily small. It follows from the maximum principle that

$$(2.14) \qquad |V(x)| \leq M \exp\Big(-m_3 \varepsilon^{-1} d(x)\Big), \ x \in \bar{\Omega},$$

where $m_3 \leq m_1$ and $d(x)$ is the distance from x to $\partial\Omega$.

With the change of variables $\xi = \xi(x), \ \xi_r = \varepsilon^{-1} x_r, r = 1, ..., n$, (2.14) leads to

$$\left| \frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}} \tilde{V}(\xi) \right| \leq M \exp\Big(-m_3 \tilde{d}(\xi)\Big), \ \xi \in \bar{\tilde{\Omega}}.$$

where $\tilde{d}(\xi)$ is the distance from ξ to $\partial\tilde{\Omega}$, and $\tilde{\Omega} = \xi(\Omega)$ is a region of diameter $O(1/\varepsilon)$. From this estimate the inequality

$$(2.15) \qquad \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} V(x) \right| \leq M \varepsilon^{-|\alpha|} \exp\Big(-m_3 \varepsilon^{-1} d(x)\Big), \ x \in \bar{\Omega}$$

follows.

The bounds on the derivatives $\Big(\partial^{|\alpha|}/\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}\Big) V(x)$ for $\alpha_1 < |\alpha|$ are now made more precise. Differentiating the equation in (2.12) with respect to $x_r, \ r = 2, ..., n$, and putting $V^{(r)} = (\partial/\partial x_r)V$, it follows that

$$(2.16) \qquad \mathcal{L}\Big(g_u(x,v)\Big) V^{(r)}(x) = F_2\Big(x,u,U,V\Big), \ x \in \Omega,$$

where F_2 comprises the terms of the derivative of (2.12) not contained in $\mathcal{L}\Big(g_u(x,v)\Big) V^{(r)}(x)$. From (2.11) and (2.15) the inequality

$$\left| F_2\Big(x, u, U, V \Big) \right| \leq M \exp\Big(-m_3 \varepsilon^{-1} d(x) \Big), \ x \in \bar{\Omega}$$

is obtained. Furthermore, similarly to (2.15), the derivatives

$$\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \right) V^{(r)}(x) = \left(\frac{\partial^{|\alpha|+1}}{\partial x_r \partial x^\alpha} \right) V(x)$$

satisfy

$$(2.17) \qquad \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} V^{(r)}(x) \right| \leq M \left[\varepsilon^{-\alpha_1} + \varepsilon^{1-|\alpha|} \right] \exp\left(-m_3 \varepsilon^{-1} d(x) \right), \ x \in \bar{\Omega}$$

Thus the following theorem holds

THEOREM 2.3. *Suppose that the data of Problem (2.1) satisfy the bound (2.2) and the smoothness conditions (2.4) and (2.7) with $\ell \geq 5$ and $M_2 = M_3$. Then, for $\varepsilon \in (0, 1]$, the bounds (2.11) and (2.17), with $0 \leq |\alpha| \leq 3$, hold for the regular and singular components U, V of the solution u defined in (2.6).*

3. Nonlinear finite difference methods. Using the results of Theorem 2.2 finite difference methods are now constructed for Problem (2.1). On $\bar{\Omega}$ the mesh

$$(3.1) \qquad \bar{\Omega}_N = \bar{\omega}_1 \times \omega_2 \times \ldots \times \omega_n$$

is introduced, where $\bar{\omega}_1$ is some mesh on the interval $[0, d]$ of the x_1-axis and ω_r is a uniform mesh on the x_r-axis with step size h_r, $r = 2, \ldots, n$. Define $h_1^i = x_1^i - x_1^{i-1}$, x_1^{i-1}, $x_1^i \in \bar{\omega}_1$, $h_1 = \max_i h_1^i$, $h = \max_r h_r$, $r = 1, \ldots, n$. Let N_1 denote the number of nodes in the mesh ω_1, $N_r = 1/h_r$, $r = 2, \ldots, n$, $N = \min_r N_r$, $r = 1, \ldots, n$, and $M = Nh < \infty$. It is assumed hereafter that

$$(3.2) \qquad a_{1,r}(x) \equiv 0, \ x \in \bar{\Omega}, \ r = 2, \ldots, n.$$

Hence no mixed finite differences involving partial differences in the x_1 direction occur.

On the uniform meshes ω_r, $r = 2, \ldots, n$, the following first order finite differences are defined

$$D_r^+ z(x) = \Big(z(x + e_r h_r) - z(x) \Big)/h_r,$$
$$D_r^- z(x) = \Big(z(x) - z(x - e_r h_r) \Big)/h_r,$$

where e_r is the unit vector along the x_r-axis. The following second order finite differences are also required on these meshes

$$\delta_r^2 z(x) = (D_r^+ - D_r^-)z(x)/h_r,$$
$$D_r^+ D_s^+ z(x) = D_r^+ \left(D_s^+ z(x) \right),$$
$$D_r^+ D_s^- z(x) = D_r^+ \left(D_s^- z(x) \right),$$
$$D_r^- D_s^+ z(x) = D_r^- \left(D_s^+ z(x) \right),$$
$$D_r^- D_s^- z(x) = D_r^- \left(D_s^- z(x) \right).$$

On the non-uniform mesh ω_1 first and second order finite differences, for each $x = x^i = \left(x_1^i, x_2, ..., x_n \right) \in \Omega_N$, are defined by

$$D_1^+ z(x) = \left(z(x^{i+1}) - z(x^i) \right) / h_1^i$$
$$D_1^- z(x) = \left(z(x^i) - z(x^{i-1}) \right) / h_1^{i-1}$$
$$\delta_1^2 z(x) = 2 \left(D_1^+ - D_1^- \right) z(x) / \left(h_1^i + h_1^{i-1} \right)$$

The finite difference method for Problem 2.1, on the mesh $\bar{\Omega}_N$, is then defined by

(3.3) $$L^h u_N(x) = 0, \ x \in \Omega_N, \ u_N(x) = \phi(x), \ x \in \partial \Omega_N$$

where $\Omega_N = \Omega \cap \bar{\Omega}_N$ and

$$L^h z(x) \equiv \left\{ \varepsilon^2 L_2^h + \varepsilon L_1^h \right\} z(x) - g(x, z(x))$$

Here

$$L_2^h \equiv \sum_{r=1}^{n} a_{rr}(x) \delta_r^2 + \sum_{r,s=2, r \neq s}^{n} 2^{-1} \left[a_{rs}^+(x)(D_r^+ D_s^+ + D_r^- D_s^-) \right.$$

(3.4) $$\left. + a_{rs}^-(x)(D_r^+ D_s^- + D_r^- D_s^+) \right] + \sum_{r=1}^{n} \left[b_r^+(x) D_r^+ + b_r^- D_r^- \right] - c(x)$$

$$L_1^h \equiv \sum_{r=1}^{n} \left[b_r^{1+}(x) D_r^+ + b_r^{1-}(x) D_r^- \right] - c^1(x)$$

where $v^+(x) = \left(v(x) + |v(x)| \right)/2$ and $v^-(x) = \left(v(x) - |v(x)| \right)/2$.

By analogy with $\mathcal{L}$, we define the formal notation

$$\mathcal{L}^h\Big(f(x,w)\Big)z(x) = (\varepsilon^2 L_2^h + \varepsilon L_1^h)z(x) - f(x,w(x)).$$

and it is clear that

$$L^h z(x) = \mathcal{L}^h(g(x,z))z(x).$$

Under the assumptions (3.2) and the additional assumption

(3.5) $\qquad \dfrac{|a_{rs}(x)|}{h_r h_s} < \dfrac{1}{n-1} \min\left\{ \dfrac{a_{rr}(x)}{h_r^2}, \dfrac{a_{ss}(x)}{h_s^2} \right\}, \ x \in \bar{\Omega}, \ r,s = 1, ..., n, \ r \neq s$

the operator L_2^h is monotone on Ω_N for any distribution of nodes in $\bar{\omega}_1$.

The finite difference method (3.3) is now examined. The error $w(x) = u_N(x) - u(x)$, $x \in \bar{\Omega}_N$, satisfies

(3.6) $\qquad \mathcal{L}^h\Big(\hat{g}_u(x, u_N, u)\Big)w(x) = F_1^h\Big(x, u\Big), \ x \in \Omega_N, \ w(x) = 0, \ x \in \partial\Omega_N$

where

$$F_1^h\Big(x, u\Big) \equiv \left\{ \varepsilon^2 \left(L_2^h - L_2 \right) + \varepsilon \left(L_1^h - L_1 \right) \right\} u(x)$$

and $\hat{g}$ is as in (2.9). The operator $\mathcal{L}^h\Big(\hat{g}_u(x, u_N, u)\Big)$ is monotone on $\bar{\Omega}_N$, for any distribution of nodes in $\bar{\omega}_1$, provided that conditions (3.2) and (3.5) are fulfilled.

However, these conditions are not sufficient to guarantee that the finite difference method (3.3) is ε-uniform. To obtain an ε-uniform numerical method a special mesh $\bar{\Omega}_N^\star$, which is refined or condensed in the boundary layer, is required. A piecewise uniform mesh having this property is now constructed. For $r = 2, ..., n$ the uniform mesh ω_r is used, as before. In the interval $[0, d]$ of the x_1-axis a special mesh $\bar{\omega}_1^\star$ is defined as follows. The interval $[0, d]$ is partitioned into three subintervals $[0, \sigma], [\sigma, d-\sigma], [d-\sigma, d]$ where $\sigma \equiv \min\left[\frac{1}{4}d, M\varepsilon \ln N_1\right]$, $M > \left[a^0(g_0)^{-1}\right]^{\frac{1}{2}}$. A uniform mesh is constructed using $N_1/4$ points in each of $[0, \sigma], [d-\sigma, d]$ and $N_1/2$ points in $[\sigma, d-\sigma]$.

Using the technique in [23] it is possible to prove that the finite difference operator L^h in (3.3) on the mesh $\bar{\Omega}_N^\star$ gives an ε-uniform method. In particular the following ε-uniform error estimate holds

(3.7) $\qquad\qquad\qquad |u(x) - u_N(x)| \leq M N^{-1} \ln N, \ x \in \bar{\Omega}_N^\star.$

The above results are summarized in the following theorem :

THEOREM 3.1. *Suppose that the data of Problem 2.4 satisfy the conditions (2.2), (2.4) and (2.7) and the assumptions (3.2) and (3.5). Suppose, also, that with $\varepsilon \in (0, 1]$ and $0 \leq |\alpha| \leq 3$ the bound (2.5) holds for the solution u and its derivatives and that the bounds (2.11), (2.17) hold for its regular and singular components U, V respectively. Then, the numerical method comprising the nonlinear finite difference operator L^h in (3.3) on the mesh $\bar{\Omega}_N^\star$ is ε-uniform and the ε-uniform error estimate (3.7) holds.*

This theorem shows that in principle the problem is solved. In practice, however we are still faced with the problem of solving a nonlinear system of algebraic equations. This is dealt with in the next section.

4. Iterative finite difference methods. Throughout this section it is assumed that (3.2) is satisfied. Analogously to $\mathcal{L}^h$ we define

$$\mathcal{L}_t^h\Big(f(x,w),p\Big)z(x,t) \equiv \Big(\varepsilon^2 L_2^h + \varepsilon L_1^h - pD_t^+\Big)z(x,t) - f\left(x, w(x,t)\right).$$

where the positive parameter t, which plays the part of a continuation parameter, will be determined later, and D_t^+ is defined by

$$D_t^+ z(x,t_j) = (t_j - t_{j-1})^{-1}(z(x,t_j) - z(x,t_{j-1})).$$

We now consider the iterative finite difference method

(4.1) $\mathcal{L}_t^h\Big(g(x,u_o),p\Big)u_N(x,t) = 0, \ (x,t) \in G_N^*, \ u_N(x,t) = \phi(x,t), \ (x,t) \in \partial G_N^*,$

where $u_o(x,t)$ is the *old* value of u_N defined by $u_o(x,t_j) = u_N(x,t_{j-1})$, and where the mesh G_N^* is defined by

(4.2) $$\bar{G}_N^* = \bar{\Omega}_N^* \times \bar{\omega}_t$$

with $\bar{\omega}_t$ being some mesh having $K+1$ points in the interval $[0,T]$, with $t_0 = 0$ and $t_K = T$. The boundary of this mesh is defined by

$$\partial G_N^* = \partial G_{0,N}^* \cup \partial G_{d,N}^* \cup \partial G_N^{*,0}$$

where

$$\partial G_N^{*,0} = \{(x,t) \in \bar{G}_N^*, t = 0\}, \qquad \partial G_{0,N}^* = \{(x,t) \in \bar{G}_N^*, x_1 = 0\}$$

and

$$\partial G_{d,N}^* = \{(x,t) \in \bar{G}_N^*, x_1 = d\}$$

The following boundary and initial conditions are imposed

$$\phi(x,t) = \phi(x), (x,t) \in \partial G_{0,N}^* \cup \partial G_{d,N}^*, \quad \phi(x,t) = \phi^0(x), (x,t) \in \partial G_N^{*,0},$$

where $\phi^0(x)$ is a bounded function and, in general, $\phi \neq \phi^0$.

The finite difference method (4.1) is implicit. At each time level, the function $u_N(x,t)$ is the solution of a discrete linear boundary value problem. Note that the operator $\varepsilon^2 L_2^h + \varepsilon L_1^h - pD_t^+$ is monotone on the mesh G_N^* for any distribution of the mesh points in $\bar{\omega}_t$.

To investigate the convergence of this method, note that the error function $w(x,t) = u_N(x,t) - u(x), (x,t) \in \overline{G}_N^*$ is the solution of the discrete problem

(4.3) $\mathcal{L}_t^h\Big(\hat{g}_u(x,u_0,u), P(x,t,u_o(x,t),u(x))\Big)w(x,t) = F_1^h(x,u), \ (x,t) \in G_N^*,$

$$w(x,t) = \phi(x,t) - u(x) = \phi^0(x) - u(x), \ (x,t) \in \partial G_N^{*,0},$$

$$w(x,t) = 0, \ (x,t) \in \partial G^*_{0,N} \cup \partial G^*_{d,N},$$

where F_1^h is as in (3.6), $P(x,t,v_1,v_2) = p - \tau_j \hat{g}_u(x,v_1,v_2)$, and $\tau_j = t_j - t_{j-1}$. The operator $\mathcal{L}_t^h \left(\hat{g}_u(x,u_0,u), P(x,t,u_o(x,t),u(x)) \right)$ is monotone if

$$\min_{\bar{G}^*_N} P(x,t,u_o(x,t),u(x)) \geq 0.$$

The error $w(x,t)$ is estimated using a discrete maximum principle (see for example [23]). This yields the discrete analogue of (2.3)

$$|u_N(x,t)| \leq \max \left[g_0^{-1} \max_{\bar{\Omega}^*} |g(x,0)|, \max_{\partial G^*_N} |\phi(x,t)| \right].$$

From this inequality and (2.3), we obtain

(4.4) $$|\hat{g}_u(x,u_o(x,t),u(x))| \leq M, \ (x,t) \in \overline{G}^*_N.$$

The mesh $\bar{\omega}_t$ is chosen to be uniform. Then $\tau_j = h_t$ for all j. The parameter p is chosen sufficiently large or the time step h_t is chosen sufficiently small so that

(4.5) $$P(x,t,u_o(x,t),u(x)) = p - h_t \hat{g}_u(x,u_o(x,t),u(x)) \geq p_0 > 0, \ (x,t) \in G^*_N.$$

Using a maximum principle, the following error estimate is obtained from (4.3)

(4.6) $$|w(x,t)| \leq M(N^{-1} \ln N + e^{-m_4 t}), (x,t) \in \overline{G}^*_N,$$

if the constant m_4 is sufficiently small to ensure that

$$[\min_{\bar{G}^*_N} \hat{g}_u(x,u_o(x,t),u(x))] - h_t^{-1}(exp(m_4 h_t) - 1) \max_{\bar{G}^*_N} P(x,t,u_o(x,t),u(x))$$

(4.7) $$\geq g_0 - h_t^{-1}(exp(m_4 h_t) - 1)(p + h_t M) \geq 0.$$

Thus we have

(4.8) $$|u(x) - u_N(x,t)| \leq M(N^{-1} \ln N + q^{t/h_t}), (x,t) \in G^*_N,$$

where $q = exp(-m_4 h_t)$ and hence

(4.9) $$|u(x) - u_N(x,T_0)| \leq M(N^{-1} \ln N), x \in \bar{\Omega}^*_N,$$

where

$$T_0 \geq m_4^{-1} \ln(N(\ln N)^{-1}).$$

The number of iterations required is

$$(4.10) \qquad K = T_0/h_t = \frac{\ln(N(\ln N)^{-1})}{h_t m_4} \le M \ln N.$$

The above results are summarized in the following theorem.

THEOREM 4.1. *Assume that the estimates (2.11), (2.17) for $|\alpha| \le 3$ are satisfied by the solution of problem (2.1). Assume also that the conditions (2.3), (3.2) and (3.5) are satisfied. Then the solutions of (4.1) converge ε-uniformly to the solution of Problem 2.1. The error estimate (4.9) is valid. The number of iterations required is given in (4.10).*

5. Numerical Results. In this section numerical results are given for the following one dimensional examples of the semilinear problem (2.1)

$$(5.1) \qquad \varepsilon^2 \frac{d^2}{dx^2} u(x) - g(x, u(x)) = 0, \; x \in (0,1),$$

$$u(0) = A \quad u(1) = B$$

where either

$$(5.2) \qquad g(x, u) = u - u^3$$

or

$$(5.3) \qquad g(x, u) = u - u^2.$$

Various choices of A and B in the boundary conditions are examined.

The central difference operator

$$L^h u_N \equiv \varepsilon^2 \delta_x^2 u_N - g(x, u_N) = 0, u_N(0) = u(0) \; u_N(1) = u(1)$$

on a special piecewise-uniform mesh $\overline{\omega}_1^*$ is used. The mesh $\overline{\omega}_1^*$ is constructed by dividing the interval into three subintervals $[0, \sigma], [\sigma, 1 - \sigma], [1 - \sigma, 1]$ where

$$\sigma \equiv \min\{0.25, \varepsilon \ln N\}$$

A uniform mesh is used on each subinterval, taking $N/4, N/2$, and $N/4$ points in the respective intervals.

The nonlinear finite difference method $\{L^h, \omega_1^*\}$ is linearized using the continuation method (with the parameter $p = 1$) given in §4. That is

$$L_t^h u_N \equiv \varepsilon^2 \delta_x^2 u_N(x, t_j) - g(x, u_N(x, t_{j-1})) - D_t^+ u_N(x, t_j) = 0, \; j = 1, \cdots K$$

$$u_N(0, t_j) = u(0), \; u_N(1, t_j) = u(1) \text{ for all } j,$$

$$u_N(x,0) = u_{init}(x)$$

Various starting values $u_{init}(x)$ are chosen. The number of iterations K and the choice of uniform time step $h_t = t_j - t_{j-1}$ are discussed below. With the definition

$$(5.4) \qquad e(j) \equiv \max_{1 \leq i \leq N} |u_N(x_i, t_j) - u_N(x_i, t_{j-1})|/h_t, \text{ for } j = 1, 2, \cdots, K,$$

the time step h_t is chosen sufficiently small so that

$$(5.5) \qquad e(j) \leq e(j-1), \text{ for } 1 < j \leq K$$

and the number of iterations K is chosen such that

$$(5.6) \qquad e(K) \leq \text{ TOL}$$

where TOL is some prescribed small tolerance.

The numerical solution is obtained as follows:

Start with $h_t = 0.0625$. If, at some value of j, (5.5) is not satisfied then halve the time step until (5.5) is satisfied. Continue the iterations until either (5.6) is satisfied or until $K = 90$. If (5.6) is not satisfied, then repeat the entire process starting with $h_t = 0.03125$.

The resulting values of $u_N(x, K)$ are taken as approximations to the solution of the continuous problem (5.1).

The problem is solved on a sequence of meshes, with $N = 8, 16, 32, 64, 128, 256, 512, 1024$ and for $\varepsilon = 2^{-n}, n = 1, 2, \cdots j_{red}$, where j_{red} is chosen so that ε is a value at which the rate of convergence stabilizes, which normally occurs when, to machine accuracy, we are solving the reduced problem.

The errors $|u_N(x_i, K) - u(x_i)|$ are approximated on each mesh for successive values of ε by $e_{\varepsilon,N}(i) = |u_N(x_i, K) - u^I(x_i, K)|$, where $u^I(x, K)$ is defined by linear interpolation on each subinterval $[y_{j-1}, y_j]$ by

$$u^I(x, K) = u^*(y_{j-1}, K) + (u^*(y_j, K) - u^*(y_{j-1}, K))\frac{x - y_{j-1}}{y_j - y_{j-1}}, \ 1 \leq j \leq 1024$$

where the nodal values $\{u^*(y_j, K)\}_{j=0}^{1024}$ are obtained from the solution of the finite difference method $\{L_t^h, \omega_1^* \times \omega_t\}$ with $N = 1024$. For each ε and each N the maximum nodal error is approximated by

$$E_{\varepsilon,N} = \max_i e_{\varepsilon,N}(i)$$

For each N, the ε-uniform maximum nodal error is approximated by

$$E_N = \max_\varepsilon E_{\varepsilon,N}$$

In what follows all calculations were carried out in double-precision FORTRAN 77 on an Hewlett-Packard/Apollo 730.

A numerical method for solving (5.1) is ε-uniform of order p on the mesh $\Omega_N = \{x_i, i = 0, 1, \ldots, N\}$ if

$$\sup_{0 < \varepsilon \le 1} \max_{\Omega_N} |u(x) - u_N(x, K)| \le CN^{-p},$$

where u is the solution of (5.1), u_N is the numerical approximation to u, C and $p > 0$ are independent of ε and N. An approximation to p, the ε-uniform rate of convergence, was determined using a variation of the double mesh method described in [5]. This involves calculating the double mesh error

$$D_{\varepsilon,N} = \max_{\Omega_N} |u_N(x_i, K) - u_{2N}^I(x_i, K)|,$$

which is the difference between the values of the solution on a mesh of N points and the interpolated value for the solution, at the same point, on a mesh of $2N$ points. For each value of N the quantities

$$D_N = \max_\varepsilon D_{\varepsilon,N}, \quad p_N = \log_2\left(\frac{D_N}{D_{2N}}\right),$$

are computed. The values of p_N are the approximations to p.

Tables 5.1-5.6 and 5.8 present numerical results for centered differences on the special mesh G_n^* for the problem (5.1),(5.2). Table 5.1 gives the errors $E_{\varepsilon,N}$ and E_N for problem (5.1),(5.2) with boundary conditions $u(0) = 1$, $u(1) = 1$ and initial guess $u_{init} = 0$. Tables 5.2-5.6 present summary results for problem (5.1),(5.2), for various boundary values and initial guesses.   In all of these examples, it is clear that the central difference scheme on the special mesh G_N^* yields an ε-uniform method. It should be noted that, in the case of the numerical examples presented here, $g(u, x)$ does not satisfy the condition (2.2), and thus the method may in practice be ε-uniformly convergent even in cases where this condition is not satisfied. Moreover, the computed rate of ε-uniform convergence is approximately 1, independent of the choice of the initial guess and of the boundary conditions. This indicates that the theoretical result given in (4.10) may be a conservative estimate of the rate of ε-uniform convergence. To verify this further we give in Table 5.7 below the local theoretical rates of ε-uniform convergence, corresponding to those in Tables 5.2-5.6. It is clear that the actual computed rates in Tables 5.2-5.6 are substantially better, in all cases than the theoretical rates.

We should also remark that, as shown in [7, pp. 124-127], there are multiple solutions to this problem for the boundary conditions, $u(0) = 1$, $u(1) = 1$. The stable solution has boundary layers at both end-points and, for sufficiently small ε, approaches the reduced solution $u(x) \equiv 0$ in the interior. On the other hand, it is clear that $u(x) \equiv 1$ is also a solution, in this case an unstable one. We remark that for all the initial guesses discussed above, the numerical solutions converge to the stable solution, and furthermore the computed rate of ε-uniform convergence obtained is the same. With the initial guess $u_{init} = 1$ however, it converges to the unstable solution after one iteration as would be expected. On the other hand, it is of some interest, that for initial guesses as close to 1 as $u_{init} = .98$, the solutions converge to the stable solution, with essentially the same ε-uniform rate as the above examples. The results for this case are given in Table 5.8.

Boundary Conditions: $u(0) = 1$, $u(1) = 1$						
Initial Guess : $u_{init} = 0$						
	Number of Mesh Points N						
ε	8	16	32	64	128	256	512
1/ 2	.000055	.000013	.000003	.000001	.000000	.000000	.000000
1/ 4	.000037	.000634	.000057	.000014	.000003	.000001	.000000
1/ 8	.006885	.001755	.000440	.000110	.000027	.000006	.000001
1/ 16	.006907	.002070	.000444	.000111	.000027	.000007	.000001
1/ 32	.011231	.003309	.000878	.000219	.000054	.000013	.000003
1/ 64	.012372	.006810	.001777	.000446	.000111	.000026	.000005
1/ 128	.016902	.011381	.003402	.000899	.000223	.000053	.000011
1/ 256	.033105	.011972	.005121	.001780	.000443	.000106	.000021
1/ 512	.056008	.012978	.005182	.001950	.000673	.000226	.000043
1/ 1024	.077146	.021205	.005260	.001952	.000677	.000230	.000081
1/ 2048	.094695	.031616	.007446	.001971	.000679	.000230	.000081
1/ 4096	.108414	.041192	.012061	.002426	.000682	.000230	.000081
1/ 8192	.118767	.049207	.016901	.004247	.000736	.000231	.000081
1/ 16384	.126412	.055532	.021361	.006463	.001373	.000231	.000081
1/ 32768	.131978	.060343	.025123	.008769	.002272	.000409	.000082
1/ 65536	.135994	.063915	.028117	.010900	.003350	.000728	.000108
1/ 131072	.138874	.066527	.030410	.012707	.004469	.001170	.000202
1/ 262144	.140930	.068418	.032123	.014153	.005503	.001694	.000348
1/ 524288	.142394	.069775	.033379	.015264	.006380	.002232	.000544
1/ 1048576	.143433	.070745	.034289	.016095	.007080	.002723	.000767
1/ 2097152	.144171	.071436	.034943	.016704	.007614	.003131	.000985
1/ 4194304	.144693	.071926	.035410	.017143	.008010	.003450	.001173
1/ 8388608	.145062	.072274	.035742	.017458	.008297	.003688	.001321
1/ 16777216	.145323	.072519	.035977	.017681	.008503	.003861	.001433
1/ 33554432	.145508	.072693	.036143	.017840	.008649	.003985	.001514
E_N	.145508	.072693	.036143	.017840	.008649	.003985	.001514

TABLE 5.1

Errors $E_{\varepsilon,N}$ and E_N for problem (5.1),(5.2)

Boundary Conditions: $u(0) = 1$, $u(1) = 1$						
Initial Guess : $u_{init} = 0$						
N	8	16	32	64	128	256
E_N	.145508	.072693	.036143	.017840	0.08649	.003985
p_N	0.94	1.02	1.02	1.03	1.04	1.09

TABLE 5.2

Maximum errors E_N and computed rates of convergence p_N

The final tables 5.9-13 give the corresponding numerical results for the problem (5.1),(5.3). Again, it may be observed that for all the initial guesses discussed, convergence to the true

Boundary Conditions: $u(0) = 1$, $u(1) = 1$					
Initial Guess : $u_{init} = .5$					
N	8	16	32	64	128	256
E_N	.145674	.072779	.036186	.017861	.008660	.003990
p_N	.94	1.02	1.02	1.03	1.04	1.09

TABLE 5.3

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = .5$, $u(1) = .7$					
Initial Guess : $u_{init} = 0$					
N	8	16	32	64	128	256
E_N	.093222	.046517	.023123	.011413	.005533	.002549
p_N	.97	1.03	1.02	1.03	1.04	1.09

TABLE 5.4

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = .5$, $u(1) = .7$					
Initial Guess : $u_{init} = .5$					
N	8	16	32	64	128	256
E_N	.093268	.046541	.023135	.011419	.005536	.002551
p_N	.97	1.03	1.02	1.03	1.04	1.09

TABLE 5.5

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = .5$, $u(1) = .7$					
Initial Guess : $u_{init} = u(0) + (u(1) - u(0))x$					
N	8	16	32	64	128	256
E_N	.093257	.046535	.023132	.011417	.005536	.002550
p_N	.97	1.03	1.02	1.03	1.04	1.09

TABLE 5.6

Maximum errors E_N and computed rates of convergence p_N

N	8	16	32	64	128	256
p_N	.4150	.5850	.6781	.7370	7776	.8074

TABLE 5.7

Theoretical rates of convergence p_N from (4.10)

solution occurs and the computed rate of ε-uniform convergence of the scheme is essentially the same, independent of the initial guess.

Boundary Conditions: $u(0) = 1$, $u(1) = 1$					
Initial Guess : $u_{init} = .98$					
N	8	16	32	64	128	256
E_N	.145661	.072771	.036182	.017859	.008659	.003989
p_N	.94	1.02	1.02	1.03	1.04	1.09

TABLE 5.8

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = 1$, $u(1) = 1$					
Initial Guess : $u_{init} = 0$					
N	8	16	32	64	128	256
E_N	.097937	.048050	.023691	.011648	.005637	.002595
p_N	1.06	1.08	1.04	1.04	1.05	1.09

TABLE 5.9

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = 1$, $u(1) = 1$					
Initial Guess : $u_{init} = .5$					
N	8	16	32	64	128	256
E_N	.097873	.048014	.023672	.011638	.005632	.002593
p_N	1.06	1.08	1.04	1.04	1.05	1.09

TABLE 5.10

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = .5$, $u(1) = .7$					
Initial Guess : $u_{init} = 0$					
N	8	16	32	64	128	256
E_N	.073090	.036008	.017791	.008755	.004239	.001952
p_N	1.05	1.07	1.04	1.03	1.05	1.09

TABLE 5.11

Maximum errors E_N and computed rates of convergence p_N

Boundary Conditions: $u(0) = .5$, $u(1) = .7$					
Initial Guess : $u_{init} = .5$					
N	8	16	32	64	128	256
E_N	.073034	.035977	.017774	.008747	.004235	.001950
p_N	1.05	1.07	1.04	1.03	1.05	1.09

TABLE 5.12

Maximum errors E_N and computed rates of convergence p_N

REFERENCES

| Boundary Conditions: $u(0) = .5$, $u(1) = .7$ | | | | | |
| Initial Guess : $u_{init} = u(0) + (u(1) - u(0))x$ | | | | | |
N	8	16	32	64	128	256
E_N	.073063	.035993	.017782	.008751	.004237	.001951
p_N	1.05	1.07	1.04	1.03	1.05	1.09

TABLE 5.13

Maximum errors E_N and computed rates of convergence p_N

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